On global L_p-solutions for some Oldroyd models on bounded domains[1]

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[1] Joint work with Matthias Geißert and Yoshihiro Shibata

Oldroyd-B fluid model

Aim: prove existence and uniqueness of global solutions of

$$
(1)\qquad
\begin{array}{rcll}
\partial_t u + u.\nabla u - \Delta u + \nabla \pi & = & \operatorname{Div} \tau & \text{in } (0,\infty)\times\Omega \\
\operatorname{div} u & = & 0 & \text{in } (0,\infty)\times\Omega \\
\partial_t \tau + u.\nabla \tau + \gamma\tau & = & \delta E u + b(\nabla u, \tau) & \text{in } (0,\infty)\times\Omega \\
u|_{\partial\Omega} & = & 0 & \text{on } (0,\infty)\times\Omega \\
u(0) & = & u_0 & \text{in } \Omega \\
\tau(0) & = & \tau_0 & \text{in } \Omega
\end{array}
$$

- Unknown functions:
  u : velocity field, π: pressure, τ: elastic part of the stress
  $Eu = \frac{1}{2}(\nabla u + \nabla u^T)$: symmetric part of gradient
- Given quantities:
  $b : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ bilinear
  $\gamma \geq 0$: $\gamma = 0 \Leftrightarrow$ infinite Weissenberg number
  $\gamma = 0 \Rightarrow$ for τ_0 with $\operatorname{Div}\tau_0 = \nabla\pi_0$ is $(u,\pi,\tau) = (0,\pi_0,\tau_0)$ stat. sol. of (1)
  $\delta \geq 0$: Coupling constant (no smallness condition!)

Some literature (on strong solutions in 3D)

L_p-solutions (on arbitrary, but finite time intervals)

- small $\delta \geq 0$
  - bounded domains: Fernández-Cara, Guillén, Ortega (1998)
  - exterior domains, layers, half spaces . . . : Geissert, Götz, N. (2012)

L_2-solutions

- $\gamma > 0$ and small $\delta > 0$
  - bounded domains: Guillopé, Saut (1990)
  - exterior domains: Hieber, Naito, Shibata (2012)
- $\gamma > 0$ and arbitrary $\delta > 0$
  - bounded domains: Molinet, Talhouk (2004)
  - exterior domains: Fang, Hieber, Zi (2012)
- $\gamma = 0$ and arbitrary $\delta > 0$
  - bounded domains: Lin, Zhang (2008)
  - whole space: Lei, Liu, Zhou (2008)

Now: w.l.o.g. $\delta = 1$

Lagrangian formulation

- Transformation: $X(t,\xi) = \xi + \int_0^t u(s, X(s,\xi))ds$
- Transformed unknowns: $(v,\theta,\eta)(t,\xi) = (u,\pi,\tau)(t, X(t,\xi))$
- Transformed system:

$$
(2) \qquad \begin{array}{rcll}
\partial_t v - \Delta v + \nabla\theta - \operatorname{Div}\eta & = & F(v,\theta,\eta) & \text{in } (0,\infty)\times\Omega \\
\operatorname{div} v & = & F_d(v) & \text{in } (0,\infty)\times\Omega \\
\partial_t \eta + \gamma\eta - Ev & = & b(\nabla v, \eta) + B(v,\eta) & \text{in } (0,\infty)\times\Omega \\
v|_{\partial\Omega} & = & 0 & \text{on } (0,\infty)\times\partial\Omega \\
v(0) & = & u_0 & \Omega \\
\eta(0) & = & \tau_0 & \Omega
\end{array}
$$

  and for $N \in \{F, F_d, b(\nabla\cdot,.), B\}$ holds $N(0) = 0$ and $DN(0) = 0$ in the "right space"
- Advantages: transport term $u.\nabla\tau$ vanishes
- Sufficient to solve nonlinear problem: maximal regularity of linearization

Main result

Theorem

Let:

- $n \in \mathbb{N}$, $n \geq 2$, $1 < p < \infty$, $n < q < \infty$
- Ω *bounded* C^2*-domain*
- $\gamma \geq 0$
- $(u_0, \tau_0) \in B^{2-2/p}_{q,p}(\Omega) \times H^1_{q,\mathrm{sym}}(\Omega)$ *satisfy compatibility conditions*

Then: $\exists \kappa > 0$, $\mu < 0$, *and a unique strong solution* (v, θ, η) *of* (2) *in*

$$v \in H^1_{p,\mu}(0,\infty; L_q) \cap L_{p,\mu}(0,\infty; H^2_q)$$
$$\theta \in L_{p,\mu}(0,\infty; \widehat{H}^1_q) + \{\theta \in BC([0,\infty); \widehat{H}^1_q) \colon \partial_t\theta \in L_{p,\mu}(0,\infty; \widehat{H}^1_q)\}$$
$$\eta \in H^1_{p,\mu}(0,\infty; H^1_{q,\mathrm{sym}}) + \{\eta \in BC([0,\infty); H^1_{q,\mathrm{sym}}) \colon \partial_t\eta \in L_{p,\mu}(0,\infty; H^1_{q,\mathrm{sym}})\}$$

provided $\|u_0\|_{B^{2-2/p}_{q,p}(\Omega)} + \|\tau_0\|_{H^1_q(\Omega)} \leq \kappa$

Note:

- $f \in L_{p,\mu}(0,\infty; X) \Leftrightarrow e^{-\mu\cdot} f \in L_p(0,\infty; X)$
- If $\gamma > 0$, we do not need the second addenda

Sketch of the proof

Associate linearization

- Associate linearization

$$
(3)\qquad
\begin{array}{rcll}
\partial_t v - \Delta v + \nabla\theta - \operatorname{Div}\eta & = & f & \text{in } (0,\infty)\times\Omega \\
\operatorname{div} v & = & f_d & \text{in } (0,\infty)\times\Omega \\
\partial_t\eta + \gamma\eta - Ev & = & b & \text{in } (0,\infty)\times\Omega \\
v & = & 0 & \text{on } (0,\infty)\times\partial\Omega \\
v(0) & = & u_0 & \text{in } \Omega \\
\eta(0) & = & \tau_0 & \text{in } \Omega
\end{array}
$$

- Define corresponding operator

$$
\mathcal{A}_\gamma = \begin{pmatrix} A_q & P_q \operatorname{Div} \\ -E & \gamma \end{pmatrix} : D(\mathcal{A}_\gamma) \subset X \to X
$$

defined on

$$
X = L_{q,\sigma}(\Omega) \times H^1_{q,\mathrm{sym}}(\Omega) \quad \text{with domain} \quad D(\mathcal{A}_\gamma) = D(A_q) \times H^1_{q,\mathrm{sym}}(\Omega)
$$

P_q: Helmholtz projection
A_q: Stokes operator ($A_q = -P_q\Delta$)

Sketch of the proof

Oldroyd operator $\mathcal{A}_\gamma$

Proposition

Let:

- $2 \leq q < \infty$
- $\gamma \geq 0$

Then:

(a) $\mathcal{A}_\gamma$ *admits maximal* L_p*-regularity on* $(0, T)$ *for* $0 < T < \infty$ *and* $1 < p < \infty$

(b) $\sigma(\mathcal{A}_\gamma) \subset [\min\{\gamma, C_P^{-2}\}, \infty)$ $\quad C_P$: *Poincaré constant*
If $\gamma > 0$, *then* $0 \in \rho(\mathcal{A}_\gamma)$

(c) *If* $\gamma > 0$, *there exits* $\mu_- < 0$, *such that* $\mathcal{A}_\gamma + \mu_-$ *admits maximal* L_p*-regularity on* $(0, \infty)$ *for* $1 < p < \infty$

(d) *If* $\gamma = 0$, *then* 0 *is an isolated eigenvalue of* $\mathcal{A}_0$ *and*

$$X = N(\mathcal{A}_0) \oplus R(\mathcal{A}_0)$$
$$D(\mathcal{A}_0) = N(\mathcal{A}_0) \oplus (R(\mathcal{A}_0) \cap D(\mathcal{A}_0))$$

Sketch of the proof

Maximal regularity of associate linearization (3) for $\gamma = 0$

Proposition

Let:

- $1 < p < \infty$, $2 \leq q < \infty$
- $\gamma = 0$
- $(u_0, \tau_0) \in B^{2-2/p}_{q,p}(\Omega) \times H^1_{q,\mathrm{sym}}(\Omega)$ *satisfy compatibility conditions*

Then: $\exists \mu < 0$, *s.t. for*

$$f \in L_{p,\mu}(0,\infty; L_q) + \mathrm{Div}\left\{F \in BC([0,\infty); H^1_q) \colon \partial_t F \in L_{p,\mu}(0,\infty; H^1_q)\right\}$$

$$f_d \in H^1_{p,\mu}(0,\infty; \widehat{H}^{-1}_{q,0}) \cap L_p(0,\infty; \widehat{H}^1_q)$$

$$b \in L_{p,\mu}(0,\infty; H^1_{q,\mathrm{sym}})$$

$\exists!$ *unique solution* (v, θ, η) *of* (3) *in*

$$v \in L_{p,\mu}(0,\infty; H^2_q) \cap H^1_{p,\mu}(0,\infty; L_q)$$

$$\theta \in L_{p,\mu}(0,\infty; \widehat{H}^1_q) + \left\{\theta \in BC([0,\infty); \widehat{H}^1_q) \colon \partial_t \theta \in L_{p,\mu}(0,\infty; \widehat{H}^1_q)\right\}$$

$$\eta \in H^1_{p,\mu}(0,\infty; H^1_{q,\mathrm{sym}}) + \left\{\eta \in BC([0,\infty); H^1_{q,\mathrm{sym}}) \colon \partial_t \eta \in L_{p,\mu}(0,\infty; H^1_{q,\mathrm{sym}})\right\}$$

THANK YOU FOR YOUR ATTENTION