Linear Stability Analysis of Falling Films with Chiba's Method

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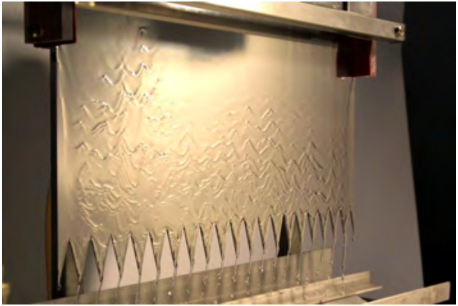
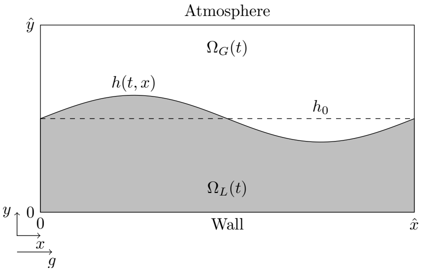
Motivation: Onset of waves on a falling film



- Applied for the transfer of heat or mass
- Surface Waves increase transfer rates
- Understand onset of waves by linear stability analysis

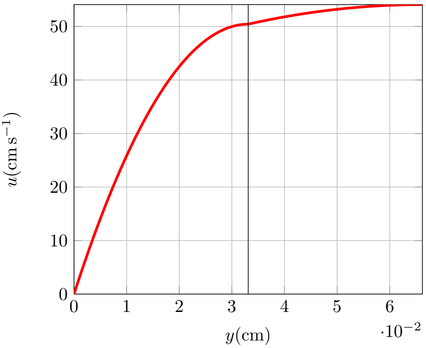
Hydrodynamic Model



- Incompressible, isothermal, Newtonian two-phase flow
- No phase transition
- Constant surface tension

$$
\begin{aligned}
\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \eta\Delta\mathbf{u} + \nabla p &= \rho\mathbf{g}, && \Omega_L \cup \Omega_G \\
\nabla\cdot\mathbf{u} &= 0, && \Omega_L \cup \Omega_G \\
[\![u]\!] &= 0, && y = h_0(x) + h(t,x) \\
[\![p\mathbf{I} - \eta(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)]\!]\mathbf{n}_\Sigma &= \sigma\kappa\mathbf{n}_\Sigma, && y = h_0(x) + h(t,x) \\
\mathbf{u} &= 0, && y = 0 \\
\frac{\partial\mathbf{u}}{\partial y} &= 0, && y = \hat{y}
\end{aligned}
$$

Steady State



$$
\begin{aligned}
p_N &= \text{const.}, h_0 = \text{const.} > 0, h = 0,\\
u_N(y) &= \frac{\rho_L g}{2\mu_L}(2h_0 y - y^2)\\
&\quad + \frac{\rho_G g}{\eta_L}(\hat{y} - h_0)y, 0 \le y \le h_0\\
u_N(y) &= \frac{\rho_G g}{2\eta_G}(2\hat{y}y - y^2)\\
&\quad + h_0^2 g\left(\frac{\rho_L}{2\eta_L} + \rho_G(\frac{1}{2\eta_G} - \frac{1}{\eta_L})\right)\\
&\quad + g h_0 \hat{y}(\frac{\rho_G}{\eta_L} - \frac{\rho_G}{\eta_G}), h_0 \le y \le \hat{y},\\
v_N &= 0,
\end{aligned}
$$

Stability for dynamical systems

Consider a dynamical system on a finite dimensional phase space $\mathbf{X}$:

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{f}(\mathbf{x}),$$
$$\mathbf{x}(0) = \mathbf{x}_{\text{ini}}.$$

Stability for dynamical systems

Consider a dynamical system on a finite dimensional phase space $\mathbf{X}$:

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{f}(\mathbf{x}),$$
$$\mathbf{x}(0) = \mathbf{x}_{\text{ini}}.$$

Let $\mathbf{x}_0$ be a steady state, i.e. $\mathbf{f}(\mathbf{x}_0) = 0$. The linearized stability equations are written as

$$\frac{\partial \zeta(t)}{\partial t} = \mathbf{A}\zeta(t),$$
$$\zeta(0) = \zeta_0,$$

where $\mathbf{A} = \frac{\partial \mathbf{f}(\mathbf{x}_0)}{\partial \mathbf{x}}$ is the Jacobian.

Chiba's Method

Solution of the linearized stability equations is given by

$$\zeta(t) = \exp(\mathbf{A}t)\zeta_0.$$

The asymptotic stability is determined by the spectrum of $\mathbf{A}$ whose dimension can be extremely large.

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Idea: Compute low dimensional approximation to the evolution operator $\mathbf{B} := \exp(\mathbf{A}T)$ for some fixed $T > 0$, and compute approximative spectrum of $\mathbf{A}$ via the spectral mapping theorem.

Chiba's Method

- Iterative Algorithm that computes a subspace

$$\tilde{\mathbf{X}} = \text{span}\{\zeta_1, \dots, \zeta_M\} \subset \mathbf{X}$$

  such that the most amplified eigenmodes can be well approximated in $\tilde{\mathbf{X}}$.
- Need a solver for the nonlinear problem, a steady state $\mathbf{x}_0$, and an initial perturbation ζ_1.
- Choose $\zeta_1 \in \mathbf{X}$ randomly.

Chiba's Method for dynamical systems

Compute random initial perturbation ζ_1, $\|\zeta_1\| = 1$

for $k = 1$ to M:

Simulate dynamical system for time T, starting with two different initial values $\mathbf{x}_0 + \epsilon\zeta_k$ and $\mathbf{x}_0 - \epsilon\zeta_k$. Store the results as $\mathbf{x}_{k+}$ and $\mathbf{x}_{k-}$.

$$\mathbf{B}\zeta_k = (\mathbf{x}_{k+} - \mathbf{x}_{k-})/2\epsilon$$

Compute ζ_{k+1} with Gram-Schmidt:

$$\mathbf{B}\zeta_k = \sum c_{k,l}\zeta_l, l = 1 \dots k+1$$

$$< \zeta_i, \zeta_j > = \delta_{i,j}.$$

next k

Chiba's Method

Last step:

$$
\begin{aligned}
\mathbf{B}\zeta_1 &= c_{1,1}\zeta_1 + c_{2,1}\zeta_2 \\
&\dots \\
\mathbf{B}\zeta_{M-1} &= c_{1,M-1}\zeta_1 + \ldots + c_{M,M-1}\zeta_M. \\
\mathbf{B}\zeta_M &= c_{1,M}\zeta_1 + \ldots + c_{M,M}\zeta_M + c_{M+1,M}\zeta_{M+1}.
\end{aligned}
$$

If $|c_{M+1,M}|$ is small, we can ignore that term.

Chiba's Method

Last step:

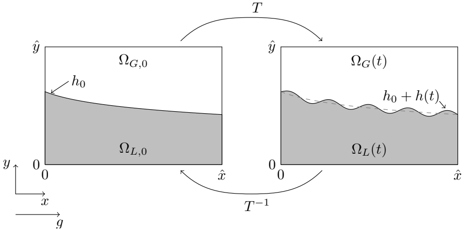
$$
\begin{aligned}
\tilde{\mathbf{B}}\zeta_1 &= c_{1,1}\zeta_1 + c_{2,1}\zeta_2 \\
&\dots \\
\tilde{\mathbf{B}}\zeta_{M-1} &= c_{1,M-1}\zeta_1 + \dots + c_{M,M-1}\zeta_M. \\
\tilde{\mathbf{B}}\zeta_M &= c_{1,M}\zeta_1 + \dots + c_{M,M}\zeta_M.
\end{aligned}
$$

$\tilde{\mathbf{B}}$ is a square matrix, and we can discuss stability on the basis of this.

Chiba's Method for free boundary problems

- Problem: In order to apply Chiba's Method, we have to perform linear algebra, but for falling films, the phase space is not a vector space.
- Solution: Transform velocity field to a fixed reference domain.

Chiba's Method for free boundary problems



- Diffeomorphism between fixed and physical domain:

$$\mathbf{T}(x', y') = \begin{pmatrix} x' \\ \frac{h(t,x')}{h_0(x')^2 - \hat{y}\,h_0(x')} y'^2 + \frac{h_0(x')^2 - \hat{y}\,h_0(x') - \hat{y}\,h(t,x')}{h_0(x')^2 - \hat{y}\,h_0(x')} y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

- Transform velocity in a divergence free way:

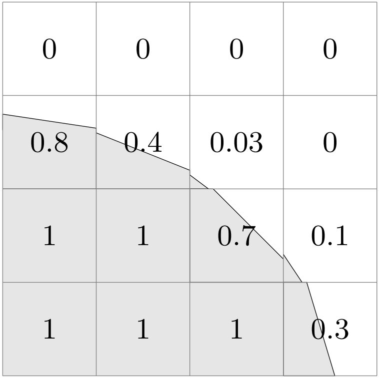
$$\begin{pmatrix} u \\ v \end{pmatrix}(x, y) = \left[\frac{1}{\det D\mathbf{T}} D\mathbf{T} \begin{pmatrix} u' \\ v' \end{pmatrix} \right](x', y') \text{ where } \mathbf{T}(x', y') = (x, y).$$

Chiba's Method for free boundary problems

- With this transformation, $\begin{pmatrix} u' \\ v' \end{pmatrix}$ solenoidal if and only if $\begin{pmatrix} u \\ v \end{pmatrix}$ solenoidal.
- Phase space consists of solenoidal velocity fields on the fixed domain, combined with an interfacial perturbation.
- Pressure is treated implicitly.

Simulating the evolution of a perturbation

- Utilize volume of fluid solver *FS3D*.
- Interface advection via *PLIC*.
- Surface tension via *Balanced CSF* with height functions.
- The solver is well validated for falling films, see [ARB12].



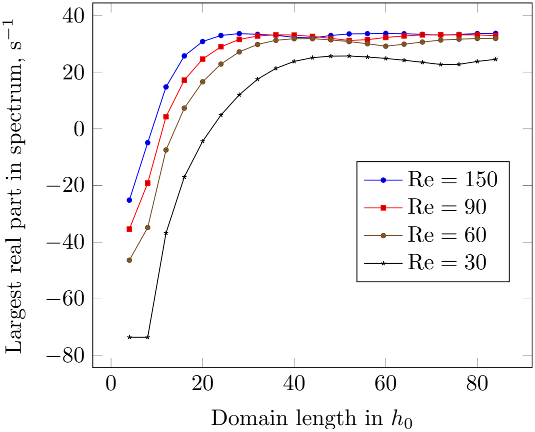
Periodic Case

- Boundary conditions in streamwise direction:

$$\mathbf{u}(t, 0, y) = \mathbf{u}(t, \hat{x}, y)$$
$$h(t, 0) = h(t, \hat{x}).$$

- Analyzed stability of ten water/air falling films at Reynolds numbers from 15 to 150, with different domain lengths.
- Stability behavior in this setup equivalent to results achieved by Orr-Sommerfeld equations.

Stability in dependence of domain length



- For small domain lengths, real shear modes dominate.
- With growing domain length, films become more unstable due surface modes of maximal wavelength.
- After local minimum: Modes with smaller than maximal wavelength dominate.

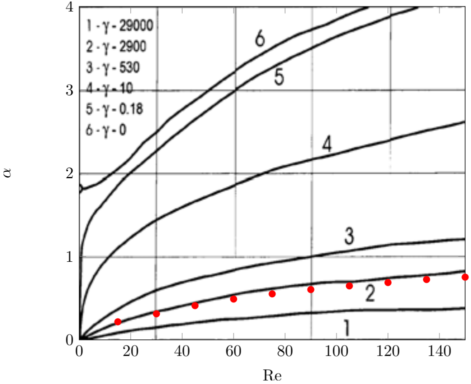
Neutral stability curve

- Define a dimensionless wave number α as

$$\alpha = \frac{2\pi h_0}{\hat{x}},$$

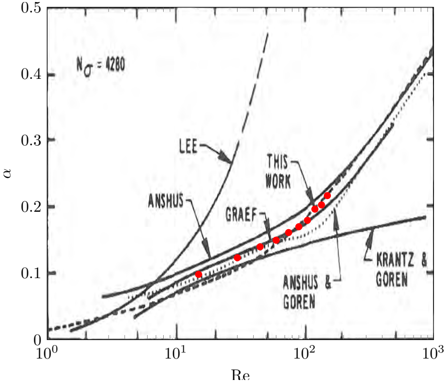
and look for α at which the film is neutrally stable.

- Comparison to [CD02]:



Wave number of most amplified mode

- Compare wavenumber of the most unstable mode [PW77]:

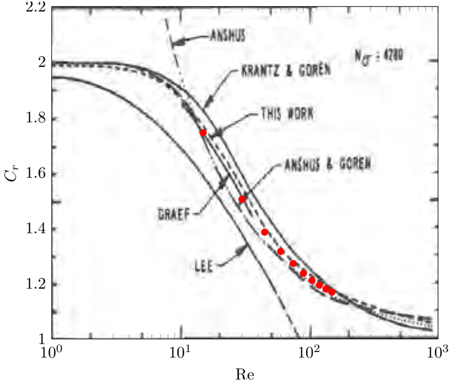


Wave velocity of most amplified mode

- Define a dimensionless wave velocity C_r as

$$C_r = \frac{\mathfrak{Im}\lambda_1 l}{2\pi u_{max}},$$

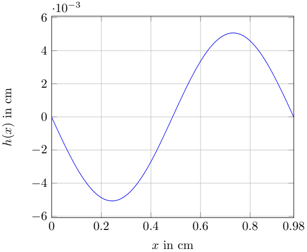
- Compare to [PW77]

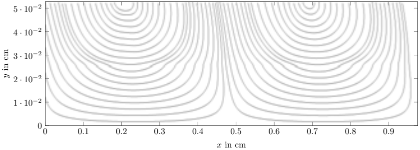


Surface Modes

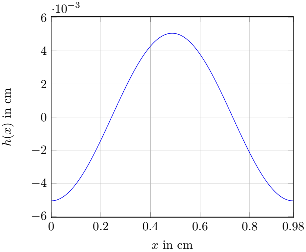
Most amplified mode at Re $= 90$,
$\hat{x} = 37h_0, \lambda = (33.20 + 271.7i)\mathrm{s}^{-1}$

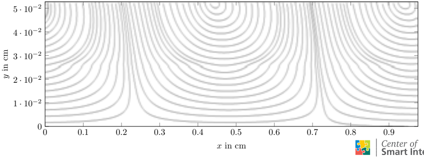
Real part





Imaginary part





Center of Smart Interfaces

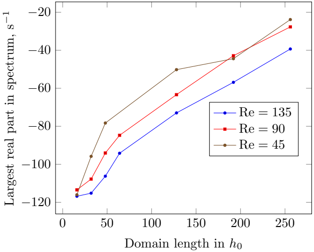
Inflow/Outflow case

- Boundary conditions in streamwise direction:

$$
\begin{aligned}
\mathbf{u} &= \mathbf{u}_N && \text{at } x = 0, \\
\frac{\partial \mathbf{u}}{\partial x} &= 0 && \text{at } x = \hat{x}, \\
h_0 + h &= h_0 && \text{at } x = 0, \\
\frac{\partial (h_0 + h)}{\partial x} &= 0 && \text{at } x = \hat{x}.
\end{aligned}
$$

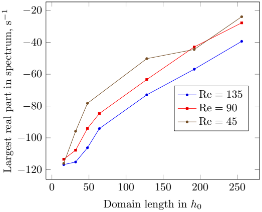
- Standard boundary conditions for VOF simulations of falling films; deliver good agreement with experiments.
- Analyzed stability of water/air falling films at Reynolds numbers from 45, 90, and 135, and at domain lengths up to $256 h_0$.

Stability in dependence of domain length



- Film is asymptotically stable for all considered setups.
- Becomes more unstable with a longer domain.
- For a fixed dimensionless domain length, the film becomes mores stable with a higher Reynolds number.

Stability in dependence of domain length



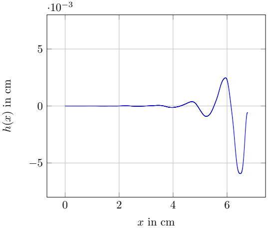
- Length of the domain: $\hat{x} = n(\frac{2\eta_L^2}{\rho_L^2 g})^{1/3}\,\mathrm{Re}^{1/3}$.
- Lower bound for wave inception line according to [PW77]: $L_e = (\frac{2\eta_L^2}{\rho_L^2 g})^{1/3}\,\mathrm{Re}^{4/3}$.

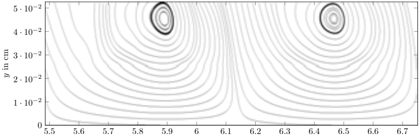
Complex Modes

Most amplified mode at Re = 90,

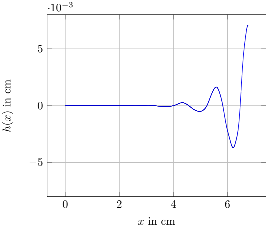
$$\hat{x} = 256h_0, \lambda = (-27.69 + 213.7i)\mathrm{s}^{-1}, d_r = 0.0927$$

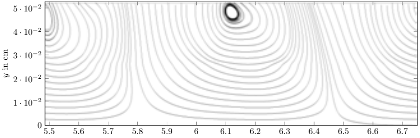
Real part





Imaginary part





- Critical wavelength for $\mathrm{Re} = 90$ in the periodic setting is about $10.5h_0$; wavelength of the most amplified mode is $37h_0$.
- No amplified modes on a domain of length $256h_0$ in the inflow/outflow-setting.
- Calculations on longer domains are needed.

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Thank you for your attention!

References I

[ARB12] Albert, Christoph ; Raach, Henning ; Bothe, Dieter: Influence of surface tension models on the hydrodynamics of wavy laminar falling films in Volume of Fluid-simulations. In: *International Journal of Multiphase Flow* 43 (2012), Nr. 0, 66 - 71. http://dx.doi.org/10.1016/j.ijmultiphaseflow.2012.02.011. – DOI 10.1016/j.ijmultiphaseflow.2012.02.011. – ISSN 0301–9322

[CD02] Chang, H.-C. ; Demekhin, E. A. ; Möbius, Reinhard Dietmar; M. Dietmar; Miller (Hrsg.): *Complex Wave Dynamics on Thin Films*. Elsevier, 2002 (Studies in Interface Science)

References II

[PW77] Pierson, F. W. ; Whitaker, Stephen: Some Theoretical and Experimental Observations of the Wave Structure of Falling Liquid Films. In: *Industrial & Engineering Chemistry Fundamentals* 16 (1977), Nr. 4, 401-408. http://dx.doi.org/10.1021/i160064a002. – DOI 10.1021/i160064a002

Scalar product

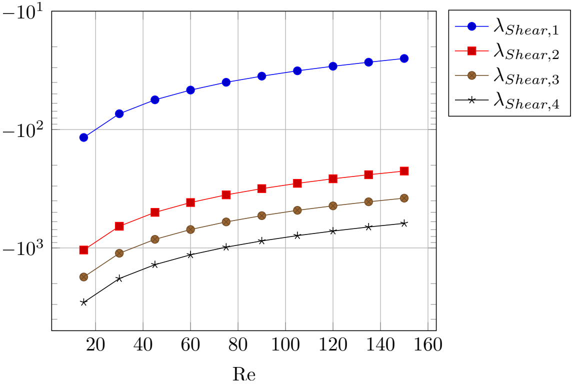
- Outcome of stability analysis independent of norm.
- Convenient to use a physically reasonable scalar product:

$$
\begin{aligned}
< (\mathbf{u}_\alpha, h_\alpha), (\mathbf{u}_\beta, h_\beta) > = & \frac{1}{2}\int_0^{\hat{x}}\int_0^{\hat{y}} \rho(x,y)\mathbf{u}_\alpha\overline{\mathbf{u}_\beta}dxdy \\
& + \frac{1}{2}\int_0^{\hat{x}} \sigma(\partial_x h_\alpha)\overline{(\partial_x h_\beta)}dx.
\end{aligned}
$$

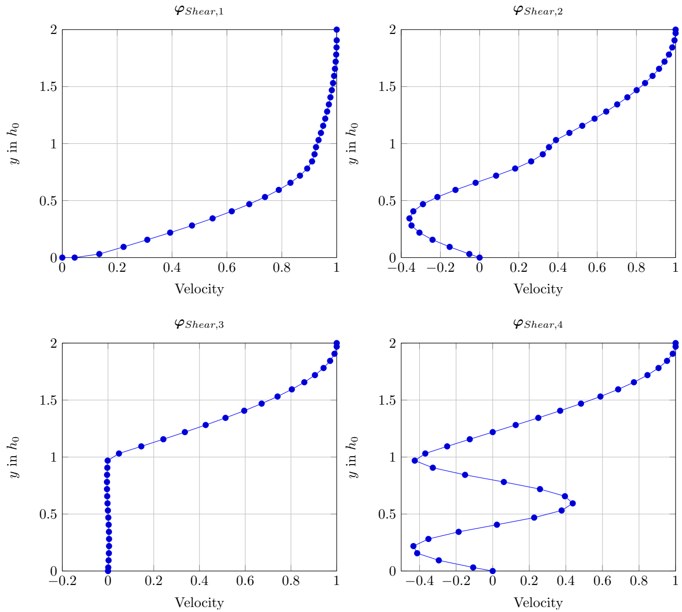
- Sum of kinetic energy and energy of interface disturbance.

Shear Modes

- Real modes with no interfacial perturbation and no velocity component in y-direction.
- Independent of domain length
- If written as $u(y/h_0)$, velocity independent of Re.
- Growth rate increases with Re.



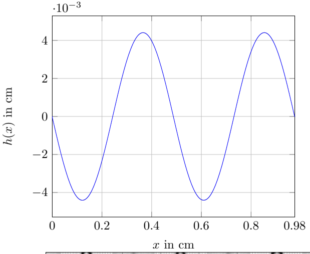
Shear Modes

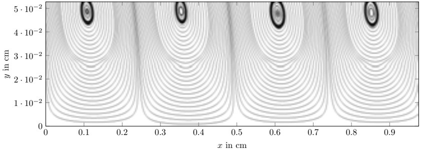

$\varphi_{Shear,1}$
$\varphi_{Shear,2}$
$\varphi_{Shear,3}$
$\varphi_{Shear,4}$
y in h_0
Velocity
0 0.5 1 1.5 2
0 0.2 0.4 0.6 0.8 1
−0.4 −0.2 0 0.2 0.4 0.6 0.8 1
−0.2 0 0.2 0.4 0.6 0.8 1

Surface Modes

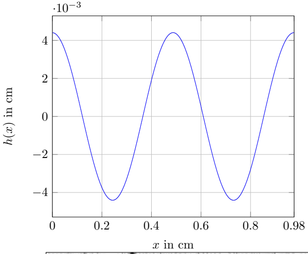
Second most amplified mode at $\mathrm{Re} = 90$,
$\hat{x} = 37h_0, \lambda = (22.29 + 596.6i)\mathrm{s}^{-1}$

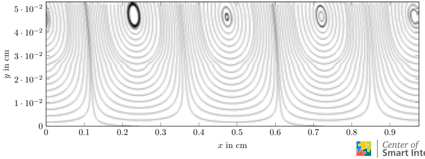
Real part





Real part





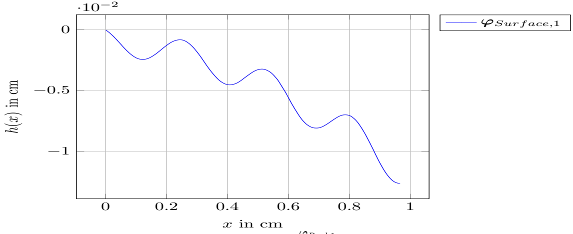
Verification

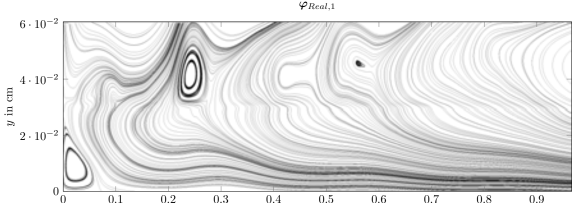
- In order verify to that the found modes are actually eigenmodes, we initialized the solver with steady state + mode, and let the simulation run for time $T = -1/\mathfrak{Re}(\lambda)$, and calculated the expected outcome due to linear theory.
- Define a relative error by

$$d_r = 2\frac{\|\mathbf{B}\varphi_{1,numerical} - \mathbf{B}\varphi_{1,theoretical}\|}{\|\mathbf{B}\varphi_{1,numerical}\| + \|\mathbf{B}\varphi_{1,theoretical}\|}$$

Real modes

- Also in this case there are real modes present, although they can not be characterized as shear modes.
- Example: $\mathrm{Re} = 135, \hat{x} = 32h_0, \lambda = (-115.2)\mathrm{s}^{-1}, d_r = 0.0391$:

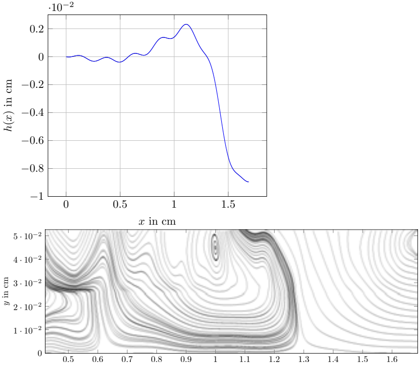




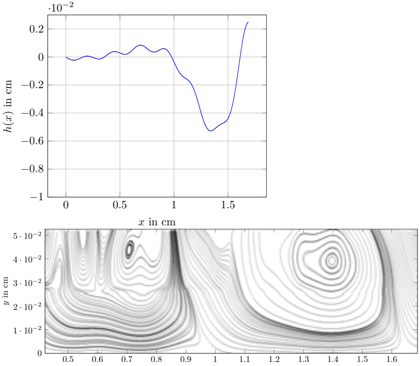
Complex Modes

Most amplified mode at Re $= 90$,
$\hat{x} = 64h_0, \lambda = (-84.74 + 195.1i)\mathrm{s}^{-1}, d_r = 0.0224$

Real part



Imaginary part

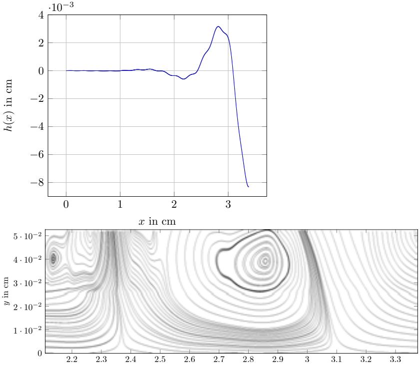


Complex Modes

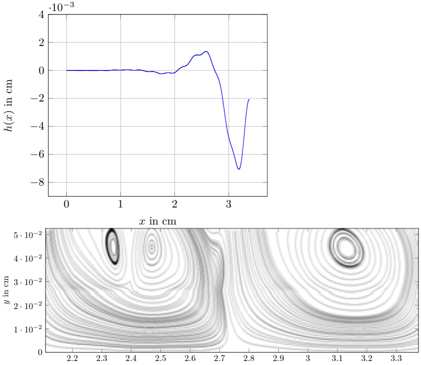
Most amplified mode at Re $= 90$,

$$\hat{x} = 128h_0, \lambda = (-63.35 + 195.2i)\mathrm{s}^{-1}, d_r = 0.0121$$

Real part



Imaginary part



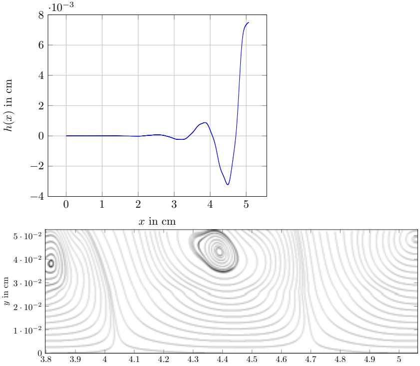
Center of Smart Interfaces

Complex Modes

Most amplified mode at Re $= 90$,

$$\hat{x} = 192h_0, \lambda = (-42.87 + 207.4i)\mathrm{s}^{-1}, d_r = 0.0218$$

Real part



Imaginary part