The local existence and blow-up criterion for the compressible Navier-Stokes system with a Yukawa-potential in Besov spaces

Noboru Chikami

Mathematical Institute Tohoku University

2012/11/6

The 7th Japanese-German International Workshop on Mathematical Fluid Dynamics

Compressible Navier-Stokes-Yukawa system

$N \geq 2$

$\rho(t, x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: density

$u(t, x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$: velocity

$\psi(t, x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: potential

Compressible Navier-Stokes-Yukawa system

$N \geq 2$

$\rho(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: density

$u(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$: velocity

$\psi(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: potential

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \\ \\ \end{cases}$$

Compressible Navier-Stokes-Yukawa system

$N \geq 2$

$\rho(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: density

$u(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$: velocity

$\psi(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: potential

$$
\begin{cases}
\partial_t \rho + \text{div }(\rho u) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\partial_t(\rho u) + \text{div }(\rho u \otimes u) + \nabla P(\rho) & \\
\qquad = \mu \Delta u + (\mu + \lambda)\nabla \text{div } u + \rho \nabla \psi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,
\end{cases}
$$

Compressible Navier-Stokes-Yukawa system

$N \geq 2$

$\rho(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: density

$u(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$: velocity

$\psi(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: potential

$$
\begin{cases}
\partial_t \rho + \mathrm{div}\,(\rho u) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P(\rho) & \\
\qquad = \mu \Delta u + (\mu + \lambda) \nabla \mathrm{div}\, u + \rho \nabla \psi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
-\Delta \psi + \psi = \rho - \bar{\rho}, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N,
\end{cases}
$$

Compressible Navier-Stokes-Yukawa system

$N \geq 2$

$\rho(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: density

$u(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$: velocity

$\psi(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: potential

$$
\begin{cases}
\partial_t \rho + \text{div } (\rho u) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\partial_t(\rho u) + \text{div } (\rho u \otimes u) + \nabla P(\rho) & \\
\qquad = \mu \Delta u + (\mu + \lambda) \nabla \text{div } u + \rho \nabla \psi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
-\Delta \psi + \psi = \rho - \bar{\rho}, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\rho(0,x) = \rho_0(x), \ u(0,x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
$$

Compressible Navier-Stokes-Yukawa system

$N \geq 2$
$\rho(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: density
$u(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$: velocity
$\psi(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: potential

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P(\rho) & \\ \qquad = \mu \Delta u + (\mu + \lambda)\nabla \mathrm{div}\, u + \rho \nabla \psi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ -\Delta \psi + \psi = \rho - \bar{\rho}, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \rho(0,x) = \rho_0(x),\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

μ, λ : Lamé constant, $\mu > 0$, $\lambda + \mu > 0$,
$\bar{\rho} > 0$: Constant,

$P(\rho) = \rho^\alpha$, $\alpha \geq 1$: pressure
Physical model for the motion of an "exotic" nuclei.

Compressible Navier-Stokes system

$N \geq 2$
$\rho(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: density
$u(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$: velocity
$\psi(t,x)$: $\mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$: potential

$$
\begin{cases}
\partial_t \rho + \text{div }(\rho u) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\partial_t(\rho u) + \text{div }(\rho u \otimes u) + \nabla P(\rho) & \\
\qquad = \mu \Delta u + (\mu + \lambda)\nabla \text{div } u + \rho \nabla \psi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
-\Delta \psi + \psi = \rho - \bar{\rho}, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
\rho(0,x) = \rho_0(x), \ u(0,x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
$$

μ, λ : Lamé constant, $\mu > 0$, $\lambda + \mu > 0$,
$\bar{\rho} > 0$: Constant,

$P(\rho) = \rho^\alpha$, $\alpha \geq 1$: pressure
Physical model for the general barotropic compressible flow.

Scaling property

$$\begin{cases} \rho_\nu(t,x) := \rho(\nu^2 t, \nu x) \\ u_\nu(t,x) := \nu u(\nu^2 t, \nu x), \end{cases} \qquad \nu > 0.$$

Scaling property

$$\begin{cases} \rho_\nu(t,x) := \rho(\nu^2 t, \nu x) \\ u_\nu(t,x) := \nu u(\nu^2 t, \nu x), \end{cases} \quad \nu > 0.$$

Known results for compressible N-S

$\bar{\rho} > 0$: a positive constant.

- $(\rho - \bar{\rho}, u) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$, Matsumura-Nishida, 1980.

Scaling property

$$\begin{cases} \rho_\nu(t,x) := \rho(\nu^2 t, \nu x) \\ u_\nu(t,x) := \nu u(\nu^2 t, \nu x), \end{cases} \quad \nu > 0.$$

Known results for compressible N-S

$\bar{\rho} > 0$: a positive constant.

- $(\rho - \bar{\rho}, u) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$, Matsumura-Nishida, 1980.
- Scaling invariant Sobolev space :
  $(\rho - \bar{\rho}, u) \in \dot{H}^{\frac{N}{2}}(\mathbb{R}^N) \times \dot{H}^{\frac{N}{2}-1}(\mathbb{R}^N)$,

Scaling property

$$\begin{cases} \rho_\nu(t,x) := \rho(\nu^2 t, \nu x) \\ u_\nu(t,x) := \nu u(\nu^2 t, \nu x), \end{cases} \quad \nu > 0.$$

Known results for compressible N-S

$\bar{\rho} > 0$: a positive constant.

- $(\rho - \bar{\rho}, u) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$, Matsumura-Nishida, 1980.
- Scaling invariant Sobolev space :
  $(\rho - \bar{\rho}, u) \in \dot{H}^{\frac{N}{2}}(\mathbb{R}^N) \times \dot{H}^{\frac{N}{2}-1}(\mathbb{R}^N)$, No result.

Scaling property

$$\begin{cases} \rho_\nu(t,x) := \rho(\nu^2 t, \nu x) \\ u_\nu(t,x) := \nu u(\nu^2 t, \nu x), \end{cases} \quad \nu > 0.$$

Known results for compressible N-S

$\bar{\rho} > 0$: a positive constant.

- $(\rho - \bar{\rho}, u) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$, Matsumura-Nishida, 1980.
- Scaling invariant Sobolev space :
  $(\rho - \bar{\rho}, u) \in \dot{H}^{\frac{N}{2}}(\mathbb{R}^N) \times \dot{H}^{\frac{N}{2}-1}(\mathbb{R}^N)$, No result.
- $(\rho - \bar{\rho}, u) \in \dot{B}^{\frac{N}{2}}_{2,1}(\mathbb{R}^N) \times \dot{B}^{\frac{N}{2}-1}_{2,1}(\mathbb{R}^N)$, $N \geq 2$ Danchin, 2001.

Scaling property

$$\begin{cases} \rho_\nu(t,x) := \rho(\nu^2 t, \nu x) \\ u_\nu(t,x) := \nu u(\nu^2 t, \nu x), \end{cases} \quad \nu > 0.$$

Known results for compressible N-S

$\bar\rho > 0$: a positive constant.

- $(\rho - \bar\rho, u) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$, Matsumura-Nishida, 1980.
- Scaling invariant Sobolev space :
  $(\rho - \bar\rho, u) \in \dot H^{\frac{N}{2}}(\mathbb{R}^N) \times \dot H^{\frac{N}{2}-1}(\mathbb{R}^N)$, No result.
- $(\rho - \bar\rho, u) \in \dot B^{\frac{N}{2}}_{2,1}(\mathbb{R}^N) \times \dot B^{\frac{N}{2}-1}_{2,1}(\mathbb{R}^N)$, $N \geq 2$ Danchin, 2001.

Remark 1

$\dot B^{\frac{N}{2}}_{2,1}(\mathbb{R}^N) \subset \dot H^{\frac{N}{2}}(\mathbb{R}^N)$ *and* $\dot B^{\frac{N}{2}}_{2,1}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$

Scaling property

$$\begin{cases} \rho_\nu(t,x) := \rho(\nu^2 t, \nu x) \\ u_\nu(t,x) := \nu u(\nu^2 t, \nu x), \end{cases} \quad \nu > 0.$$

Known results for compressible N-S

$\bar{\rho} > 0$: a positive constant.

- $(\rho - \bar{\rho}, u) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$, Matsumura-Nishida, 1980.
- Scaling invariant Sobolev space : $(\rho - \bar{\rho}, u) \in \dot{H}^{\frac{N}{2}}(\mathbb{R}^N) \times \dot{H}^{\frac{N}{2}-1}(\mathbb{R}^N)$, No result.
- $(\rho - \bar{\rho}, u) \in \dot{B}^{\frac{N}{2}}_{2,1}(\mathbb{R}^N) \times \dot{B}^{\frac{N}{2}-1}_{2,1}(\mathbb{R}^N)$, $N \geq 2$ Danchin, 2001.

Remark 1

$\dot{B}^{\frac{N}{2}}_{2,1}(\mathbb{R}^N) \subset \dot{H}^{\frac{N}{2}}(\mathbb{R}^N)$ *and* $\dot{B}^{\frac{N}{2}}_{2,1}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$

Known result for compressible NSY

- Derivation of the system, global weak sol. for the model of 1-D nuclear slab, Ducomet, 2001

Besov space

$\{\widehat{\phi_j}\}_{j\in\mathbb{Z}}$: a dyadic decomposition of an unity in Fourier sp.

Definition 1 (Inhomogeneous Besov space $B^s_{p,\sigma}$)

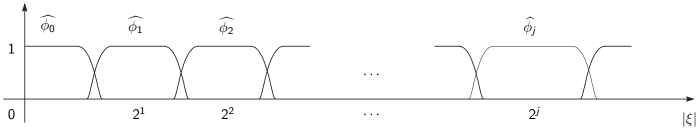
$s \in \mathbb{R}$, $1 \le p \le \infty$, $1 \le \sigma < \infty$.

$$B^s_{p,\sigma}(\mathbb{R}^N) := \{u \in \mathcal{S}';\ \|u\|_{B^s_{p,\sigma}} < \infty\}$$

$$\|u\|_{B^s_{p,\sigma}} := \|\Phi * u\|_{L^p} + \Big(\sum_{j\ge 1} 2^{js\sigma}\|\phi_j * u\|^{\sigma}_{L^p}\Big)^{\frac{1}{\sigma}}.$$

Besov space

$\{\widehat{\phi_j}\}_{j\in\mathbb{Z}}$: a dyadic decomposition of an unity in Fourier sp.



Definition 1 (Inhomogeneous Besov space $B^s_{p,\sigma}$)

$s\in\mathbb{R}$, $1\le p\le\infty$, $1\le\sigma<\infty$.

$$B^s_{p,\sigma}(\mathbb{R}^N):=\{u\in\mathcal{S}';\ \|u\|_{B^s_{p,\sigma}}<\infty\}$$

$$\|u\|_{B^s_{p,\sigma}}:=\|\Phi*u\|_{L^p}+\Big(\sum_{j\ge1}2^{js\sigma}\|\phi_j*u\|^{\sigma}_{L^p}\Big)^{\frac{1}{\sigma}}.$$

Definition 2 (Chemin-Lerner space $\widetilde{L^q_T(B^s_{p,\sigma})}$)

$$\|u\|_{\widetilde{L^q_T(B^s_{p,\sigma})}}:=\Big(\int_0^T\|\Phi*u\|^q_{L^p}dt\Big)^{\frac{1}{q}}+\Big\{\sum_{j\ge1}2^{js\sigma}\Big(\int_0^T\|\phi_j*u\|^q_{L^p}dt\Big)^{\frac{\sigma}{q}}\Big\}^{\frac{1}{\sigma}}.$$

Problem

$a := \rho^{-1} - 1$: *specific volume*, $a_0 := \rho_0^{-1} - 1$

Problem

$a := \rho^{-1} - 1$: *specific volume*, $a_0 := \rho_0^{-1} - 1$

(NSY2) $$\begin{cases} \partial_t a + u \cdot \nabla a = (1+a)\mathrm{div}\ u, \\ \partial_t u - (1+a)\mathcal{L}u \\ \qquad = -u \cdot \nabla u - \nabla(Q(a)) - \nabla(Id - \Delta)^{-1}\dfrac{a}{1+a}. \end{cases}$$

$$\mathcal{L} := \mu\Delta + (\lambda+\mu)\nabla\mathrm{div}\ ,\ Q(a) := -\int_0^t \frac{P'((1+z)^{-1})}{(1+z)^2}dz.$$

Problem

$a := \rho^{-1} - 1$: *specific volume*, $a_0 := \rho_0^{-1} - 1$

$$\text{(NSY2)} \qquad \begin{cases} \partial_t a + u \cdot \nabla a = (1+a)\mathrm{div}\, u, \\ \partial_t u - (1+a)\mathcal{L}u \\ \qquad = -u \cdot \nabla u - \nabla(Q(a)) - \nabla(Id - \Delta)^{-1} \dfrac{a}{1+a}. \end{cases}$$

$$\mathcal{L} := \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}\ ,\ Q(a) := -\int_0^t \frac{P'((1+z)^{-1})}{(1+z)^2} dz.$$

Definition 3 (Inhomogeneous critical space)

$a \in L^\infty(0, T; B_{p,1}^{\frac{N}{p}}),\ u \in L^\infty(0, T; B_{p,1}^{\frac{N}{p}-1})$.

Problem

$a := \rho^{-1} - 1$: *specific volume*, $a_0 := \rho_0^{-1} - 1$

$$\text{(NSY2)} \qquad \begin{cases} \partial_t a + u \cdot \nabla a = (1+a)\mathrm{div}\ u, \\ \partial_t u - (1+a)\mathcal{L}u \\ \qquad = -u \cdot \nabla u - \nabla(Q(a)) - \nabla(Id - \Delta)^{-1} \dfrac{a}{1+a}. \end{cases}$$

$$\mathcal{L} := \mu\Delta + (\lambda + \mu)\nabla \mathrm{div}\ ,\ Q(a) := -\int_0^t \frac{P'((1+z)^{-1})}{(1+z)^2} dz.$$

Definition 3 (Inhomogeneous critical space)

$a \in L^\infty(0, T; B^{\frac{N}{p}}_{p,1})$, $u \in L^\infty(0, T; B^{\frac{N}{p}-1}_{p,1})$.

> Aim :
> To find a local-in-time solution to (NSY2) in the critical space and show the corresponding blow-up criterion.

Local existence and uniqueness of (NSY2)

Theorem 1

$N \geq 2$, $1 < p \leq N$. $a_0 \in B^{\frac{N}{p}}_{p,1}$, $u_0 \in B^{\frac{N}{p}-1}_{p,1}$.

$$1 + a_0 \geq \underline{a} > 0$$

$\Rightarrow$ $^{\exists}T > 0$ *s.t.* $^{\exists}$ (ρ, u, ψ): *unique solution of (NSY2)*

$$a \in C([0,T); B^{\frac{N}{p}}_{p,1}),$$

$$u \in \left(C([0,T); B^{\frac{N}{p}-1}_{p,1}) \cap L^1(0,T; B^{\frac{N}{p}+1}_{p,1})\right)^N,$$

$$\psi \in C([0,T); B^{\frac{N}{p}+2}_{p,1}).$$

Local existence and uniqueness of (NSY2)

Theorem 1

$N \geq 2$, $1 < p \leq N$. $a_0 \in B^{\frac{N}{p}}_{p,1}$, $u_0 \in B^{\frac{N}{p}-1}_{p,1}$.

$$1 + a_0 \geq \underline{a} > 0$$

$\Rightarrow {}^{\exists} T > 0$ *s.t.* ${}^{\exists}$ (ρ, u, ψ): *unique solution of (NSY2)*

$$a \in C([0,T]; B^{\frac{N}{p}}_{p,1}),$$

$$u \in \left(C([0,T]; B^{\frac{N}{p}-1}_{p,1}) \cap L^1(0,T; B^{\frac{N}{p}+1}_{p,1})\right)^N,$$

$$\psi \in C([0,T]; B^{\frac{N}{p}+2}_{p,1}).$$

The proof is based on a standard approximation scheme. See for details papers such as Danchin(2001, 2006).

Blow-up criterion for (NSY)

Theorem 2

Let $1 < p < N$. If the solution of (NSY2)

$$(a, u, \psi) \in C([0, T); B^{\frac{N}{p}}_{p,1} \times (B^{\frac{N}{p}-1}_{p,1})^N \times B^{\frac{N}{p}+2}_{p,1})$$

satisfies

(i) $a \in L^\infty(0, T; B^{\frac{N}{p}}_{p,1})$, $1 + a \geq \exists \underline{a} > 0$,

(ii) $\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}) \in L^1(0, T)$,

then (a, u, ψ) may be continued beyond T.

Remark 2

Danchin (2007) obtained a blow-up criterion of the same type with a condition $\nabla u \in L^1(0, T; L^\infty)$. $L^\infty \subset BMO \subset \dot{B}^0_{\infty,\infty}$.

Key *a priori* estimate for the blow-up criterion

$$\text{(LPV)} \qquad \begin{cases} \partial_t u - (1+a)(\mu\Delta u + (\lambda+\mu)\nabla \text{div } u) = -u\cdot\nabla u + h, \\ u(0,x) = u_0(x). \end{cases}$$

Proposition 3

$N \geq 2$, $1 < p < \infty$, $\frac{N}{p} \geq s > 0$. Let $a \in \widetilde{L^\infty(0,T; B^{\frac{N}{p}}_{p,1})}$ with $1 + a \geq \underline{b}$, and u be the solution of (LPV).
$\Rightarrow^{\exists}$ some constant C^* (depending on N, s, p, μ, λ, T, $\|u_0\|_{B^s_{p,1}}$, $\|h\|_{L^1(0,T;B^s_{p,1})}$ and $\int_0^T \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}})dt$) such that we have for all $t \in [0,T]$,

$$\|u\|_{\widetilde{L^\infty_t(B^s_{p,1})}} \leq C^*.$$

Sketch of proof of Proposition 3

Standard energy-estimate procedure in Besov sp. (LPV):

$$\begin{cases} \partial_t u - (1+a)(\mu\Delta u + (\lambda+\mu)\nabla \mathrm{div}\, u) = - u\cdot\nabla u + h \\ u(0,x) = u_0(x), \end{cases}$$

Sketch of proof of Proposition 3

Standard energy-estimate procedure in Besov sp. We apply $\phi_j *$ to (LPV):

(2)

$$\begin{cases} \partial_t u_j - (1+a)(\mu \Delta u_j + (\lambda+\mu)\nabla \mathrm{div}\ u_j) = -\phi_j * (u \cdot \nabla u) + h_j + \mathit{Error} \\ u_j(0,x) = \phi_j * u_0(x), \end{cases}$$

Sketch of proof of Proposition 3

Standard energy-estimate procedure in Besov sp. We apply $\phi_j *$ to (LPV):
(2)

$$\begin{cases} \partial_t u_j - (1+a)(\mu\Delta u_j + (\lambda+\mu)\nabla \mathrm{div}\ u_j) = -\phi_j * (u\cdot\nabla u) + h_j + Error \\ u_j(0,x) = \phi_j * u_0(x), \end{cases}$$

$$\begin{aligned} \phi_j * (u\cdot\nabla u) &= u\cdot\nabla(\phi_j * u) + \{\phi_j * (u\cdot\nabla u) - u\cdot\nabla(\phi_j * u)s\} \\ &= u\cdot\nabla u_j + [\phi_j * , u\cdot\nabla]u, \end{aligned}$$

where $u_j = \phi_j * u$.

Sketch of proof of Proposition 3

Standard energy-estimate procedure in Besov sp. We apply $\phi_j *$ to (LPV):
(2)

$$\begin{cases} \partial_t u_j - (1+a)(\mu\Delta u_j + (\lambda+\mu)\nabla \mathrm{div}\ u_j) = -\phi_j * (u\cdot\nabla u) + h_j + Error \\ u_j(0,x) = \phi_j * u_0(x), \end{cases}$$

$$\begin{aligned} \phi_j * (u\cdot\nabla u) &= u\cdot\nabla(\phi_j * u) + \{\phi_j * (u\cdot\nabla u) - u\cdot\nabla(\phi_j * u)s\} \\ &= u\cdot\nabla u_j + [\phi_j * \ , u\cdot\nabla]u, \end{aligned}$$

where $u_j = \phi_j * u$. Multiply (2) by $|u_j|^{p-2}u_j$ and integrate over $\mathbb{R}^N$:

Sketch of proof of Proposition 3

Standard energy-estimate procedure in Besov sp. We apply $\phi_j *$ to (LPV):
(2)
$$\begin{cases} \partial_t u_j - (1+a)(\mu\Delta u_j + (\lambda+\mu)\nabla \mathrm{div}\ u_j) = -\phi_j * (u\cdot\nabla u) + h_j + Error \\ u_j(0,x) = \phi_j * u_0(x), \end{cases}$$

$$\begin{aligned} \phi_j * (u\cdot\nabla u) &= u\cdot\nabla(\phi_j * u) + \{\phi_j * (u\cdot\nabla u) - u\cdot\nabla(\phi_j * u)s\} \\ &= u\cdot\nabla u_j + [\phi_j * , u\cdot\nabla]u, \end{aligned}$$

where $u_j = \phi_j * u$. Multiply (2) by $|u_j|^{p-2}u_j$ and integrate over $\mathbb{R}^N$:

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u_j\|_{L^p}^p + C2^{2j}\|u_j\|_{L^p}^p \\ &\qquad \le \left|\int_{\mathbb{R}^N}(u\cdot\nabla u_j)\cdot(|u_j|^{p-2}u_j)dx\right| + \|u_j\|_{L^p}^{p-1}\|[u\cdot\nabla,\phi_j]u\|_{L^p} \\ &\qquad\qquad + \|u_j\|_{L^p}^{p-1}\big(\|h_j\|_{L^p} + \|Error\|_{L^p}\big). \end{aligned}$$

Sketch of proof of Proposition 3

Standard energy-estimate procedure in Besov sp. We apply $\phi_j *$ to (LPV):

(2)

$$\begin{cases} \partial_t u_j - (1+a)(\mu\Delta u_j + (\lambda+\mu)\nabla \mathrm{div}\ u_j) = -\phi_j * (u\cdot\nabla u) + h_j + Error \\ u_j(0,x) = \phi_j * u_0(x), \end{cases}$$

$$\begin{aligned} \phi_j * (u\cdot\nabla u) &= u\cdot\nabla(\phi_j * u) + \{\phi_j * (u\cdot\nabla u) - u\cdot\nabla(\phi_j * u)s\} \\ &= u\cdot\nabla u_j + [\phi_j * \, , u\cdot\nabla]u, \end{aligned}$$

where $u_j = \phi_j * u$. Multiply (2) by $|u_j|^{p-2}u_j$ and integrate over $\mathbb{R}^N$:

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|u_j\|_{L^p}^p + C2^{2j}\|u_j\|_{L^p}^p \\ &\qquad \le \left|\int_{\mathbb{R}^N}(u\cdot\nabla u_j)\cdot(|u_j|^{p-2}u_j)dx\right| + \|u_j\|_{L^p}^{p-1}\|[u\cdot\nabla,\phi_j]u\|_{L^p} \\ &\qquad\qquad + \|u_j\|_{L^p}^{p-1}\big(\|h_j\|_{L^p} + \|Error\|_{L^p}\big). \end{aligned}$$

$$\left|\int_{\mathbb{R}^N}(u\cdot\nabla u_j)\cdot(|u_j|^{p-2}u_j)dx\right| = \left|\int_{\mathbb{R}^N}(\mathrm{div}\ u)|u_j|^p dx\right| \le \|\nabla u\|_{L^\infty}\|u_j\|_{L^p}^p.$$

Sketch of proof of Proposition 3

Splitting frequencies of u as follows

$$u = \sum_{j<-K} u_k + \sum_{|k|\leq K} u_k + \sum_{k>K} u_k =: u_L + u_M + u_H,$$

we may estimate $\|\nabla u_L\|_{L^\infty}$ and $\|\nabla u_M\|_{L^\infty}$ by logarithmic Sobolev interpolation ineq. (Kozono-Ogawa-Taniuchi.) without problem.

Sketch of proof of Proposition 3

Splitting frequencies of u as follows

$$u = \sum_{j<-K} u_k + \sum_{|k|\le K} u_k + \sum_{k>K} u_k =: u_L + u_M + u_H,$$

we may estimate $\|\nabla u_L\|_{L^\infty}$ and $\|\nabla u_M\|_{L^\infty}$ by logarithmic Sobolev interpolation ineq. (Kozono-Ogawa-Taniuchi.) without problem.
On the other hand, we have problem bounding the high-frequency part ∇u_H since it can only be controlled as follows:

$$\|\nabla u_H\|_{L^\infty} \le \sum_{j>K} \|\nabla u_j\|_{L^\infty} = \sum_{j>K} 2^{-\varepsilon j} 2^{\varepsilon j} \|\nabla u_j\|_{L^\infty} \le C 2^{-\varepsilon K} \|\nabla u\|_{\dot{B}^{\varepsilon}_{\infty,\infty}}$$

where $\varepsilon > 0$ is arbitrary. We don't want the extra regularity ε.

Sketch of proof of Proposition 3

Splitting frequencies of u as follows

$$u = \sum_{j<-K} u_k + \sum_{|k|\le K} u_k + \sum_{k>K} u_k =: u_L + u_M + u_H,$$

we may estimate $\|\nabla u_L\|_{L^\infty}$ and $\|\nabla u_M\|_{L^\infty}$ by logarithmic Sobolev interpolation ineq. (Kozono-Ogawa-Taniuchi.) without problem.
On the other hand, we have problem bounding the high-frequency part ∇u_H since it can only be controlled as follows:

$$\|\nabla u_H\|_{L^\infty} \le \sum_{j>K} \|\nabla u_j\|_{L^\infty} = \sum_{j>K} 2^{-\varepsilon j} 2^{\varepsilon j} \|\nabla u_j\|_{L^\infty} \le C 2^{-\varepsilon K} \|\nabla u\|_{\dot{B}^{\varepsilon}_{\infty,\infty}}$$

where $\varepsilon > 0$ is arbitrary. We don't want the extra regularity ε.
If we could replace $\|\nabla u_H\|_{L^\infty}$ by $\|\nabla |\nabla|^{-\varepsilon} u_H\|_{L^\infty}$, then we could get rid of the extra regularity: $\|\nabla |\nabla|^{-\varepsilon} u_H\|_{L^\infty} \le C 2^{-\varepsilon K} \|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}}$.

Sketch of proof of Proposition 3

Splitting frequencies of u as follows

$$u = \sum_{j<-K} u_k + \sum_{|k|\le K} u_k + \sum_{k>K} u_k =: u_L + u_M + u_H,$$

we may estimate $\|\nabla u_L\|_{L^\infty}$ and $\|\nabla u_M\|_{L^\infty}$ by logarithmic Sobolev interpolation ineq. (Kozono-Ogawa-Taniuchi.) without problem.
On the other hand, we have problem bounding the high-frequency part ∇u_H since it can only be controlled as follows:

$$\|\nabla u_H\|_{L^\infty} \le \sum_{j>K} \|\nabla u_j\|_{L^\infty} = \sum_{j>K} 2^{-\varepsilon j} 2^{\varepsilon j} \|\nabla u_j\|_{L^\infty} \le C 2^{-\varepsilon K} \|\nabla u\|_{\dot{B}^{\varepsilon}_{\infty,\infty}}$$

where $\varepsilon > 0$ is arbitrary. We don't want the extra regularity ε.
If we could replace $\|\nabla u_H\|_{L^\infty}$ by $\|\nabla|\nabla|^{-\varepsilon} u_H\|_{L^\infty}$, then we could get rid of the extra regularity: $\|\nabla|\nabla|^{-\varepsilon} u_H\|_{L^\infty} \le C 2^{-\varepsilon K} \|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}}$. This can be done by some trick.

Sketch of proof of Proposition 3

In order to complete the proof, we have to establish following lemma

Lemma 4

Let $1 \leq p, \sigma \leq \infty$ *and* $0 \leq \varepsilon < 1$*. If* $s > 0$ *we have*

$$\left\| \{2^{js} \| [u \cdot \nabla, \phi_j * \,] u \|_{L^p} \}_{j \in \mathbb{Z}_+} \right\|_{\ell^\sigma(\mathbb{Z}_+)} \leq C \|u\|_{B^{s+\varepsilon}_{p,\sigma}} \|\nabla |\nabla|^{-\varepsilon} u\|_{L^\infty}.$$

The same estimate holds for $[u \cdot \nabla, \Phi * \,]u$ *as well.*

The above Lemma is known for $\varepsilon = 0$.

Sketch of proof of Proposition 3

In order to complete the proof, we have to establish following lemma

Lemma 4

Let $1 \le p, \sigma \le \infty$ *and* $0 \le \varepsilon < 1$. *If* $s > 0$ *we have*

$$\left\| \{2^{js} \|[u \cdot \nabla, \phi_j *]u\|_{L^p}\}_{j \in \mathbb{Z}_+} \right\|_{\ell^\sigma(\mathbb{Z}_+)} \le C \|u\|_{B^{s+\varepsilon}_{p,\sigma}} \|\nabla |\nabla|^{-\varepsilon} u\|_{L^\infty}.$$

The same estimate holds for $[u \cdot \nabla, \Phi *]u$ *as well.*

The above Lemma is known for $\varepsilon = 0$. After estimating low frequency, we obtain

$$X(t) \le C(1+t)\Big\{ \|u_0\|_{B^s_{p,1}} + \|h\|_{L^1(0,t;B^s_{p,1})} + C \int_0^t \{ Y(t) + 2^{2m}(1 + \|a\|^2_{B^{\frac{N}{p}}_{p,1}})\} X(\tau) d\tau \Big\},$$

where $X(t) := \|u\|_{\widetilde{L^\infty_t(B^s_{p,1})}} + \|u\|_{L^1(0,t;B^{s+2}_{p,1})}$

and $Y(t) := \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}) \log(e + \|u\|_{B^s_{p,1}})$. The final estimate follows by using Gronwall's lemma twice.

Details

Interpolation by differentiability:

$$\left|\int_{\mathbb{R}^N}(u\cdot\nabla u_j)\cdot(|u_j|^{p-2}u_j)dx\right|$$

$$\leq \left|\frac{1}{p}\int_{\mathbb{R}^N} u_L\cdot\nabla|u_j|^p dx\right| + \left|\frac{1}{p}\int_{\mathbb{R}^N} u_M\cdot\nabla|u_j|^p dx\right| + \int_{\mathbb{R}^N}|u_H\cdot\nabla u_j|\ |u_j|^{p-1}dx$$

$$\leq \frac{1}{p}\int_{\mathbb{R}^N}|u_j|^p|\mathrm{div}\ u_L|dx + \frac{1}{p}\int_{\mathbb{R}^N}|u_j|^p|\mathrm{div}\ u_M|dx + \|u_H\cdot\nabla u_j\|_{L^p}\|u_j\|_{L^p}^{p-1}$$

$$\leq \left(\|\nabla u_L\|_{L^\infty} + \|\nabla u_M\|_{L^\infty}\right)\|u_j\|_{L^p}^p + C\|u_H\|_{L^\infty}2^j\|u_j\|_{L^p}^p$$

$$\leq C\left\{\|\nabla u\|_{\dot B^0_{\infty,\infty}}\left(1+\log(e+\|u\|_{B^s_{p,1}})\right) + K\|\nabla u\|_{\dot B^0_{\infty,\infty}}\right\}\|u_j\|_{L^p}^p$$

$$+ C2^{-K}\|\nabla u\|_{\dot B^0_{\infty,\infty}}2^j\|u_j\|_{L^p}^p$$

$$\leq C\|\nabla u\|_{\dot B^0_{\infty,\infty}}\log(e+\|\nabla u\|_{\dot B^0_{\infty,\infty}})\log(e+\|u\|_{B^s_{p,1}})\|u_j\|_{L^p}^p$$

$$+ \tilde c 2^j\|u_j\|_{L^p}^p,$$

where we used Brezis-Gallauet-Wainger type ineq. and K is taken large enough so that $C2^{-K}\|\nabla u\|_{\dot B^0_{\infty,\infty}} < \tilde c$ (for small enough $\tilde c$).