On the sectorial $\mathcal{R}$-boundedness of the Stokes operator for the compressible viscous fluid flow in a general domain

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Joint work with Prof. Yoshihiro Shibata

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Outline

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   - known results
   - definition of $\mathcal{R}$-boundedness

2. Main theorem
   - $\mathcal{R}$-boundedness for the solution operator to the resolvent problem
   - the generation of analytic semigroup
   - the maximal L_p-L_q regularity

Linearized problem

We consider the linearized problem in general domain with slip boundary condition.

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\text{(P)} \quad \begin{cases} \dfrac{\partial \rho}{\partial t} + \gamma \operatorname{div} u = f & \text{in } \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial t} - \alpha \Delta u - \beta \nabla \operatorname{div} u + \gamma \nabla \rho = g & \text{in } \Omega,\ t > 0, \\ S(u, \rho)\nu|_{\tan} = h|_{\tan} & \text{on } \Gamma,\ t > 0, \\ u \cdot \nu = 0 & \text{on } \Gamma,\ t > 0, \\ (\rho, u)|_{t=0} = (0, 0). & \end{cases}
$$

- $u = (u_1, \dots, u_N)$: unknown velocity field $(N \geq 2)$,
  ρ : unknown density

$n|_{\tan} = n - \langle n, \nu \rangle \nu$

- α, β, γ : constant. $\alpha, \gamma > 0$, $\alpha + \beta > 0$.
- $S(u, \rho) = 2\alpha D(u) + [(\beta - \alpha) \operatorname{div} u - \gamma \rho] I$: stress tensor
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2\alpha [D(u)\nu - \langle D(u)\nu, \nu \rangle \nu] = h - \langle h, \nu \rangle \nu & \text{on } \Gamma,\ t > 0, \\
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Definition (general domain)

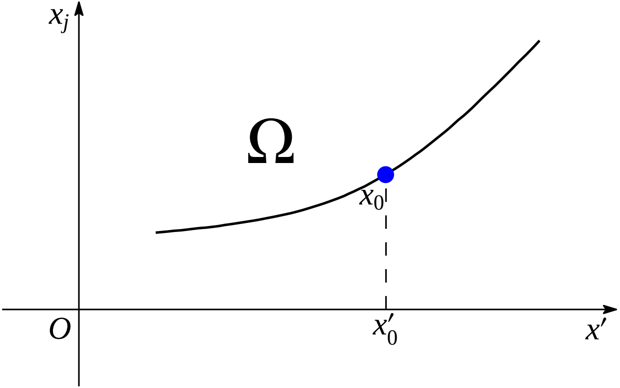
Let $1 < r < \infty$ and let Ω be a domain in $\mathbb{R}^N$ with boundary Γ. We call Ω a uniform $W_r^{3-1/r}$ domain if there exist positive constants α, β and K such that for any $x_0 = (x_{01}, \ldots, x_{0N}) \in \Gamma$ there exist a coordinate number j and $h \in W_r^{3-1/r}(B'_\alpha(x'_0))$ ($x'_0 = (x_{01}, \ldots, \hat{x}_{0j}, \ldots x_{0N})$) and $\|h\|_{W_r^{3-1/r}(B'_\alpha(x'_0))} \leq K$ satisfying

$$\Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x') \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0),$$
$$\Gamma \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x') \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0),$$

where $B'_\alpha(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < \alpha\}$,
$B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\}$ and $W_r^{3-1/r}(B'_\alpha(x'_0))$ denotes the set of all functions $h \in W_r^2(B'_\alpha(x'_0))$ such that

$$< D_k D_l h >_{1-1/r, r, B'_\alpha(x'_0)} = \left\{ \iint_{B'_\alpha(x_0) \times B'_\alpha(x_0)} \frac{|D_k D_l h(x') - D_k D_l h(y')|^r}{|x' - y'|^{N-2+r}} dx' dy' \right\}^{\frac{1}{r}} < \infty$$

for $k, l \neq j$ with $D_k D_l h = \partial^2 h / \partial x_k \partial x_l$.



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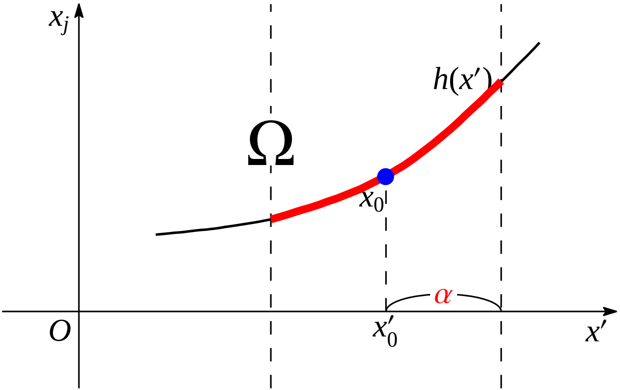

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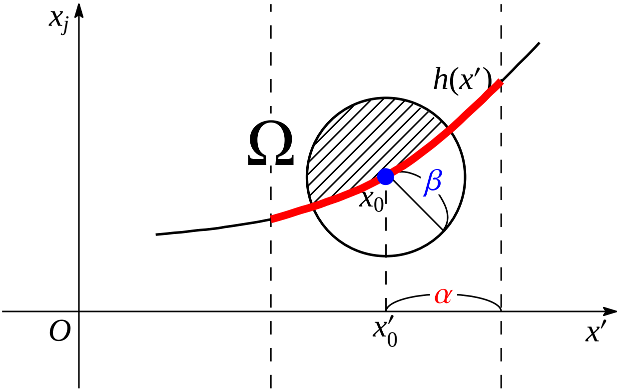

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Example of general domain

- boundary : bounded $\Rightarrow$bounded domain, exterior domain
- boundary : unbounded $\Rightarrow$half space, tube, layer

tube

$T = \{x = (x', x_N) \mid x' \in D, x_N \in \mathbb{R}\}$,
$D : W_r^{3-1/r}$ bounded

$x' = (x_1, \ldots x_{N-1})$

asymptotically flat layer

$$L = \left\{ x = (x', x_N) \,\middle|\, \begin{array}{l} a(x') < x_N < 1 + b(x'), \\ a(x') \leq b(x') \end{array} \right\}$$

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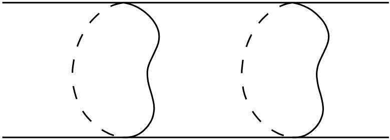

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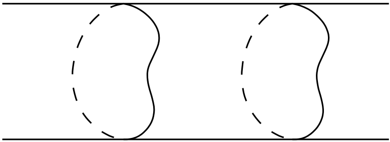

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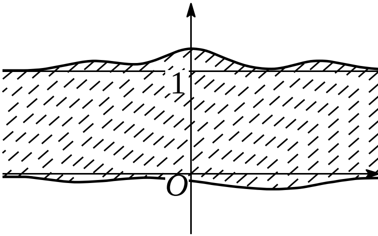

Known result and motivation

Enomoto and Shibata (2012, submitted)$\Rightarrow$
· the generation of analytic semigroup and the maximal regularity by the $\mathcal{R}$-boundedness of Stokes operator

The difference between "result of Enomoto and Shibata" and "our result" is the following:

• Enomoto and Shibata
$\Rightarrow$ boundary condition (Dirichlet) : homogeneous
• our result
$\Rightarrow$ boundary condition (slip) : derivative term, non-homogeneous

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Resolvent problem

The corresponding resolvent problem:

(RP)
$$\begin{cases} \lambda\rho + \gamma \operatorname{div} u = f & \text{in } \Omega, \\ \lambda u - \alpha\Delta u - \beta\nabla \operatorname{div} u + \gamma\nabla\rho = g & \text{in } \Omega, \\ S(u,\rho)\nu|_{\tan} = h|_{\tan} & \text{on } \Gamma, \\ u \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

In order to show the generation of analytic semigroup, we define a linear operator $\mathcal{A}$ by

$$\mathcal{A}(\rho,u) = (-\gamma \operatorname{div} u, \alpha\Delta u + \beta\nabla \operatorname{div} u - \gamma\nabla\rho) \text{ for } (\rho,u) \in \mathcal{D}(\mathcal{A}),$$

$$\mathcal{D}(\mathcal{A}) = \{(\rho,u) \in W_q^{1,2}(\Omega) \mid S(u,\rho)\nu|_{\tan} = 0, u\cdot\nu = 0 \text{ on } \Gamma\},$$

where we set $W_q^{m,l}(\Omega) = \{(f,g) \mid f \in W_q^m(\Omega), g \in W_q^l(\Omega)^N\}$.

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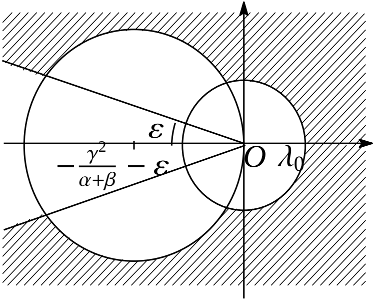
$$\Lambda_{\varepsilon,\lambda_0} = \Sigma_{\varepsilon,\lambda_0} \cap K_\varepsilon.$$



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The solution formula in $\mathbb{R}^N$:

$$u_j = \frac{1}{\alpha} \sum_{k=1}^{N} \mathcal{F}_\xi^{-1} \left[\frac{\delta_{jk} - \xi_j \xi_k |\xi|^{-2}}{\alpha^{-1}\lambda + |\xi|^2} \hat{f}_k \right](x) + \frac{1}{\alpha + \eta_\lambda} \sum_{k=1}^{N} \mathcal{F}_\xi^{-1} \left[\frac{\xi_j \xi_k |\xi|^{-2}}{(\alpha + \eta_\lambda)^{-1}\lambda + |\xi|^2} \hat{f}_k \right](x).$$

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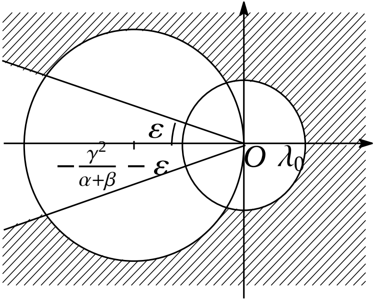

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Aim and key point

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- $\mathcal{A}$ generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $W_q^{1,0}(\Omega)$.
- the maximal L_p-L_q regularity for (P).

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$\mathcal{R}$-boundedness

Definition ($\mathcal{R}$-boundedness)

A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for each $m \in \mathbb{N}$, $T_j \in \mathcal{T}$, $f_j \in X$ $(j = 1, \ldots, m)$ for all sequences $\{r_j(u)\}_{j=1}^m$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, there holds the inequality :

$$\int_0^1 \Big\|\sum_{j=1}^m r_j(u) T_j f_j\Big\|_Y^p du \leq C \int_0^1 \Big\|\sum_{j=1}^m r_j(u) f_j\Big\|_X^p du.$$

Remark

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X, Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

For any Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y. $\mathcal{L}(X) = \mathcal{L}(X, X)$.

Main theorem

Main theorem

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$. Then, there exist a $\lambda_0 \geq 1$ depending on ε, q, N and an operator $R(\lambda) \in \mathcal{L}(W_q^{1,0}(\Omega) \times L_q(\Omega)^{N^2+N}, W_q^{1,2}(\Omega))$ such that the following two assertions hold :

(i) For any $(f, g) \in W_q^{1,0}(\Omega),\ h \in W_q^1(\Omega)^N$ and $\lambda \in \Lambda_{\varepsilon,\lambda_0}$, $(\rho, u) = R(\lambda)(f, g, \nabla h, \lambda^{1/2}h) \in W_q^{1,2}(\Omega)$ solves the problem (RP) uniquely.

(ii) There exist $\gamma_0 > 0$ such that

$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega)\times L_q(\Omega)^{N^2+N},\ W_q^{1,0}(\Omega))}(\{\lambda R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \gamma_0,$$
$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega)\times L_q(\Omega)^{N^2+N},\ L_q(\Omega)^{N^2})}(\{\lambda^{1/2}\nabla P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \gamma_0,$$
$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega)\times L_q(\Omega)^{N^2+N},\ L_q(\Omega)^{N^3})}(\{\nabla^2 P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \gamma_0,$$

where we set $P_v R(\lambda)(f, g, \nabla h, \lambda^{1/2}h) = u$.

The generation of analytic semigroup

Since the definition of $\mathcal{R}$-boundedness with $m = 1$ implies the usual boundedness, it follows from that

$$
\begin{aligned}
&|\lambda| \|(\rho, u)\|_{W_q^{1,0}(\Omega)} + |\lambda|^{1/2} \|\nabla u\|_{L_q(\Omega)} + \|\nabla^2 u\|_{L_q(\Omega)} \\
&\qquad \le C(\|(f, g)\|_{W_q^{1,0}(\Omega)} + \|(\lambda^{1/2} h, \nabla h)\|_{L_q(\Omega)}).
\end{aligned}
$$

Theorem

Let $1 < q < \infty$.Then, the operator $\mathcal{A}$ generates an analytic semigroup $\{T(t)\}_{t \ge 0}$ on $W_q^{1,0}(\Omega)$.

Theorem(the maximal L_p-L_q regularity)

Let $1 < p, q < \infty$. Then, there exists a constant $\gamma_1 > 0$ satisfying

(i) For any $f \in L_{p,\gamma_1,0}(\mathbb{R}, W_q^1(\Omega))$, $g \in L_{p,\gamma_1,0}(\mathbb{R}, L_q(\Omega)^N)$ and $h \in L_{p,\gamma_1,0}(\mathbb{R}, W_q^1(\Omega)^N) \cap H^{1/2}_{p,\gamma_1,0}(\mathbb{R}, L_q(\Omega)^N)$, the problem (P) admits a unique solution (ρ, u) satisfying

$$\begin{aligned} &\rho \in W^1_{p,\gamma_1,0}(\mathbb{R}, W_q^1(\Omega)), \\ &u \in L_{p,\gamma_1,0}(\mathbb{R}, W_q^2(\Omega)^N) \cap W^1_{p,\gamma_1,0}(\mathbb{R}, L_q(\Omega)^N) \end{aligned}$$

(ii)
$$\begin{aligned} &\|e^{-\gamma t}(\rho_t, \gamma\rho)\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \|e^{-\gamma t}(u_t, \gamma u, \Lambda_\gamma^{1/2}\nabla u, \nabla^2 u)\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\qquad \le C\|e^{-\gamma t}(f, \nabla f, \Lambda_\gamma^{\frac{1}{2}} h, \nabla h, g)\|_{L_p(\mathbb{R}, L_q(\Omega))} \; (\forall \gamma \ge \gamma_1). \end{aligned}$$

$$\begin{aligned} L_{p,\gamma_1}(\mathbb{R}, X) &= \{ f(t) \in L_{p,loc}(\mathbb{R}, X) \mid e^{-\gamma_1 t} f(t) \in L_p(\mathbb{R}, X)\}, \\ L_{p,\gamma_1,0}(\mathbb{R}, X) &= \{ f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid f(t) = 0, \quad t < 0\}, \\ H^{1/2}_{p,\gamma_1}(\mathbb{R}, X) &= \{ f \in L_p(\mathbb{R}, X) \mid e^{-\gamma t}\Lambda_\gamma^{1/2}[f](t) \in L_p(\mathbb{R}, X), \ \forall \gamma \ge \gamma_1\}. \\ \Lambda_\gamma^{1/2}[f](t) &= \mathcal{L}_\lambda^{-1}[|\lambda|^{1/2}\mathcal{L}[f](\lambda)](t). \end{aligned}$$

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$$\leq C\|e^{-\gamma t}(f, \nabla f, \Lambda_\gamma^{\frac{1}{2}} h, \nabla h, g)\|_{L_p(\mathbb{R}, L_q(\Omega))} \ (\forall \gamma \geq \gamma_1).$$

$$L_{p,\gamma_1}(\mathbb{R}, X) = \{ f(t) \in L_{p,loc}(\mathbb{R}, X) \mid e^{-\gamma_1 t} f(t) \in L_p(\mathbb{R}, X)\},$$
$$L_{p,\gamma_1,0}(\mathbb{R}, X) = \{ f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid f(t) = 0, \quad t < 0\},$$
$$H^{1/2}_{p,\gamma_1}(\mathbb{R}, X) = \{ f \in L_p(\mathbb{R}, X) \mid e^{-\gamma t}\Lambda_\gamma^{1/2}[f](t) \in L_p(\mathbb{R}, X), \ \forall \gamma \geq \gamma_1\}.$$
$$\Lambda_\gamma^{1/2}[f](t) = \mathcal{L}_\lambda^{-1}[|\lambda|^{1/2}\mathcal{L}[f](\lambda)](t).$$

Thank you for your attention !

Sketch of proof

1. whole-space and half-space
   ⇒calculate solution formula
2. bent half-space
   ⇒perturbation method
3. general domain
   ⇒by unit decomposition, connect solution operator in $\mathbb{R}^N$ and bent half-space

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Sketch of proof in general domain

(RP) $$\begin{cases} \lambda\rho + \gamma \operatorname{div} u = f & \text{in } \Omega, \\ \lambda u - \alpha\Delta u - \beta\nabla \operatorname{div} u + \gamma\nabla\rho = g & \text{in } \Omega, \\ S(u,\rho)\nu|_{\tan} = h|_{\tan} & \text{on } \Gamma, \\ u \cdot \nu = 0 & \text{on } \Gamma. \end{cases}$$

Setting $\rho = \lambda^{-1}(f - \gamma \operatorname{div} u)$ and $\eta_\lambda = \beta + \gamma^2\lambda^{-1}$,

$$\lambda u - \alpha\Delta u - \eta_\lambda\nabla \operatorname{div} u = g - \gamma\lambda^{-1}\nabla f =: f \quad \text{in } \Omega.$$

(RP') $$\begin{cases} \lambda u - \alpha\Delta u - \eta_\lambda\nabla \operatorname{div} u = f & \text{in } \Omega, \\ Bu = \begin{pmatrix} [2\alpha D(u)\nu + (\eta_\lambda - \alpha)\operatorname{div} u\nu]|_{\tan} \\ u \cdot \nu \end{pmatrix} = h = \begin{pmatrix} h|_{\tan} \\ 0 \end{pmatrix} & \text{on } \Gamma. \end{cases}$$

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Theorem (general domain)

Let $1 < q < \infty$, $N < r < \infty$ and $0 < \varepsilon < \pi/2$. Let Ω be a uniform $W_r^{2-1/r}$ domain in $\mathbb{R}^N$. Assume that $\max(q, q') \le r$. Then, there exist a constant $\lambda_0 \ge 1$ and an operator $\mathcal{U}_j(\lambda) \in \mathcal{L}(L_q(\Omega)^N, W_q^2(\Omega)^N)$ $(\lambda \in \Lambda_{\varepsilon,\lambda_0},\ j = 0, \ldots, N+1)$ satisfying the following assertions:

(i) For any $f \in L_q(\Omega)^N$ and $h \in W_q^1(\Omega)^N$,

$$u = \mathcal{U}_0(\lambda) f + \sum_{j=1}^{N} \mathcal{U}_j(\lambda)(D_j h) + |\lambda|^{\frac{1}{2}} \mathcal{U}_{N+1}(\lambda) h$$

solves the equations (RP') uniquely.

(ii) There exist $\gamma_1 > 0$ such that

$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N)}(\{\lambda \mathcal{U}_j(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_1$,

$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N,\ L_q(\Omega)^{N^2})}(\{|\lambda|^{\frac{1}{2}} \nabla \mathcal{U}_j(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_1$,

$\mathcal{R}_{\mathcal{L}(L_q(\Omega)^N,\ L_q(\Omega)^{N^3})}(\{\nabla^2 \mathcal{U}_j(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \le \gamma_1$. $(j = 0, \ldots, N+1)$

Main theorem

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$. Then, there exist a $\lambda_0 > 0$ depending on ε, q, N and an operator $R(\lambda) \in \mathcal{L}(W_q^{1,0}(\Omega) \times L_q(\Omega)^{N^2+N}, W_q^{1,2}(\Omega))$ such that the following two assertions hold :

(i) For any $(f, g) \in W_q^{1,0}(\Omega),\ h \in W_q^1(\Omega)^N$ and $\lambda \in \Lambda_{\varepsilon,\lambda_0}$, $(\rho, u) = R(\lambda)(f, g, \nabla h, |\lambda|^{\frac{1}{2}} h) \in W_q^{1,2}(\Omega)$ solves the equations (RP) uniquely.

(ii) There exist $\gamma_0 > 0$ such that

$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega)\times L_q(\Omega)^{N^2+N},\ W_q^{1,0}(\Omega))}(\{\lambda R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \gamma_0,$$

$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega)\times L_q(\Omega)^{N^2+N},\ L_q(\Omega)^{N^2})}(\{|\lambda|^{\frac{1}{2}} \nabla P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \gamma_0,$$

$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega)\times L_q(\Omega)^{N^2+N},\ L_q(\Omega)^{N^3})}(\{\nabla^2 P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \gamma_0,$$

Theorem (operator-valued Fourier multiplier theorem)

Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X, Y))$ such that

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_0 < \infty,$$
$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{\tau M'(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_1 < \infty.$$

If we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by the formula:

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \ (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)).$$

Then, the operator T_M is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$||T_M||_{\mathcal{L}(L_P(\mathbb{R},X),L_P(\mathbb{R},Y))} \leq C(\kappa_0 + \kappa_1)$$

for some constant $C > 0$ depending on p, X and Y.