Some recent results on regularity criteria for weak solutions of the Navier-Stokes equations

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1. Regularity up to the boundary with no–slip boundary condition

1.1. Regularity up to the whole boundary under Serrin's condition

Theorem 1 (regularity up to the boundary). *Let Ω be a domain in $\mathbb{R}^3$ with the uniformly C^2–boundary $\partial\Omega$. Let $\mathbf{v}$ be a weak solution to the Navier–Stokes initial–boundary value problem with the no–slip boundary condition $\mathbf{v} = \mathbf{0}$ (on $\partial\Omega \times (0,T)$) and with $\mathbf{f} \equiv \mathbf{0}$. Suppose, in addition, that*

$$\mathbf{v} \in L^r(0,T;\, \mathbf{L}^s(\Omega)) \quad \textit{for some } r, s \textit{ satisfying } \frac{2}{r} + \frac{3}{s} = 1, \;\; 3 < s \leq \infty.$$

Then $\partial_t^k \mathbf{v} \in L^2(\epsilon, T;\, \mathbf{W}^{2,2}(\Omega))$ for all $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ for any $0 < \epsilon < T$.

If, in addition, $\partial\Omega$ is uniformly of the class C^m then $\partial_t^k \mathbf{v} \in L^2(\epsilon, T;\, \mathbf{W}^{m,2}(\Omega))$ for all $k \in \mathbb{N}_0$.

Remarks.

- The statement on the regularity of solution $\mathbf{v}$ holds up to the initial time $t = 0$ if the initial velocity $\mathbf{v}_0$ is "smooth".
- The theorem was successively proved by **Leray** (1934) (the case $\Omega = \mathbb{R}^3$), **Sohr** (1983) (the case of bounded $\partial\Omega$), **von Wahl** (1983, 1986), **Giga** (1986).

1.2. Regularity up to a part of the boundary under Serrin's condition

- **S. Takahashi (1992, 1994):** assumed that $(\mathbf{x}_0, t_0) \in \partial\Omega \times (0, T)$,

$$\mathbf{v} \in L^q(t_0 - r^2, t_0 + r^2; \mathbf{L}^s(B_r(\mathbf{x}_0) \cap \Omega)), \qquad q, s \in (0, \infty), \ \ \frac{3}{s} + \frac{2}{q} \le 1$$

then $\mathbf{v} \in \mathbf{L}^\infty\big([B_{r_1}(\mathbf{x}_0) \cap \Omega] \times (t_0 - r_1^2, t_0 + r_1^2)\big)$ for any $r_1 \in (0, r)$ provided that $B_r(\mathbf{x}_0) \cap \partial\Omega$ is a part of a plane.

- **H. J. Choe** (1998): proved that a suitable weak solution is bounded locally near the boundary if it satisfies Serrin's conditions near the boundary and the trace of the pressure is bounded on the boundary.

- **K. Kang** (2004): assumed that Ω is a half–space and has shown that if a weak solution satisfies Serrin's conditions in a neighbourhood of $\partial\Omega$ then it is Hölder–continuous up to the boundary.

- **Z. Skalák** (2005): Ω is a domain in $\mathbb{R}^3$ with a smooth boundary, $T > 0$, $D_r(\mathbf{x}_0) := B_r(\mathbf{x}_0) \cap \Omega$, $\Gamma_r := B_r(\mathbf{x}_0) \cap \partial\Omega$, $Q_r := D_r(\mathbf{x}_0) \times (t_0 - r^2, t_0 + r^2)$

Theorem 2 (Skalák 2005). *Let* $\mathbf{u}$ *be a weak solution of the Navier–Stokes initial–boundary value problem,* $(\mathbf{x}_0, t_0) \in \partial\Omega \times (0, T)$, $r > 0$. *Suppose that* $\mathbf{u} \in L^q(t_0 - r^2, t_0 + r^2;\ \mathbf{L}^p(D_r(\mathbf{x}_0)))$ *for some* $p, q \in (1, \infty)$, *satisfying* $2/q + 3/p = 1$. *Then*

$$\mathbf{u} \in L^\infty(t_0 - \rho^2, t_0 + \rho^2;\ \mathbf{C}^\beta(\overline{D_\rho(\mathbf{x}_0)}\,))$$

for every $\beta \in (0, 1)$ *and* $\rho \in (0, r)$.

Principle of the proof.

a) Localization to the neighbourhood of point $(\mathbf{x}_0, t_0) \in \partial\Omega \times (0,T)$: Let η be a C^∞ cut off function such that $0 \le \eta \le 1$, $\eta = 0$ in $Q_T \smallsetminus Q_{2r/3+\rho/3}$, and $\eta = 1$ in $Q_{r/3+2\rho/3}$.

Put $\mathbf{v} = \eta\mathbf{u} - \mathbf{V}$, where $\mathbf{V}$ is a correction that guarantees the equation $\operatorname{div}\mathbf{v} = 0$. Function $\mathbf{V}$ can be constructed so that

$$\operatorname{supp}\mathbf{V} \subset Q_{2r/3+\rho/3} \cup \big\{\text{the cluster points of } Q_{2r/3+\rho/3} \text{ on } \partial\Omega \times (0,T)\big\}.$$

Function $\mathbf{v}$ satisfies the localized system

$$\partial_t\mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} = -\nabla(\eta\phi) + \nu\Delta\mathbf{v} + \mathbf{h} \quad \text{in } Q_T, \tag{1.1}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

$$\mathbf{v} = \mathbf{0} \quad \text{in } \Omega \times \{0\}, \tag{1.4}$$

where

$$\begin{aligned} \mathbf{h} = &-\nu(\Delta\eta)\mathbf{u} - 2\nu\nabla\eta\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla(\eta\mathbf{u}) - \phi\nabla\eta - \partial_t\mathbf{V} + \nu\Delta\mathbf{V} \\ &- \mathbf{u}\cdot\nabla\mathbf{V} - (\partial_t\eta)\,\mathbf{u}. \end{aligned}$$

b) One can show that $\mathbf{h} \in L^{l'}(0,T;\, \mathbf{L}^l(\Omega))$ for $l' \in (1,2)$, $l \in (\frac{3}{2}, 3)$ such that

$$\frac{2}{l'} + \frac{3}{l} = 3.$$

Here, it is necessary to apply estimates of the weak solution $\mathbf{u}$ and function ϕ derived by Y. Giga and H. Sohr in 1991, and the estimates of function $\mathbf{V}$, following from its construction (see e.g. the book by G. P. Galdi).

c) The next step is the proof and application of a "very technical" lemma on the linearized Navier–Stokes problem

$$\begin{aligned}
\partial_t \mathbf{v} + \mathbf{b} \cdot \nabla \mathbf{v} &= -\nabla\phi + \nu\Delta\mathbf{v} + \mathbf{h} && \text{in } Q_T, && (1.5)\\
\operatorname{div} \mathbf{v} &= 0 && \text{in } Q_T, && (1.6)\\
\mathbf{v} &= \mathbf{0} && \text{on } \partial\Omega \times (0,T), && (1.7)\\
\mathbf{v} &= \mathbf{0} && \text{in } \Omega \times \{0\}. && (1.8)
\end{aligned}$$

Lemma 1. *Let (i)* $1 < p, q < \infty$, $\dfrac{2}{q} + \dfrac{3}{p} = 1$, $\mathbf{b} \in L^q(0,T;\,\mathbf{L}^p(\Omega))$,

(ii) $1 < \theta$, $2 < \theta'$, $\dfrac{2'}{\theta} + \dfrac{3}{\theta} = 3$, $\dfrac{p}{p-1} < \theta < 3$, $\dfrac{1}{\alpha} = \dfrac{1}{\theta} - \dfrac{1}{3}$

(iii) $r,\, r',\, l,\, l' \in (1, \infty)$, $\dfrac{1}{r} = \dfrac{1}{l} - \dfrac{1}{p}$, $\dfrac{1}{r'} = \dfrac{1}{l'} - \dfrac{1}{q}$, $r \geq \theta$, $r' \geq \theta'$,

(iv) $\mathbf{h} \in L^{l'}(0,T;\,\mathbf{L}^l(\Omega))$,

(v) $\mathbf{v} \in L^2(0,T;\,\mathbf{L}^\infty(\Omega)) \cap L^\alpha(0,T;\,\mathbf{L}^{\theta'}(\Omega))$ *with* $\nabla \mathbf{v} \in L^2(Q_T)^9 \cap L^\theta(0,T;\,L^{\theta'}(\Omega)^9)$
be a weak solution of the linearized Navier–Stokes problem (1.5)–(1.8).

Then there exists $\epsilon > 0$ *such that if* $|||\mathbf{b}|||_{q,p} < \epsilon$ *then*

$$|||\nabla \mathbf{v}|||_{r',r} \;\leq\; C\,|||\mathbf{h}|||_{l',l},$$

$$|||\nabla \mathbf{v}|||_{l',m} \;\leq\; C\,|||\mathbf{h}|||_{l',l} \quad \textit{provided that } 1 < l < 3 \textit{ and } \frac{1}{m} = \frac{1}{l} - \frac{1}{3},$$

$$|||\nabla \phi|||_{l',l} + |||\partial_t \mathbf{v}|||_{l',l} \;\leq\; C\,|||\mathbf{h}|||_{l',l}.$$

Applying Lemma 1, one obtains that

$$|||\nabla \mathbf{v}|||_{r',r}, \quad |||\nabla(\eta\phi)|||_{l',l}, \quad |||\nabla \mathbf{v}|||_{l',m}, \quad |||\partial_t \mathbf{v}|||_{l',l} \ \leq \ C\, |||\mathbf{h}|||_{l',l},$$

where r, r', m are as in Lemma 1.

d) Using these estimates of $\mathbf{v}$, and the coincidence of $\mathbf{v}$ with $\mathbf{u}$ in $Q_{r/3+2\rho/3}$, one can consider **a new "smaller" localization** so that the new cut–off function is now supported in $Q_{r/3+2\rho/3}$. Thus, one can improve the information on a new function $\mathbf{h}$ and a new function $\eta\phi$:

$$\mathbf{h} \in L^{l'}(0,T;\, \mathbf{L}^m(\Omega)), \qquad \eta\phi \in L^{l'}(0,T;\, L^m(\Omega)), \qquad \text{where } \frac{1}{m} = \frac{1}{l} - \frac{1}{3}.$$

e) Using the bootstrapping argument (i.e. several further "smaller" localizations), one can finally arrive at

$$\mathbf{h} - \mathbf{u}\cdot\nabla\mathbf{v} \in L^{l'}(0,T;\, \mathbf{L}^m(\Omega))$$

for all m, l' such that

$$3 < m < p, \quad 1 < l' < 2, \quad \frac{2}{l'} + \frac{3}{m} < 2.$$

Then we have

$$\mathbf{v}(t) \;=\; \int_0^t \mathrm{e}^{-A_m(t-s)}\, P_\sigma^m[\mathbf{h}(s) - \mathbf{u}(s)\cdot\nabla\mathbf{v}(s)]\, \mathrm{d}s.$$

If $0 < \alpha < \frac{1}{2}$ then one can choose l' such that $\alpha l'/(l'-1) < 1$ and obtain the estimate

$$\begin{aligned} \|A_m^\alpha \mathbf{v}(t)\|_m \;&\leq\; \int_0^t \|A_m^\alpha\, \mathrm{e}^{-A_m(t-s)}\, P_\sigma^m[\mathbf{h}(s) - \mathbf{u}(s)\cdot\nabla\mathbf{v}(s)]\, \mathrm{d}s \\ &\leq\; \int_0^t \frac{\|\mathbf{h}(s) - \mathbf{u}(s)\cdot\nabla\mathbf{v}(s)\|_m}{(t-s)^\alpha}\, \mathrm{d}s \\ &\leq\; \left(\int_0^t \frac{1}{(t-s)^{\alpha l'/(l'-1)}}\right)^{\frac{l'-1}{l'}} |\!|\!|\mathbf{h} - \mathbf{u}\cdot\nabla\mathbf{v}|\!|\!|_{l',m} \;\leq\; C. \end{aligned}$$

We use the imbedding $D(A_m^\alpha) \hookrightarrow \mathbf{W}^{2\alpha,m}(\Omega)$ (following from the interpolation theory) and the imbedding $\mathbf{W}^{2\alpha,m}(\Omega) \hookrightarrow \mathbf{C}^\beta(\overline{\Omega})$ (for $\beta = 2\alpha - 3/m > 0$).

By a suitable choice of α and m, one can get $\beta < 1-3/p$ arbitrarily close to $1-3/p$.

- **G. A. Seregin (2005):** shows that an $L^{3,\infty}$–weak solution is smooth in a neighbourhood of a flat part of the boundary. Concretely:

  $B_r^+ := B_r(\mathbf{0}) \cap \{x_3 > 0\}, \quad Q_r^+ := B_r^+ \times (-1, 0)$

  Let functions $\mathbf{v}$ *and* p *have these integrability properties:*

$$\mathbf{v} \in L^\infty(-1,0;\ \mathbf{L}^2(B_1^+)) \cap L^2(-1,0;\ \mathbf{W}^{1,2}(B_1^+)),$$
$$\mathbf{v},\ \nabla\mathbf{v},\ \nabla^2\mathbf{v},\ \partial_t\mathbf{v},\ p,\ \nabla p\ \in\ L^{9/8}(-1,0;\ L^{3/2}(B_1^+)).$$

  Let $\mathbf{v}$, p *satisfy the Navier–Stokes equations in* Q_1^+*, and the boundary condition*

$$\mathbf{v} = \mathbf{0} \qquad \textit{for}\ \ |\mathbf{x}| < 1,\ \ x_3 = 0,\ \ -1 < t < 0.$$

  Let, in addition, $\mathbf{v} \in L^\infty(-1,0;\ \mathbf{L}^3(B_1^+))$*.*

  Then $\mathbf{v}$ *is Hölder–continuous in the closure of* $Q_{1/2}^+$*.*

1.3. CKN–type regularity conditions on the boundary

- **G. A. Seregin (2002, 2003):** introduced the notion of a *boundary–suitable weak solution,* as a suitable weak solution that possesses the regularity

$$\mathbf{v} \in L^\infty(0,T;\, \mathbf{L}^2(\Omega)) \cap L^2(0,T;\, \mathbf{W}^{1,2}(\Omega)), \quad p \in L^{3/2}(Q_T),$$
$$\nabla^2\mathbf{v},\ \nabla p \in L^{9/8}(0,T;\, L^{3/2}(\Omega)).$$

Seregin gives reasons for this definition: if $\mathbf{f} \in \mathbf{L}^2(Q_T)$ then one can prove that Hopf's weak solution has the same properties on any time interval (δ, T), $\delta > 0$ (referring to a previous paper by Ladyzhenskaya and Seregin).

Furthe, Seregin has proven the existence of a suitable weak solution with the same integrability properties for $t \in (\delta, T)$ for any $\delta > 0$.

Finally, he derives a condition for the local Hölder continuity of a boundary–suitable weak solution near a flat boundary. The condition has the form of the C-K-N condition on velocity for the local essential boundedness of suitable weak solutions. The difference, in comparison to C-K-N, is that the condition can be used in the interior points as well as in the boundary points.

- **G. A. Seregin, T. N. Shilkin and V. A. Solonnikov (2004):**

  The authors assume that $\partial\Omega$ is uniformly of the class C^2 and prove that there are absolute constants ϵ_1, $\epsilon_0 > 0$ such that, if a boundary–suitable weak solution satisfies for some point $(\mathbf{x}_0, t_0)$ with $\mathbf{x}_0 \in \partial\Omega$ and $0 < t_0 < T$ the condition

$$\frac{1}{\rho^2} \int_{t_0-\rho^2}^{t_0} \int_{\Omega\cap B_\rho(x_0)} \left(|\mathbf{v}|^3 + |p|^{3/2}\right) \mathrm{d}\mathbf{y}\, \mathrm{d}t \; < \; \epsilon_0$$

  for some (small) ρ or the condition

$$\limsup_{\rho\to 0} \frac{1}{\rho} \int_{t_0-\rho^2}^{t_0} \int_{\Omega\cap B_\rho(x_0)} |\nabla\mathbf{v}|^2\, \mathrm{d}\mathbf{y}\, \mathrm{d}t \; < \; \epsilon_1$$

  then $\mathbf{v}$ is Hölder–continuous in a neighbourhood of the point $(\mathbf{x}_0, t_0)$.

  The problem, in a neighbourhood of point $(\mathbf{x}_0, t_0)$, is transformed to the problem in the neighbourhood of a flat boundary. In this process the coefficients of the Navier–Stokes system are changed. The main work is to develop a theory for the new (perturbed Navier–Stokes) system in appropriate spaces via linearization.

Corollary. *The set of singular points of a boundary–suitable weak solution, including singular points on the boundary, has the 1–dimensional Hausdorff measure equal to zero.*

- **S. Gustafson, K. Kang and T.-P. Tsai (2006):** Certain extensions of the results of Seregin (2002).

  The authors study the 3D Navier-Stokes equations near a flat boundary. They are able to prove Hölder continuity of suitable weak solutions near a flat boundary (as well as in the interior) for solutions with vanishing Dirichlet boundary conditions.

  In particular, if the external force $\mathbf{f}$ is reasonably smooth, $\mathbf{x}_0 \in \partial\Omega$ ($\partial\Omega$ is flat in the neighbourhood of $\mathbf{x}_0$), they show that for every pair (s, r) satisfying

$$1 \leq \frac{3}{s} + \frac{2}{r} \leq 2, \quad 2 < r \leq \infty, \quad (s, r) \neq (\frac{3}{2}, \infty)$$

  and

$$\limsup_{\rho \to 0} \, \rho^{1-(3/s+2/r)} \, \|\mathbf{v}\|_{L^r(t-\rho^2, t;\, L^s(B_\rho^+(\mathbf{x}_0))} \, \leq \, \epsilon$$

  for some $\epsilon > 0$ depending only on r, s, and $\mathbf{f}$, then $(\mathbf{x}_0, t_0)$ is a regular point.

- **J. Wolf:** considered a suitable weak solution in the half space $\mathbb{R}^3_+$. The main result is a direct proof of the partial regularity up to the flat boundary, based on a new decay estimate, which implies the regularity in the cylinder $Q^+_\rho(x_0, t_0)$ provided

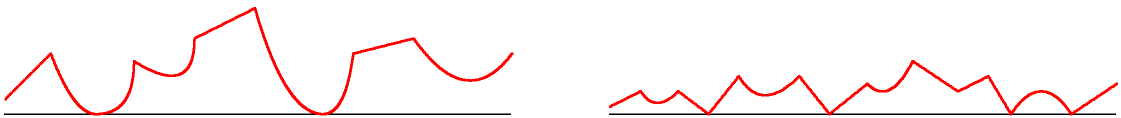
$$\limsup_{\rho\to 0+} \frac{1}{\rho}\int_{Q^+_\rho(\mathbf{x}_0,t_0)} |\mathbf{curl}\,\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;\le\; \epsilon$$

with ϵ sufficiently small.

2. Conditions on a fixed material boundary

- **The no–slip boundary condition:** $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \times (0, T)$

  This condition is supported by a series of recent papers (e.g. by Bucur, Feireisl, Nečasová), where the authors consider a rugous boundary, velocity field satisfying the impermeability boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$, and assume that the rugosity varies so that it becomes "smaller" and "denser", up to the limit case when the rugosity vanishes and the boundary becomes smooth.



  If $\operatorname{div} \mathbf{v} = 0$ and $\mathbf{v}$ is "smooth" then the condition $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \times (0, T)$ is equivalent to the three conditions

$$\mathbf{v} \cdot \mathbf{n} = 0, \qquad \mathbf{curl}\, \mathbf{v} \cdot \mathbf{n} = 0, \qquad \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{n} = 0. \tag{2.1}$$

- **Navier's boundary conditions:**

$$\text{(a)}\;\; \mathbf{v}\cdot\mathbf{n}=0, \qquad \text{(b)}\;\; [\mathbb{T}\cdot\mathbf{n}]_\tau+\gamma\mathbf{v}=\mathbf{0} \qquad \text{on } \partial\Omega\times(0,T), \tag{2.2}$$

where $\mathbb{T}$ is the stress tensor.

The second condition says that *the tangential component of the force with which the fluid acts on the boundary is proportional to the tangential velocity.*

In the incompressible Newtonian fluid with the density $\rho=1$, we have $\mathbb{T}=-p\mathbb{I}+2\nu\mathbb{D}$, where $\mathbb{D}$ is the rate of deformation tensor $\ldots$ $\mathbb{D}=(\nabla\mathbf{v})_{\mathrm{sym}}$.

$\gamma\geq 0\ldots$ coefficient of friction between the fluid and the boundary

- **Navier–type boundary conditions:**

$$\text{(a)}\;\; \mathbf{v}\cdot\mathbf{n}=0, \qquad \text{(b)}\;\; \mathbf{curl}\,\mathbf{v}\times\mathbf{n}=\mathbf{0} \qquad \text{on } \partial\Omega\times(0,T) \tag{2.3}$$

Condition (2.3b) comes from Navier's condition (2.2b): assuming that $\gamma=0$, using the formulas $\mathbb{T}=2\nu\mathbb{D}$ and

$$\nu\,\mathbf{curl}\,\mathbf{v}\times\mathbf{n} \;=\; [\mathbb{T}\cdot\mathbf{n}]_\tau+2\nu\mathbf{v}\cdot\nabla\mathbf{n},$$

and neglecting the curvature of the boundary. we obtain (2.3b).

- **The generalized impermeability boundary conditions:**

$$\text{(a)}\ \ \mathbf{v}\cdot\mathbf{n}=0, \quad \text{(b)}\ \ \mathbf{curl}\,\mathbf{v}\cdot\mathbf{n}=0, \quad \text{(c)}\ \ \mathbf{curl}^2\mathbf{v}\cdot\mathbf{n}=0 \tag{2.4}$$

on $\partial\Omega\times(0,T)$.

We observe that these conditions in fact differ from the series of boundary conditions (2.1a) only in the third condition (2.4c).

The third condition (2.4c) says that $\mathbf{n}\cdot\mathbb{T}\cdot\mathbf{n}=0$ which means that the normal component of the viscous stress acting on $\partial\Omega$ equals zero. On the other hand, since $\nu\,\mathbf{curl}^2\mathbf{v}=-\nu\Delta\mathbf{v}=-\mathrm{Div}\,\mathbb{T}$, condition (2.4c) can be written in the form $\mathrm{Div}\,\mathbb{T}\cdot\mathbf{n}=0$. It says that the normal component of the intensity of production of the viscous stress on $\partial\Omega$ equals zero.

- **Serrin's proposal:** $\mathbf{v}\cdot\mathbf{n}=0$ and

$$\mathbf{v}=0 \qquad \text{if} \quad |(\mathbb{T}\cdot\mathbf{n})_\tau|\ \leq\ K\,|(\mathbb{T}\cdot\mathbf{n})_n|, \tag{2.5}$$

$$[\mathbb{T}\cdot\mathbf{n}]_\tau+\gamma\mathbf{v}=\mathbf{0} \qquad \text{if} \quad |(\mathbb{T}\cdot\mathbf{n})_\tau|\ >\ K\,|(\mathbb{T}\cdot\mathbf{n})_n|. \tag{2.6}$$

Remarks.

- Conditions (2.1) and (2.3) can also be used in inhomogeneous versions, when one studies an inflow or outflow from domain Ω.
- Conditions (2.3), (2.4) guarantee that $\Delta\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega \times (0, T)$, which e.g. means that the Helmholtz projection and the Laplace operator commute.

3. A regularity criterion based on the eigenvalues of tensor $\mathbb{D}$, assuming Navier's boundary conditions

We assume that Ω is a bounded smooth domain.

$$\begin{aligned}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \nu \Delta \mathbf{v} && \text{in } Q_T, && (3.1)\\
\operatorname{div} \mathbf{v} &= 0 && \text{in } Q_T, && (3.2)\\
\mathbf{v} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \times (0,T), && (3.3)\\
[\mathbb{T} \cdot \mathbf{v}]_\tau + \gamma \mathbf{v} &= \mathbf{0} && \text{on } \partial\Omega \times (0,T), && (3.4)\\
\mathbf{v} &= \mathbf{v}_0 && \text{in } \Omega \times \{0\}. && (3.5)
\end{aligned}$$

Multiplying equation (3.1) by $P_\sigma \Delta \mathbf{v}$ and integrating in Ω, we obtain

$$\int_\Omega \partial_t \mathbf{v} \cdot P_\sigma \Delta \mathbf{v} \, \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \cdot P_\sigma \Delta \mathbf{v} \, \mathrm{d}\mathbf{x} \;=\; \nu \, \| P_\sigma \Delta \mathbf{v} \|_2^2. \tag{3.6}$$

We further assume, for simplicity, that $\nu = 1$, $\gamma = 1$.

The first integral on the left hand side can be treated as follows:

$$\begin{aligned}
\int_\Omega \partial_t \mathbf{v} \cdot P_\sigma \Delta \mathbf{v} \,\mathrm{d}\mathbf{x} &= \int_\Omega \partial_t \mathbf{v} \cdot \Delta \mathbf{v} \,\mathrm{d}\mathbf{x} = 2 \int_\Omega \partial_t \mathbf{v} \cdot \mathrm{Div}\,(\nabla \mathbf{v})_{\mathrm{sym}} \,\mathrm{d}\mathbf{x} \\
&= 2 \int_{\partial\Omega} \partial_t \mathbf{v} \cdot \big[(\nabla \mathbf{v})_{\mathrm{sym}} \cdot \mathbf{n}\big] \,\mathrm{d}S - 2 \int_\Omega \partial_t \nabla \mathbf{v} : (\nabla \mathbf{v})_{\mathrm{sym}} \,\mathrm{d}\mathbf{x} \\
&= \int_{\partial\Omega} \partial_t \mathbf{v} \cdot \big[2\mathbb{D}(\mathbf{v}) \cdot \mathbf{n}\big]_\tau \,\mathrm{d}S - \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \big|(\nabla \mathbf{v})_{\mathrm{sym}}\big|^2 \,\mathrm{d}\mathbf{x} \\
&= -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{v}\|_{2;\,\partial\Omega}^2 - \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbb{D}(\mathbf{v})\|_2^2.
\end{aligned}$$

In order to estimate the second integral on the left hand side of (3.6), we write

$$\Delta \mathbf{v} \;=\; P_\sigma \Delta \mathbf{v} + \nabla \varphi$$

where φ is a solution of the Neumann problem

$$\Delta \varphi = 0 \quad \text{in } \Omega, \qquad \frac{\partial \varphi}{\partial \mathbf{n}} = \Delta \mathbf{v} \cdot \mathbf{n} \quad \text{on } \partial\Omega. \tag{3.7}$$

The right hand side $\Delta\mathbf{v}\cdot\mathbf{n}$ in the boundary condition can be modified in this way:

$$\begin{aligned}\Delta\mathbf{v}\cdot\mathbf{n} &= -\mathbf{curl}^2\mathbf{v}\cdot\mathbf{n} = -\mathbf{curl}\left[(\mathbf{curl}\ \mathbf{v})_\tau\right]\cdot\mathbf{n} - \mathbf{curl}\left[(\mathbf{curl}\,\mathbf{v})_n\right]\cdot\mathbf{n} \\ &= -\mathbf{curl}\left[(\mathbf{curl}\,\mathbf{v})_\tau\right]\cdot\mathbf{n}.\end{aligned}$$

The vector field $\mathbf{curl}\left[(\mathbf{curl}\,\mathbf{v})_n\right]$ is tangential because $(\mathbf{curl}\,\mathbf{v})_n$ is normal. Hence the term $\mathbf{curl}\left[(\mathbf{curl}\,\mathbf{v})_n\right]\cdot\mathbf{n}$ equals zero on $\partial\Omega$.

The tangential component of $\mathbf{curl}\,\mathbf{v}$, i.e. $(\mathbf{curl}\,\mathbf{v})_\tau$, equals $\mathbf{n}\times\mathbf{curl}\,\mathbf{v}\times\mathbf{n}$. In order to express $\mathbf{curl}\,\mathbf{v}\times\mathbf{n}$, we use the formula

$$[2\mathbb{D}\cdot\mathbf{n}]_\tau = \mathbf{curl}\,\mathbf{v}\times\mathbf{n} - 2\mathbf{v}\cdot\nabla\mathbf{n}.$$

Hence, using the boundary condition (3.4), we obtain:

$$\begin{aligned}(\mathbf{curl}\,\mathbf{v})_\tau &= \mathbf{n}\times(\mathbf{curl}\,\mathbf{v}\times\mathbf{n}) = \mathbf{n}\times\left([2\mathbb{D}\cdot\mathbf{n}]_\tau + 2\mathbf{v}\cdot\nabla\mathbf{n}\right) \\ &= \mathbf{n}\times\left(-\mathbf{v} + 2\mathbf{v}\cdot\nabla\mathbf{n}\right).\end{aligned}$$

Thus, the boundary condition in (3.7) takes the form

$$\frac{\partial\varphi}{\partial\mathbf{n}} = -\mathbf{curl}\left[\mathbf{n}\times\left(-\mathbf{v}+2\mathbf{v}\cdot\nabla\mathbf{n}\right)\right]\cdot\mathbf{n}.$$

Classical theory of solution of the Neumann problem now implies that

$$\|\nabla\varphi\|_2 \;\le\; C \left\| -\mathbf{curl}\left[\mathbf{n}\times\left(-\mathbf{v}+2\mathbf{v}\cdot\nabla\mathbf{n}\right)\right]\cdot\mathbf{n}\,\right\|_{-1/2,2;\,\partial\Omega}.$$

The right hand side can be estimated by means of continuity of the linear operator, that assigns to a divergence–free function $\mathbf{u}\in\mathbf{L}^2(\Omega)$ a scalar function $\mathbf{u}\cdot\mathbf{n}\in W^{-1/2,2}(\partial\Omega)$. Thus, we finally get

$$\|\nabla\varphi\|_2 \;\le\; C \left\|\mathbf{curl}\left[\mathbf{n}\times\Big(-\mathbf{v}+2\mathbf{v}\cdot\nabla\mathbf{n}\Big)\right]\cdot\mathbf{n}\,\right\|_2 \;\le\; C\,\|\mathbf{v}\|_{1,2}. \tag{3.8}$$

Hence the second integral in (3.6) is

$$\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot P_\sigma\Delta\mathbf{v}\;\mathrm{d}\mathbf{x} \;=\; \int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{v}\;\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\nabla\varphi\;\mathrm{d}\mathbf{x}, \tag{3.9}$$

where

$$\begin{aligned}
\left|\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\nabla\varphi\;\mathrm{d}\mathbf{x}\right| \;&\le\; \|\mathbf{v}\|_\infty\,\|\nabla\mathbf{v}\|_2\,\|\nabla\varphi\|_2 \;\le\; C\,\|\mathbf{v}\|_\infty\,\|\nabla\mathbf{v}\|_2^2 \\
&\le\; C(r)\,\|\mathbf{v}\|_{1,r}\,\|\nabla\mathbf{v}\|_2^2 \qquad (\text{for } r>2) \\
&\le\; \delta\,\|P_\sigma\Delta\mathbf{v}\|_2^2 + C(\delta)\,\|\nabla\mathbf{v}\|_2^4.
\end{aligned}$$

The first integral on the right hand side of (3.9 is

$$\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{v}\,\mathrm{d}\mathbf{x} \;=\; \int_{\partial\Omega}[\mathbf{v}\cdot\nabla\mathbf{v}]\cdot\nabla\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S - \int_\Omega \nabla[\mathbf{v}\cdot\nabla\mathbf{v}]:\nabla\mathbf{v}\,\mathrm{d}\mathbf{x}$$

$$= \; I_1 - I_2 - I_3,$$

where

$$I_2 \;=\; \int_\Omega v_{j,k}\,v_{i,j}\,v_{i,k}\,\mathrm{d}\mathbf{x},$$

$$I_3 \;=\; \int_\Omega v_j\,v_{i,jk}\,v_{i,k}\,\mathrm{d}\mathbf{x} \;=\; 0,$$

$$I_1 \;=\; \int_{\partial\Omega}[\mathbf{v}\cdot\nabla\mathbf{v}]_n\cdot\nabla\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S + \int_{\partial\Omega}[\mathbf{v}\cdot\nabla\mathbf{v}]_\tau\cdot\nabla\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S \;=\; I_4 + I_5.$$

Integral I_4 can be treated as follows:

$$I_4 \;=\; \int_{\partial\Omega}[\mathbf{v}\cdot\nabla\mathbf{v}]_n\cdot(\nabla\mathbf{v}\cdot\mathbf{n})_n\,\mathrm{d}\mathbf{x} \;=\; \int_{\partial\Omega}(v_j\,v_{l,j}\,n_l)\,(v_k\,v_{m,k}\,n_m\,n_k)\,\mathrm{d}S$$

$$= \int_{\partial\Omega}[v_j\,\partial_j(v_l\,n_l) - v_j\,v_l\,n_{l,j}]\,(v_k\,v_{m,k}\,n_m\,n_k)\,\mathrm{d}S$$

$$= -\int_{\partial\Omega} (v_j\, v_l\, n_{l,j})\, (v_k\, v_{m,k}\, n_m\, n_k)\, \mathrm{d}S \;=\; -\int_{\Omega} \partial_m \big[(v_j\, v_l\, n_{l,j})\, (v_k\, v_{m,k}\, n_k\big]\, \mathrm{d}\mathbf{x}$$

$$= -\int_{\Omega} \partial_m \big[(v_j\, v_l\, n_{l,j})\, (v_k\, n_k\big]\, v_{m,k}\, \mathrm{d}\mathbf{x} \;\leq\; C(r)\, \|\mathbf{v}\|_{1,r}\, \|\nabla\mathbf{v}\|_2^2 \qquad \text{(for } r > 2)$$

$$\leq\; \delta\, \|P_\sigma \Delta\mathbf{v}\|_2^2 + C(\delta)\, \|\nabla\mathbf{v}\|_2^4.$$

Integral I_5 is

$$\begin{aligned}
I_5 &= \int_{\partial\Omega} [\mathbf{v}\cdot\nabla\mathbf{v}]_\tau \cdot \nabla\mathbf{v}\cdot\mathbf{n}\, \mathrm{d}S \\
&= \int_{\partial\Omega} [\mathbf{v}\cdot\nabla\mathbf{v}]_\tau \cdot \big\{(\nabla\mathbf{v})_{\mathrm{sym}}\cdot\mathbf{n} + (\nabla\mathbf{v})_{\mathrm{asym}}\cdot\mathbf{n}\big\}\, \mathrm{d}S \\
&= \int_{\partial\Omega} [\mathbf{v}\cdot\nabla\mathbf{v}]_\tau \cdot \big\{[(\nabla\mathbf{v})_{\mathrm{sym}}\cdot\mathbf{n}]_\tau + (\nabla\mathbf{v})_{\mathrm{asym}}\cdot\mathbf{n}\big\}\, \mathrm{d}S \\
&= \int_{\partial\Omega} [\mathbf{v}\cdot\nabla\mathbf{v}]_\tau \cdot \Big\{-\frac{1}{2}\mathbf{v} + (\nabla\mathbf{v})_{\mathrm{asym}}\cdot\mathbf{n}\Big\}\, \mathrm{d}S \\
&= \int_{\partial\Omega} \big\{\mathbf{v}\cdot\nabla\mathbf{v} - [\mathbf{v}\cdot\nabla\mathbf{v}]_n\big\} \cdot \Big\{-\frac{1}{2}\mathbf{v} + (\nabla\mathbf{v})_{\mathrm{asym}}\cdot\mathbf{n}\Big\}\, \mathrm{d}S.
\end{aligned}$$

As in the case of I_4, we finally obtain

$$|I_5| \;\leq\; \delta\,\|P_\sigma \Delta \mathbf{v}\|_2^2 + C(\delta)\,\|\nabla \mathbf{v}\|_2^4.$$

Recall that $\mathbb{D} = (\nabla \mathbf{v})_{\mathrm{sym}} = (d_{ij})$. Further, we denote by a_{ij} the components of $(\nabla \mathbf{v})_{\mathrm{asym}}$ and by ω_i the components of $\boldsymbol{\omega} \equiv \mathbf{curl}\,\mathbf{v}$. Integral I_2 can now be modified in this way:

$$\begin{aligned}
I_2 \;&=\; \int_\Omega v_{j,k}\, v_{i,j}\, v_{i,k}\; \mathrm{d}\mathbf{x} \;=\; \int_\Omega d_{jk}\, v_{i,j}\, v_{i,k}\; \mathrm{d}\mathbf{x} \;=\; \int_\Omega d_{jk}\,(d_{ij} + a_{ij})\,(d_{ik} + a_{ik})\; \mathrm{d}\mathbf{x} \\
&=\; \int_\Omega d_{jk}\, d_{ij}\, d_{ik}\; \mathrm{d}\mathbf{x} + \int_\Omega d_{jk}\, d_{ij}\, a_{ik}\; \mathrm{d}\mathbf{x} + \int_\Omega d_{jk}\, a_{ij}\, d_{ik}\; \mathrm{d}\mathbf{x} + \int_\Omega d_{jk}\, a_{ij}\, a_{ik}\; \mathrm{d}\mathbf{x} \\
&=\; \int_\Omega d_{jk}\, d_{ij}\, d_{ik}\; \mathrm{d}\mathbf{x} + \int_\Omega d_{jk}\, a_{ij}\, a_{ik}\; \mathrm{d}\mathbf{x} \\
&=\; \int_\Omega d_{jk}\, d_{ij}\, d_{ik}\; \mathrm{d}\mathbf{x} - \frac{1}{4}\int_\Omega d_{jk}\, \omega_j\, \omega_k\; \mathrm{d}\mathbf{x}.
\end{aligned}$$

Thus, we have

$$\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{v}\; \mathrm{d}\mathbf{x} \;=\; -\int_\Omega d_{jk}\, d_{ij}\, d_{ik}\; \mathrm{d}\mathbf{x} + \frac{1}{4}\int_\Omega d_{jk}\, \omega_j\, \omega_k\; \mathrm{d}\mathbf{x} + R, \tag{3.10}$$

where R denotes any expression satisfying the estimate

$$|R| \;\leq\; \delta \, \|P_\sigma \Delta \mathbf{v}\|_2^2 + C(\delta) \, \|\nabla \mathbf{v}\|_2^4. \tag{3.11}$$

Recall the inequalities, fulfilled by divergence–free vector functions $\mathbf{w}$ that satisfy Navier's boundary conditions (3.3), (3.4).

$$\begin{aligned} \|\mathbf{w}\|_2^2 \;&\leq\; C \, \|\nabla \mathbf{w}\|_2^2, \\ \|\nabla^2 \mathbf{w}\|_2^2 \;&\leq\; C \left(\|\Delta \mathbf{w}\|_2^2 + \|\mathbf{w}\|_{1,2}^2 \right) \;\leq\; C \left(\|P_\sigma \Delta \mathbf{w}\|_2^2 + \|\mathbf{w}\|_{1,2}^2 \right), \\ &\leq\; C \left(\|P_\sigma \Delta \mathbf{w}\|_2^2 + \|\nabla \mathbf{w}\|_2^2 \right), \\ \|\nabla \mathbf{w}\|_2^2 \;&\leq\; C \, \|\mathbb{D}\|_2^2. \end{aligned}$$

The integral of $\mathbf{v} \cdot \nabla \mathbf{v} \cdot \Delta \mathbf{v}$ can also be treated in this way:

$$\int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \cdot \Delta \mathbf{v} \, \mathrm{d}\mathbf{x} \;=\; -\int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \cdot \mathbf{curl}^2 \mathbf{v} \, \mathrm{d}\mathbf{x}$$

$$
\begin{aligned}
&= -\int_{\partial\Omega} \mathbf{v}\cdot\nabla\mathbf{v}\cdot(\mathbf{n}\times\mathbf{curl}\,\mathbf{v})\;\mathrm{d}S - \int_{\Omega} \mathbf{curl}\,(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{curl}\,\mathbf{v}\;\mathrm{d}\mathbf{x} \\
&= -I_6 - I_7\,.
\end{aligned}
$$

Integral I_6 can be estimated as follows:

$$
\begin{aligned}
|I_6| &= \left|\int_{\partial\Omega} \big([2\mathbb{D}\cdot\mathbf{n}]_\tau + 2\mathbf{v}\cdot\nabla\mathbf{n}\big)\cdot(\mathbf{v}\cdot\nabla\mathbf{v})\;\mathrm{d}S\right| \\
&= \left|\int_{\partial\Omega} \big(-\mathbf{v} + 2\mathbf{v}\cdot\nabla\mathbf{n}\big)\cdot(\mathbf{v}\cdot\nabla\mathbf{v})\;\mathrm{d}S\right| \\
&\leq C\int_{\partial\Omega} |\mathbf{v}|^2\,|\nabla\mathbf{v}|\;\mathrm{d}S \;\leq\; C\left(\int_{\partial\Omega} |\nabla\mathbf{v}|^2\,\mathrm{d}S\right)^{1/2}\left(\int_{\partial\Omega} |\mathbf{v}|^4\,\mathrm{d}S\right)^{1/2} \\
&\leq C\left(\|\nabla\mathbf{v}\|_2^3 + \|\nabla^2\mathbf{v}\|_2\,\|\nabla\mathbf{v}\|_2^2\right) \;\leq\; \delta\,\|P_\sigma\Delta\mathbf{v}\|_2^2 + C(\delta)\,\|\nabla\mathbf{v}\|_2^4.
\end{aligned}
$$

Integral I_7 satisfies:

$$
I_7 \;=\; -\int_{\Omega} \big(\mathbf{v}\cdot\nabla\boldsymbol{\omega}\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{v}\cdot\boldsymbol{\omega}\big)\;\mathrm{d}\mathbf{x} \;=\; \int_{\Omega} \boldsymbol{\omega}\cdot\nabla\mathbf{v}\cdot\boldsymbol{\omega}\;\mathrm{d}\mathbf{x}
$$

$$= \int_\Omega \boldsymbol{\omega} \cdot (\nabla \mathbf{v})_{\mathrm{sym}} \cdot \boldsymbol{\omega} \, \mathrm{d}\mathbf{x} \; = \; \int_\Omega d_{jk} \, \omega_j \, \omega_k \, \mathrm{d}\mathbf{x}.$$

Thus, we obtain

$$\int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \cdot \Delta \mathbf{v} \, \mathrm{d}\mathbf{x} \; = \; \int_\Omega d_{jk} \, \omega_j \, \omega_k \, \mathrm{d}\mathbf{x} + R. \tag{3.12}$$

Comparing (3.10) and (3.12), we can exclude the integral of $d_{jk} \, \omega_j \, \omega_k$ and we obtain:

$$\int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \cdot \Delta \mathbf{v} \, \mathrm{d}\mathbf{x} \; = \; -\frac{4}{3} \int_\Omega d_{jk} \, d_{ij} \, d_{ik} \, \mathrm{d}\mathbf{x} + R.$$

Substituting all these expressions and estimates to (3.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \, \|\mathbb{D}(\mathbf{v})\|_2^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \, \|\mathbf{v}\|_{2;\,\partial\Omega}^2 + \|P_\sigma \Delta \mathbf{v}\|_2^2 \; \leq \; -\frac{4}{3} \int_\Omega d_{jk} \, d_{ij} \, d_{ik} \, \mathrm{d}\mathbf{x} + R.$$

Choosing δ sufficiently small, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \, \|\mathbb{D}(\mathbf{v})\|_2^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \, \|\mathbf{v}\|_{2;\,\partial\Omega}^2 + \frac{1}{2} \, \|P_\sigma \Delta \mathbf{v}\|_2^2 \; \leq \; -\frac{4}{3} \int_\Omega d_{ij} \, d_{jk} \, d_{ki} \, \mathrm{d}\mathbf{x} + C \, \|\nabla \mathbf{v}\|_2^4$$

The product $d_{ij}\, d_{jk}\, d_{ki}$ equals the trace of the tensor $\mathbb{D}^3$. It is invariant with respect to rotations of the coordinate system. Hence we can choose, for its expression, e.g. the system in which $\mathbb{D}$ has the diagonal representation

$$\mathbb{D} \;=\; \begin{pmatrix} \lambda_1, & 0, & 0 \\ 0, & \lambda_2, & 0 \\ 0, & 0, & \lambda_3 \end{pmatrix}.$$

Here, λ_1, λ_2, λ_3 are the eigenvalues of tensor $\mathbb{D}$. The trace of $\mathbb{D}^3$ can now be expressed as

$$\mathrm{Tr}\, \mathbb{D}^3 \;=\; \lambda_1^3 + \lambda_2^3 + \lambda_3^3.$$

The eigenvalues satisfy the characteristic equation of tensor $\mathbb{D}$, i.e.

$$-\lambda_i^3 + \lambda_i^2\, E_1 - \lambda_i\, E_2 + E_3 \;=\; 0 \qquad (i = 1, 2, 3), \tag{3.13}$$

where E_1, E_2, E_3 are the principal invariants of tensor $\mathbb{D}$. Recall that $E_1 = \mathrm{Tr}\, \mathbb{D} = 0$ (due to (3.2)) and $E_3 = \det \mathbb{D}$. Thus, summing (3.13) over $i = 1, 2, 3$, one gets

$$\mathrm{Tr}\, \mathbb{D}^3 \;=\; \lambda_1^3 + \lambda_2^3 + \lambda_3^3 \;=\; 3 \det \mathbb{D} \;=\; 3\, \lambda_1\, \lambda_2\, \lambda_3\, .$$

Assume that the eigenvalues λ_1, λ_2, λ_3 are ordered so that $\lambda_1 \le \lambda_2 \le \lambda_3$. Thus, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbb{D}(\mathbf{v})\|_2^2 + \frac{1}{2}\,\|\mathbf{v}\|_{2;\,\partial\Omega}^2\right) + \frac{1}{2}\,\|P_\sigma \Delta \mathbf{v}\|_2^2 \\
&\qquad \le\; -4\int_\Omega \lambda_1\,\lambda_2\,\lambda_3\;\mathrm{d}\mathbf{x} + C\,\|\nabla \mathbf{v}\|_2\left(\|\mathbb{D}(\mathbf{v})\|_2^2 + \frac{1}{2}\,\|\mathbf{v}\|_{2;\,\partial\Omega}^2\right) \\
&\qquad \le\; 4\int_\Omega (-\lambda_1)\,(\lambda_2)_+\,\lambda_3\;\mathrm{d}\mathbf{x} + C\,\|\nabla \mathbf{v}\|_2\left(\|\mathbb{D}(\mathbf{v})\|_2^2 + \frac{1}{2}\,\|\mathbf{v}\|_{2;\,\partial\Omega}^2\right). \qquad (3.14)
\end{aligned}$$

Integrating inequality (3.14), we obtain an estimate of $\|\mathbb{D}(\mathbf{v})\|_2^2 + \frac{1}{2}\,\|\mathbf{v}\|_{2;\,\partial\Omega}^2$ in $L^\infty(\epsilon, T)$ provided that

$$\int_\Omega (-\lambda_1)\,(\lambda_2)_+\,\lambda_3\;\mathrm{d}\mathbf{x} \;\in\; L^1(0,T). \qquad (3.15)$$

Then we obtain

$$|||\mathbb{D}\mathbf{v}|||_{\infty,2}^2 + |||\mathbf{v}|||_{\infty,2;\,\partial\Omega}^2 + |||\nabla^2\mathbf{v}|||_{2,2}^2 \;\le\; C\,\int_\Omega (-\lambda_1)\,(\lambda_2)_+\,\lambda_3\;\mathrm{d}\mathbf{x}. \qquad (3.16)$$

Assuming e.g. that $(\lambda_2)_+ \in L^r(0,T;\, L^s(\Omega))$, where $\dfrac{2}{r} + \dfrac{3}{s} \leq 1$, we get

$$\int_0^T \int_\Omega (-\lambda_1)\,(\lambda_2)_+\,\lambda_3 \;\mathrm{d}\mathbf{x}\,\mathrm{d}t \;\leq\; |||(\lambda_2)_+|||_{r,s}\; |||\lambda_1\lambda_3|||_{\frac{r}{r-1},\,\frac{s}{s-1}},$$

where

$$|||u|||_{r,s} \;:=\; \left[\int_0^T \left(\int_\Omega |u|^s \;\mathrm{d}\mathbf{x}\right)^{r/s} \mathrm{d}t\right]^{1/s}.$$

Obviously,

$$|\lambda_1|,\; |\lambda_3| \;\leq\; C\,|(\nabla\mathbf{v})_{\mathrm{sym}}| \;\leq\; C\,|\nabla\mathbf{v}|.$$

Hence

$$\int_0^T \int_\Omega (-\lambda_1)\,(\lambda_2)_+\,\lambda_3 \;\mathrm{d}\mathbf{x}\,\mathrm{d}t \;\leq\; C\,|||(\lambda_2)_+|||_{r,s}\; |||\nabla\mathbf{v}|||^2_{\frac{2r}{r-1},\,\frac{2s}{s-1}}.$$

Applying the inequality

$$|||g|||_{\alpha,\beta} \;\leq\; |||g|||_{2,2}^{\frac{2}{\alpha}+\frac{3}{\beta}-\frac{3}{2}} \left(|||g|||_{\infty,2} + |||g|||_{2,6}\right)^{\frac{5}{2}-\left(\frac{2}{\alpha}+\frac{3}{\beta}\right)}$$

(which can be proved by means of the Hölder inequality and which is valid for $2 \leq \alpha \leq +\infty$, $2 \leq \beta \leq 6$ and $\frac{3}{2} \leq 2/\alpha + 3/\beta \leq \frac{5}{2}$) to the norm of $\nabla\mathbf{v}$ with $\alpha = 2r/(r-1)$ and $\beta = 2s/(s-1)$, we obtain:

$$\int_0^T \int_\Omega (-\lambda_1)\,(\lambda_2)_+\,\lambda_3\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;\le\; C\,|||(\lambda_2)_+|||_{r,s}\left(|||\nabla\mathbf{v}|||_{\infty,2)} + |||\nabla^2\mathbf{v}|||_{2,2}\right)^{\frac{2}{r}+\frac{3}{s}}.$$

In this way, we can prove the theorem

Theorem 3. *Let* $\mathbf{v}$ *be a weak solution of the problem (3.1)–(3.5), satisfying (SEI). Suppose that* $\zeta_1 \le \zeta_2 \le \zeta_3$ *are the eigenvalues of the tensor* $\mathbb{D} := (\nabla\mathbf{v})_{\mathrm{sym}}$ *and*

(i) *one of the functions* ζ_1, $(\zeta_2)_+$, ζ_3 *belongs to* $L^{s,r}(D)$ *for some* $r \in [1,\infty]$, $s \in (\frac{3}{2},\infty]$, *satisfying* $2/r + 3/s \le 2$,

then the norm $\|\nabla\mathbf{v}(\,.\,,t)\|_2$ *is bounded for* $t \in (\epsilon, T)$ *(for any* $\epsilon > 0$*).*

Remarks.

- The sketched proof, in fact, concerns the case $2/r + 3/s < 2$. However, if $2/r + 3/s = 2$ then we can work on an arbitrarily short time interval $(t_0 - \xi, t_0)$ instead of $(0,T)$. Assuming that $\|\nabla\mathbf{v}(\,.\,,t_0-\xi)\|_2 < \infty$ and ξ is "sufficiently small", one can achieve the norm $|||(\lambda_2)_+|||_{r,s}$ (which is now the norm on the time interval $(t_0 - \xi, t_0)$) to be arbitrarily small. Then the term $C\,|||(\lambda_2)_+|||_{r,s}\,|||\nabla^2 \mathbf{v}|||_{2,2}^2$ can be absorbed by the left hand side of (3.16).

- If $\mathbf{v}_0$ is “smooth” then the statement of the theorem can be extended up to the initial time t_0.
- The theorem can be modified in such a way that it holds only “locally” in the neighbourhood of a part of $\partial\Omega \times (0, T)$.
- Since the eigenvalues of tensor $\mathbb{D}$ give the rate of deformations of infinitesimally small volumes of the fluid in the principal directions of $\mathbb{D}$, Theorem 3 shows that the deformations such that the infinitesimally small volumes are stretched in one direction and compressed in two directions act for regularity. On the other hand, deformations when the infinitesimally small volumes are stretched in two directions and compressed in one direction act again the regularity. In this case, condition (i) restricts the stretching in the direction of the eigenvector associated with eigenvalue λ_2.

4. Existence of a regular solution on a time interval independent of viscosity

We assume that Ω is a bounded smooth domain. The approach, presented in this section, shows the advantage of Navier's boundary conditions. It is not known whether an analog with Dirichlet's boundary condition is also possible.

Lemma 2 (on a strong solution to the Euler problem – Kato, Temam, et al). *Let* $r > 0$, Ω *be a bounded domain in* $\mathbb{R}^3$ *with the boundary of the class* $C^{3/2+r,1}$, $\mathbf{u}^* \in \mathbf{W}^{5/2+r,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$ *and* $\mathbf{f} \in L^1\big(0,T;\, \mathbf{W}^{5/2+r,2}(\Omega)\big)$. *Then there exists* $T_0 \in (0,T]$ *and a unique solution* $\mathbf{u}^0$ *of the Euler problem*

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{f} \qquad \text{in } Q_T, \tag{4.1}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } Q_T, \tag{4.2}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{4.3}$$

$$\mathbf{u} = \mathbf{u}^* \qquad \text{in } \Omega\times\{0\}. \tag{4.4}$$

on the time interval $(0,T^*)$ *such that* $\mathbf{u}^0 \in L^\infty\big(0,T_0;\, \mathbf{W}^{5/2+r,2}(\Omega)\big)$.

Principle of the proof. We assume for simplicity that $r = \frac{1}{2}$.

We successively apply the operator ∇^j (for $j = 0, 1, 2, 3$) to equation (4.1), multiply the equation by $\nabla^j \mathbf{u}$ and integrate on Ω. Then we sum the integrals for j running from 0 to 3. The integrals with the highest derivatives are:

$$\text{a)} \ \int_\Omega \mathbf{u} \cdot \nabla(\nabla^3 \mathbf{u}) \cdot (\nabla^3 \mathbf{u}), \quad \text{b)} \ \int_\Omega \nabla^2 \mathbf{u} \cdot \nabla^3 \mathbf{u} \cdot \nabla^2 \mathbf{u}, \quad \text{c)} \ \int_\Omega \nabla^2 \mathbf{u} \cdot \nabla^2 \mathbf{u} \cdot \nabla^3 \mathbf{u},$$

$$\text{d)} \ \int_\Omega \nabla^3 \mathbf{u} \cdot \nabla \mathbf{u} \cdot \nabla^3 \mathbf{u}, \qquad \text{e)} \ \int_\Omega \nabla^4 p \cdot \nabla^3 \mathbf{u} \qquad f) \ \int_\Omega \nabla^3 \mathbf{f} \cdot \nabla^3 \mathbf{u}.$$

Applying the integration by parts and equation (4.2), we observe that the integral a) equals zero. The integral b) is less than or equal to

$$C \, \|\nabla^3 \mathbf{u}\|_2 \, \|\nabla^2 \mathbf{u}\|_4^2 \ \leq \ C \, \|\nabla^3 \mathbf{u}\|_2 \, \|\nabla^2 \mathbf{u}\|_6 \, \|\nabla^2 \mathbf{u}\|_2 \ \leq \ C \, \|\mathbf{u}\|_{3,2}^3 .$$

The integrals c) and d) can be estimated similarly. In order to estimate the integral e), we express p as a solution of the Neumann problem

$$\Delta p \ = \ -\nabla \mathbf{u} \cdot (\nabla \mathbf{u})^T \qquad \text{in } \Omega, \tag{4.5}$$

$$\frac{\partial p}{\partial \mathbf{n}} \;=\; -\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{n} \qquad\qquad \text{on } \partial\Omega. \tag{4.6}$$

Since $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$, the right hand of (4.6) equals $-\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\mathbf{n})+\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}=\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}$. Applying the known estimates of solutions of the Neumann problem and the estimates following from the theorem on continuous imbeddings and from the theorem on traces, we obtain

$$\begin{aligned}\|\nabla^4 p\|_2 \;&\leq\; C\,\|\nabla\mathbf{u}\cdot(\nabla\mathbf{u})^T\|_{2,2}+C\,\|\mathbf{u}\cdot\nabla\mathbf{n}\cdot\mathbf{u}\|_{5/2,2;\,\partial\Omega}\\ &\leq\; C\,\|\nabla\mathbf{u}\cdot(\nabla\mathbf{u})^T\|_{2,2}+C\,\|\mathbf{u}\otimes\mathbf{u}\|_{3,2} \;\leq\; C\,\|\mathbf{u}\|_{3,2}^2.\end{aligned}$$

Thus, the integral e) is less than or equal to

$$\|\nabla^4 p\cdot\nabla^3\mathbf{u}\|_1 \;\leq\; \|\nabla^4 p\|_2\|\nabla^3\mathbf{u}\|_2 \;\leq\; \|\nabla^4 p\|_2^{3/2}+C\,\|\nabla^3\mathbf{u}\|_2^3 \;\leq\; C\,\|\mathbf{u}\|_{3,2}^3.$$

Finally we obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbf{u}\|_{3,2}^2 \;\leq\; C\,\|\mathbf{u}\|_{3,2}^3+C\,\|\mathbf{f}\|_{3,2}\,\|\mathbf{u}\|_{3,2}+C. \tag{4.7}$$

Remark. **The same approach fails if we try to apply it to the Navier–Stokes equation,** with any of the mentioned boundary conditions.

The reason lies in the ,,viscous" term $\nu\Delta\mathbf{u}$: if we apply the operator ∇^3 to this term, multiply it by $\nabla^3\mathbf{u}$ and integrate in Ω, we obtain the integral

$$\nu\int_\Omega \Delta\nabla^3\mathbf{u}\cdot\nabla^3\mathbf{u}.$$

One would like to integrate by parts and to transform it e.g. to $-\nu\int_\Omega|\nabla^4\mathbf{u}|^2$, but none of the considered boundary conditions enables us to get rid of the integral on $\partial\Omega$.

We can now construct a strong solution of the Navier–Stokes problem as a perturbation of the solution of the Euler problem.

If we want a correctly formulated problem, we need to add some "complementary boundary" condition to condition (4.3). We can use a generally inhomogeneous boundary condition

$$\big[\mathbb{T}^\nu(\mathbf{u})\cdot\mathbf{n}\big]_\tau+\kappa\,\mathbf{u} \;=\; \mathbf{a}\quad \text{(where } \mathbf{a}=\mathbf{a}(\nu,\kappa)\text{)}\quad \text{on } \partial\Omega\times(0,T). \tag{4.8}$$

We use the next two assumptions:

(A1) There exists a positive constant c_3 so that coefficients ν and κ are related by the equation $\kappa = c_3\nu$, which is in coincidence with physical observations.

(A2) Function $\mathbf{a}(\nu)$ on the right hand side of (4.8) has the form

$$\mathbf{a}(\nu) \;=\; \big[\mathbb{T}^{\nu}(\mathbf{u}^0)\cdot\mathbf{n}\big]_{\tau} + c_3\nu\,\mathbf{u}^0 + \nu\boldsymbol{\phi}^{\nu} \;\equiv\; \nu\,\big\{\big[2\mathbb{D}(\mathbf{u}^0)\cdot\mathbf{n}\big]_{\tau} + c_3\mathbf{u}^0 + \boldsymbol{\phi}^{\nu}\big\},$$

where $\boldsymbol{\phi}^{\nu} \in L^2\big(0,T_0;\ \mathbf{W}^{3/2,2}(\partial\Omega)\big) \cap W^{1,2}\big(0,T_0;\ \mathbf{W}^{1/2,2}(\partial\Omega)\big)$, $\boldsymbol{\phi}^{\nu}$ depends continuously on ν for $\nu > 0$, and is tangent to $\partial\Omega$. Moreover, $\boldsymbol{\phi}^{\nu}$ is assumed to satisfy the conditions

$$\boldsymbol{\phi}^{\nu}(\,.\,,0) \;=\; \mathbf{0} \qquad \text{on } \partial\Omega, \tag{4.9}$$

$$\big(|||\boldsymbol{\phi}^{\nu}|||_{2;\,3/2,2;\,\partial\Omega} + |||\partial_t\boldsymbol{\phi}^{\nu}|||_{2;\,1/2,2;\,\partial\Omega}\big) \;\leq\; \nu^{\alpha}c_{\phi}(\nu) \qquad \text{for } \nu \geq 0, \tag{4.10}$$

where $\frac{3}{4} < \alpha \leq 1$ and $c_{\phi}(\nu)$ is non–decreasing and continuous in dependence on ν.

The necessity of having function $\mathbf{a}(\nu)$ in the form given by condition (A2) is discussed in the remark after the next theorem.

Theorem 4 (on a family of solutions of the Euler or Navier–Stokes problem). *Let functions* $\mathbf{u}^* \in \mathbf{W}^{4,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$, $\mathbf{f} \in L^1\big(0,T;\, \mathbf{W}^{4,2}(\Omega)\big)$ *be given. Let assumptions (A1) and (A2) be fulfilled. Then there exists* $T_0 \in (0,T]$, $\nu^* > 0$ *and a unique family* $\{\mathbf{u}^\nu\}$ *(for* $0 \leq \nu < \nu^*$*) of solutions of the Euler problem (if* $\nu = 0$*) or the Navier–Stokes problem (if* $0 < \nu < \nu^*$*) in* $L^\infty\big(0,T_0;\, \mathbf{W}^{1,2}_\sigma(\Omega)\big) \cap L^2\big(0,T_0, \mathbf{W}^{2,2}(\Omega)\big)$.

Solution $\mathbf{u}^\nu$ *depends continuously on* ν *in the norm* $|||\,.\,|||_{\infty;\,1,2} + |||\,.\,|||_{2;\,2,2}$.

There exist positive constants c_1, c_2, *independent of* ν, *such that*

$$|||\mathbf{u}^\nu - \mathbf{u}^0|||_{\infty;\,1,2} \;\leq\; c_1\,\nu^\alpha, \tag{4.11}$$

$$|||\mathbf{u}^\nu - \mathbf{u}^0|||_{2;\,2,2} \;\leq\; c_2\,\nu^{\alpha-1/2}. \tag{4.12}$$

(Recall that α is the number from condition (A2). Theorem 4 is due to J.N and P. Penel, to appear in JMFM.)

Remark. In the proof, we construct $\mathbf{u}^\nu$ in the form

$$\mathbf{u}^\nu = \mathbf{u}^0 + \mathbf{U}^\nu, \qquad p^\nu = p^0 + q^\nu.$$

where $\mathbf{U}^\nu$, q^ν are perturbations of $\mathbf{u}^0$, p^0, tending to zero as $\nu \to 0$.

If this is substituted to the boundary condition (4.8), we get

$$\big[\mathbb{T}^\nu(\mathbf{u}^0)\cdot\mathbf{n}\big]_\tau + c_3\nu\mathbf{u}^0 + \big[\mathbb{T}^\nu(\mathbf{U}^\nu)\cdot\mathbf{n}\big]_\tau + c_3\nu\mathbf{U}^\nu \;=\; \mathbf{a}(\nu). \tag{4.13}$$

The expressions $\big[\mathbb{T}^\nu(\mathbf{u}^0)\cdot\mathbf{n}\big]_\tau + c_3\nu\mathbf{u}^0$ and $\big[\mathbb{T}^\nu(\mathbf{U}^\nu)\cdot\mathbf{n}\big]_\tau + c_3\nu\mathbf{U}^\nu$ generally have a different decay for $\nu \to 0$: the first expression equals $O(\nu)$, while the second one equals $o(\nu)$ for $\nu \to 0$.

Thus, equation (4.13) confirms that function $\mathbf{a}(\nu)$ cannot be chosen arbitrarily:

$\mathbf{a}(\nu)$ must be equal to $\big[\mathbb{T}^\nu(\mathbf{u}^0)\cdot\mathbf{n}\big]_\tau + c_3\nu\mathbf{u}^0$, eventually plus something that equals $o(\nu)$ for $\nu \to 0$. This is the sense of assumption (A2). Then the boundary condition (4.8) is **"naturally inhomogeneous"** because (A2) expresses the only form that the right hand side of (4.8) may have if solution $\mathbf{u}^0$ is approximated by the family $\{\mathbf{u}^\nu\}$ in the considered "strong topology".