On the Stokes equations with Neumann-Robin boundary conditions in an infinite layer. The resolvent problem in the case $\lambda = 0$.

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Lorenz von Below

Joint work with Hirokazu Saito (Waseda University)

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The Navier-Stokes equations with free boundary in an infinite layer

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\partial_t u + u \cdot \nabla u - \Delta u + \nabla\theta &= f, \qquad \operatorname{div} u = 0 && \text{in } \Omega(t) \\
S(u,\theta)\nu &= 0 && \text{on } \Gamma_h(t) \\
V &= u \cdot \nu && \text{on } \Gamma_h(t) \\
u - c(\eta + h)^\alpha \partial_n u\big|_{\tan} &= 0 && \text{on } \Gamma_0 \\
u_n &= 0 && \text{on } \Gamma_0
\end{aligned}
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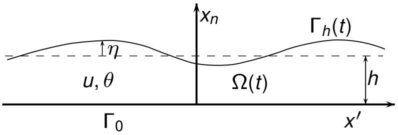
+ initial values.

- Tranformation to a fixed domain (Hanzawa transform)
- Analysis of the associated resolvent problem



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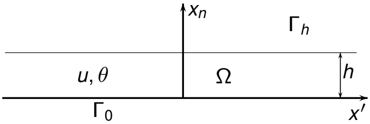

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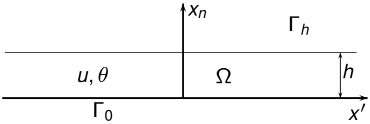

The generalised resolvent problem

The case without surface tension

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Let $\Omega = \mathbb{R}^{n-1} \times (0, h)$. We study the equation

$$\begin{cases} \lambda u - \Delta u + \nabla\theta = f, \quad \operatorname{div} u = 0 & \text{in } \Omega \\ S(u,\theta)\nu = g^+ & \text{on } \Gamma_h \\ (u - \partial_n u)\,\big|_{\tan} = g^- & \text{on } \Gamma_0 \\ u_n = 0 & \text{on } \Gamma_0. \end{cases}$$

- Now: Existence and uniqueness of solutions in the case $\lambda = 0$
- Immediate consequence: Analogous result for $|\lambda|$ small
- λ from a sector $\Sigma_\varepsilon \subset \mathbb{C}$ in the talk of H. Saito

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Main result

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Theorem

Let $1 < p < \infty$*. For every*

$$f \in L_p(\Omega), \quad g^+ \in W_p^{1-1/p}(\Gamma_h), \quad g^- \in W_p^{1-1/p}(\Gamma_0)$$

the equation

$$\begin{cases} -\Delta u + \nabla\theta = f, \qquad \operatorname{div} u = 0 & \text{in } \Omega \\ S(u,\theta)\nu = g^+ & \text{on } \Gamma_h \\ (u - \partial_n u)\big|_{tan} = g^- & \text{on } \Gamma_0 \\ u_n = 0 & \text{on } \Gamma_0 \end{cases}$$

has a unique solution $(u,\theta) \in W_p^2(\Omega) \times W_p^1(\Omega)$ *with*

$$\|(u,\theta)\|_{W_p^2(\Omega)\times W_p^1(\Omega)} \lesssim \|f\|_{L_p(\Omega)} + \|g^+\|_{W_p^{1-1/p}(\Gamma_h)} + \|g^-\|_{W_p^{1-1/p}(\Gamma_0)}.$$

Related results

- Abe, Shibata 2001: Homogeneous Dirichlet BC, $\lambda \in \Sigma_\varepsilon$
- Abe, Shibata 2003: Homogeneous Dirichlet BC, $\lambda \in \Sigma_\varepsilon \cup B(0, \delta)$
- Abe 2004: Inhomogeneous Neumann BC on Γ_h, homogeneous Dirichlet BC on Γ_0, $\lambda \in \Sigma_\varepsilon \cup B(0, \delta)$
- Abels, Abels and Wiegner ...

It suffices to consider

$$\begin{cases} -\Delta u + \nabla\theta = 0, \quad \operatorname{div} u = 0 & \text{in } \Omega \\ S(u,\theta)\nu = 0 & \text{on } \Gamma_h \\ (u - \partial_n u)\big|_{\tan} = g^- & \text{on } \Gamma_0 \\ u_n = 0 & \text{on } \Gamma_0. \end{cases}$$

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Fourier transform in tangential direction

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For $\xi' \in \mathbb{R}^{n-1}$

$$\begin{cases} (-\partial_n^2 + |\xi'|^2)\hat{u}_j + i\xi_j\hat{\theta} = 0 \quad (j = 1, \dots, n-1) & \text{in } (0, h) \\ (-\partial_n^2 + |\xi'|^2)\hat{u}_n + \partial_n\hat{\theta} = 0 & \text{in } (0, h) \\ \sum_{j=1}^{n-1} i\xi_j\hat{u}_j + \partial_n\hat{u}_n = 0 & \text{in } (0, h) \\ i\xi_j\hat{u}_n(\xi', h) + \partial_n\hat{u}_j(\xi', h) = 0 \quad (j = 1, \dots, n-1) & \\ -\hat{\theta}(\xi', h) + 2\partial_n\hat{u}_n(\xi', h) = 0 & \\ \hat{u}_j(\xi', 0) - \partial_n\hat{u}_j(\xi', 0) = \hat{g}_j^-(\xi') \quad (j = 1, \dots, n-1) & \\ \hat{u}_n(\xi', 0) = 0 & \end{cases}$$

Equations for u_n and θ

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$$\begin{cases} (-|\xi'|^2 + \partial_n^2)^2 \hat{u}_n = 0 & \text{in } (0,h) \\ (-|\xi'|^2 + \partial_n^2)\hat{\theta} = 0 & \text{in } (0,h) \end{cases}$$

+ boundary conditions and $(|\xi'|^2 - \partial_n^2)\hat{u}_n + \partial_n \hat{\theta} = 0$ in $(0, h)$.

Basic theory for linear ODEs:

$$\begin{cases} \hat{u}_n(\xi', x_n) = a_1 e^{-|\xi'|x_n} + a_2 x_n e^{-|\xi'|x_n} + a_3 e^{-|\xi'|(h-x_n)} + a_4 (h-x_n) e^{-|\xi'|(h-x_n)} \\ \hat{\theta}(\xi', x_n) = b_1 e^{-|\xi'|x_n} + b_2 e^{-|\xi'|(h-x_n)}. \end{cases}$$

Equations for u_n and θ: Boundary conditions

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Given $\xi' \in \mathbb{R}^{n-1}$ find $a_1, \dots, a_4, b_1, b_2 \in \mathbb{C}$ such that

$$\hat{u}_n(\xi', x_n) = a_1 e^{-\left|\xi'\right| x_n} + a_2 x_n e^{-\left|\xi'\right| x_n} + a_3 e^{-\left|\xi'\right|(h-x_n)} + a_4 (h - x_n) e^{-\left|\xi'\right|(h-x_n)}$$

$$\hat{\theta}(\xi', x_n) = b_1 e^{-\left|\xi'\right| x_n} + b_2 e^{-\left|\xi'\right|(h-x_n)}$$

satisfy

$$\begin{cases} (\left|\xi'\right|^2 - \partial_n^2)\hat{u}_n(\xi', h) + \partial_n \hat{\theta}(\xi', h) = 0 \\ -\hat{\theta}(\xi', h) + 2\partial_n \hat{u}_n(\xi', h) = 0 \\ (\left|\xi'\right|^2 + \partial_n^2)\hat{u}_n(\xi', h) = 0 \\ (\left|\xi'\right|^2 - \partial_n^2)\hat{u}_n(\xi', 0) + \partial_n \hat{\theta}(\xi', 0) = 0 \\ (-\partial_n + \partial_n^2)\hat{u}_n(\xi', 0) = \sum_{j=1}^{n-1} i\xi_j \hat{g}_j^-(\xi') \\ \hat{u}_n(\xi', 0) = 0. \end{cases}$$

Reduction to a linear equation in $\mathbb{C}^6$

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For $\xi' \in \mathbb{R}^{n-1}$ we obtain a linear equation

$$L(|\xi'|)\begin{pmatrix} a_1 \\ \vdots \\ a_4 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=1}^{n-1} i\xi_j \hat{g}_j^-(\xi') \\ 0 \end{pmatrix}$$

with a matrix $L(|\xi'|) \in \mathbb{C}^{6\times 6}$. Regard L as function of $z \in \mathbb{C}$.

Lemma

There is an open sector

$$\mathfrak{S}_\varepsilon = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \varepsilon\}, \qquad \varepsilon > 0,$$

such that $L(z)$ is regular for $z \in \mathfrak{S}_\varepsilon$.

The matrix $L(z)$

$$\begin{pmatrix} 0 & 2z & 0 & 2ze^{-zh} & -z & ze^{-zh} \\ 0 & 2ze^{-zh} & 0 & 2z & -ze^{-zh} & z \\ -2ze^{-zh} & 2(1-zh)e^{-zh} & 2z & -2 & -e^{-zh} & -1 \\ 2z^2e^{-zh} & -2z(1-zh)e^{-zh} & 2z^2 & -2z & 0 & 0 \\ z+z^2 & -1-2z & (-z+z^2)e^{-zh} & (1-z(2+h-zh))e^{-zh} & 0 & 0 \\ 1 & 0 & e^{-zh} & 0 & 0 & 0 \end{pmatrix}$$

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Explicit representation formulae for u_n and θ

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By Cramer's rule:

$$\hat{u}_n(\xi', x_n) = \left[\frac{L^\sharp_{1,5}(|\xi'|)}{\det L(|\xi'|)} e^{-|\xi'|x_n} + \frac{L^\sharp_{2,5}(|\xi'|)}{\det L(|\xi'|)} x_n e^{-|\xi'|x_n} + \frac{L^\sharp_{3,5}(|\xi'|)}{\det L(|\xi'|)} e^{-|\xi'|(h-x_n)} \right.$$

$$\left. + \frac{L^\sharp_{4,5}(|\xi'|)}{\det L(|\xi'|)} (h - x_n) e^{-|\xi'|(h-x_n)} \right] \sum_{j=1}^{n-1} i\xi_j \hat{g}_j^-(\xi')$$

$$\hat{\theta}(\xi', x_n) = \left[\frac{L^\sharp_{5,5}(|\xi'|)}{\det L(|\xi'|)} e^{-|\xi'|x_n} + \frac{L^\sharp_{6,5}(|\xi'|)}{\det L(|\xi'|)} e^{-|\xi'|(h-x_n)} \right] \sum_{j=1}^{n-1} i\xi_j \hat{g}_j^-(\xi')$$

with $L^\sharp_{ij}$ the entries of the cofactor matrix associated to L. Take an exemplary look at

$$\mathcal{F}^{-1}_{\xi'} e^{-|\xi'|x_n} \frac{L^\sharp_{5,5}(|\xi'|)}{\det L(|\xi'|)} i\xi_j \mathcal{F}_{x'} g_j^-$$

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Analysis of the solution operator

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$$\mathcal{F}_{\xi'}^{-1} e^{-|\xi'| x_n} \frac{|\xi'| \, L_{5,5}^{\sharp}}{\det L} i \frac{\xi_j}{|\xi'|} \mathcal{F}_{x'} g_j^-, \qquad g_j^- \in W_p^{1-1/p}(\mathbb{R}^{n-1})$$

- Riesz transform: $i\xi_j / |\xi'|$. Bounded operator on $W_p^{1-1/p}(\mathbb{R}^{n-1})$.
- The function

$$z \mapsto \frac{z L_{5,5}^{\sharp}(z)}{\det L(z)}$$

is holomorphic and bounded on a sector $\mathfrak{S}_\varepsilon$.
- Mikhlin's Multiplier Theorem + Cauchy's Integral Formula:

$$\frac{|\xi'| \, L_{5,5}^{\sharp}(|\xi'|)}{\det L(|\xi'|)} \quad \text{is a Fourier multiplier on } W_p^{1-1/p}(\mathbb{R}^{n-1}).$$

- Agmon-Douglis-Nirenberg:

$$\mathcal{F}_{\xi'}^{-1} e^{-|\xi'| x_n} \mathcal{F}_{x'} \quad \text{is bounded from } W_p^{1-1/p}(\mathbb{R}^{n-1}) \text{ to } W_p^1(\Omega).$$

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- With similar considerations:

$$\|u_n\|_{W_p^2(\Omega)} + \|\theta\|_{W_p^1(\Omega)} \lesssim \left\| g^- \right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}$$

- Similarly but easier: For $j = 1, \dots, n-1$

$$\|u_j\|_{W_p^2(\Omega)} \lesssim \left\| g^- \right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}$$

Analysis of the solution operator

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$$\mathcal{F}^{-1}_{\xi'} e^{-|\xi'|x_n} \frac{|\xi'|\, L^{\sharp}_{5,5}}{\det L} i \frac{\xi_j}{|\xi'|} \mathcal{F}_{x'} g_j^-, \qquad g_j^- \in W_p^{1-1/p}(\mathbb{R}^{n-1})$$

- $$\left\| \mathcal{F}^{-1}_{\xi'} e^{-|\xi'|x_n} \frac{|\xi'|\, L^{\sharp}_{5,5}}{\det L} i \frac{\xi_j}{|\xi'|} \mathcal{F}_{x'} g_j^- \right\|_{W_p^1(\Omega)} \lesssim \left\| g_j^- \right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}$$
- With similar considerations:

$$\|u_n\|_{W_p^2(\Omega)} + \|\theta\|_{W_p^1(\Omega)} \lesssim \left\| g^- \right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}$$

- Similarly but easier: For $j = 1, \dots, n-1$

$$\|u_j\|_{W_p^2(\Omega)} \lesssim \left\| g^- \right\|_{W_p^{1-1/p}(\mathbb{R}^{n-1})}$$

Uniqueness of solutions

For $1 < p < \infty$ let

$$A_p \colon \mathbb{E}_p \to \mathbb{F}_p, \qquad (u, \theta) \mapsto \begin{pmatrix} -\Delta u + \nabla\theta \\ S(u,\theta) \cdot \nu \\ (u - \partial_n u)\big|_{\tan} \end{pmatrix}$$

with

$$\mathbb{E}_p = \Big\{ (u, \theta) \in W_p^2(\Omega) \times W_p^1(\Omega) \colon \operatorname{div} u = 0,\ u_n\big|_{\Gamma_0} = 0 \Big\}$$
$$\mathbb{F}_p = L_p(\Omega) \times W_p^{1-1/p}(\Gamma_h) \times W_p^{1-1/p}(\Gamma_0).$$

Uniqueness of solutions

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- For $1 < p < \infty$: $A_p : \mathbb{E}_p \to \mathbb{F}_p$ is surjective. In particular semi-Fredholm.
- Förster-Günther 2011: ind A_p is independent of $p \in (1, \infty)$.
- For $p = 2$: Uniqueness of solutions via energy estimates.
- Hence $A_2 : \mathbb{E}_2 \to \mathbb{F}_2$ is an isomorphism. In particular, ind $A_2 = 0$.
- Thus, A_p is an isomorphism for $1 < p < \infty$. □

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Further results

The generalised resolvent problem

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Theorem

Let $1 < p < \infty$*. There is* $\delta > 0$ *such that for every* $\lambda \in \mathbb{C}$ *with*

$$|\lambda| \leq \delta \qquad \text{or} \qquad |\arg \lambda| < \pi - \varepsilon$$

and

$$f \in L_p(\Omega), \quad g^+ \in W_p^{1-1/p}(\Gamma_h), \quad g^- \in W_p^{1-1/p}(\Gamma_0)$$

the equation

$$\begin{cases} \lambda u - \Delta u + \nabla\theta = f, & \operatorname{div} u = 0 \quad \text{in } \Omega \\ S(u,\theta)\nu = g^+ & \text{on } \Gamma_h \\ (u - \partial_n u)\big|_{tan} = g^- & \text{on } \Gamma_0 \\ u_n = 0 & \text{on } \Gamma_0 \end{cases}$$

has a unique solution $(u, \theta) \in W_p^2(\Omega) \times W_p^1(\Omega)$*.*

Further results

Including surface tension and gravity

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Theorem

Let $1 < p < \infty$*. For* $c_g, \sigma > 0$ *and*

$$f \in L_p(\Omega),\ g^+ \in W_p^{1-1/p}(\Gamma_h),\ k^+ \in W_p^{2-1/p}(\Gamma_h),\ g^- \in W_p^{1-1/p}(\Gamma_0),$$

$$\begin{cases} -\Delta u + \nabla\theta = f, \qquad \operatorname{div} u = 0 & \text{in } \Omega \\ S(u,\theta)\nu + (c_g - \sigma\Delta')\eta\nu = g^+ & \text{on } \Gamma_h \\ -u_n = k^+ & \text{on } \Gamma_h \\ (u - \partial_n u)\big|_{tan} = g^- & \text{on } \Gamma_0 \\ u_n = 0 & \text{on } \Gamma_0. \end{cases}$$

has a unique solution $(u,\theta,\eta) \in W_p^2(\Omega) \times W_p^1(\Omega) \times W_p^{3-1/p}(\mathbb{R}^{n-1})$ *satisfying*

$$\|(u,\theta)\|_{W_p^2 \times W_p^1} + c_g \|\eta\|_{W_p^{1-1/p}} + \sigma \left\|\nabla^2\eta\right\|_{W_p^{1-1/p}} \lesssim \left\|(f, g^+, k^+, g^-)\right\|.$$