On the L_p-L_q maximal regularity of the Stokes problem with the Neumann-Robin boundary condition in an infinite layer

Lorenz von Below and Hirokazu Saito (joint work)

Waseda University

November 7th, 2012
The 7th Japanese-German International Workshop on Mathematical Fluid Dynamics

§1 Problem

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F, G^+, G^-: given functions. U: velocity, Θ: pressure: unknown. $S(U,\Theta) = -\Theta I + [\nabla U + (\nabla U)^T]$: stress tensor, I: $N \times N$ identity matrix. $\mathbf{n}$ is unit outer normal to Γ_h.

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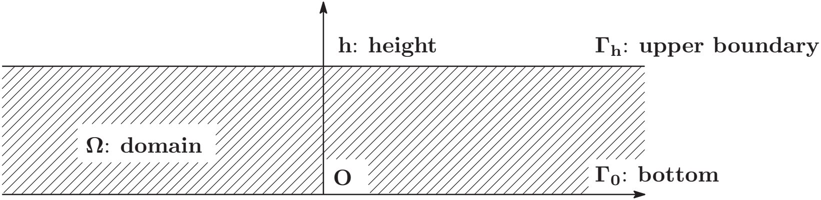

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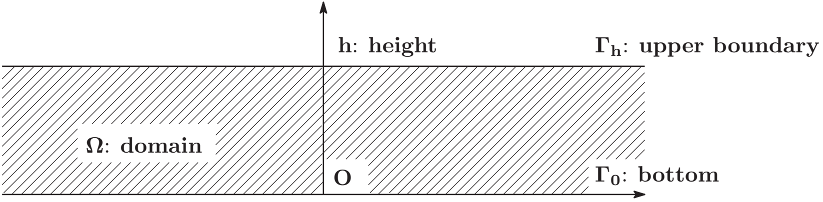

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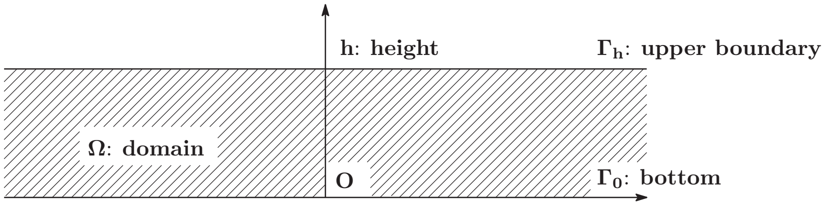

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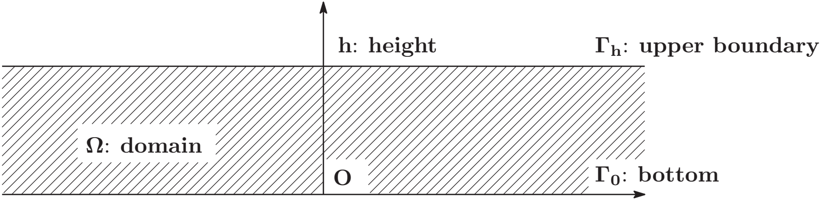

§2 Main results

Theorem (L_p-L_q maximal regularity)

Let $1 < p, q < \infty$ and $\gamma_0 > 0$. Then, for any

$$F \in L_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega))^N, \quad G^\pm \in (L_{p,\gamma_0,0}(\mathbf{R}, W_q^1(\Omega)) \cap H^{1/2}_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega)))^N,$$

(SP) admits a unique solution (U, Θ) in

$$U \in (L_{p,\gamma_0,0}(\mathbf{R}, W_q^2(\Omega)) \cap W^1_{p,\gamma_0,0}(\mathbf{R}, L_q(\Omega)))^N, \Theta \in L_{p,\gamma_0,0}(\mathbf{R}, W_q^1(\Omega)).$$

Moreover, (U, Θ) satisfies the estimate:

$$\|e^{-\gamma_0 t}(\partial_t U, U, \Lambda^{1/2}_{\gamma_0}\nabla U, \nabla^2 U, \Theta, \nabla\Theta)\|_{L_p(\mathbf{R}, L_q(\Omega))}$$
$$\leq C(N, p, q, \gamma_0, h)\|e^{-\gamma_0 t}(F, \Lambda^{1/2}_{\gamma_0} G^\pm, \nabla G^\pm)\|_{L_p(\mathbf{R}, L_q(\Omega))}.$$

Functional spaces

$$L_{p,\gamma_0,0}(\mathbf{R}, X) = \{ f : \mathbf{R} \to X \mid e^{-\gamma_0 t} f(t) \in L_p(\mathbf{R}, X),\ f(t) = 0\ (t < 0)\},$$
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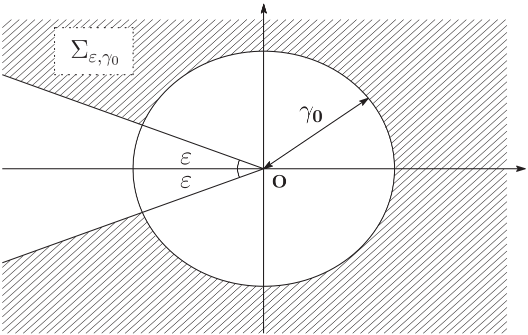

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By using Shibata-Shimizu approach, we obtain the resolvent estimate as corollary of the previous theorem.

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Corollary (Resolvent estimate)

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. For any $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, $f \in L_q(\Omega)^N$ and $g^{\pm} \in W_q^1(\Omega)^N$, (RP) has a unique solution $(u,\theta) \in W_q^2(\Omega)^N \times W_q^1(\Omega)$. Moreover, the solution (u,θ) satisfies the estimate:

$$
\|(\lambda u, |\lambda|^{1/2}\nabla u, \nabla^2 u)\|_{L_q(\Omega)} + \|\theta\|_{W_q^1(\Omega)} \leq C\|(f, |\lambda|^{1/2} g^{\pm}, \nabla g^{\pm})\|_{L_q(\Omega)}
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for some positive constant $C = C(N, q, \varepsilon, \gamma_0, h)$.

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Essentially, it is sufficient to consider the equations:

$$\begin{cases} \lambda u - \Delta u + \theta = 0 \quad \mathrm{div} u = 0 & \text{in} \quad \Omega, \\ S(u, \theta)\mathbf{n} = 0 & \text{on} \quad \Gamma_h, \\ [u - \partial_N u]_{\tan} = [g^-]_{\tan} & \text{on} \quad \Gamma_0, \\ u_N = 0 & \text{on} \quad \Gamma_0. \end{cases}$$

By the partial Fourier transform, we have the following ODEs:

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Especially, $\widehat{u}_N(\xi', x_N)$ is given by

$$\widehat{u}_N(x_N) = \left[\frac{L_{5,1}e^{-A(h-x_N)}}{\det L} + \frac{L_{5,2}e^{-Ax_N}}{\det L} + \frac{L_{5,3}e^{-B(h-x_N)}}{\det L} + \frac{L_{5,4}e^{-Bx_N}}{\det L}\right]\sum_{j=1}^{N-1} i\xi_j\widehat{g_j^-}(0),$$

where $A = |\xi'|$, $B = \sqrt{\lambda + |\xi'|^2}$, $\det L$ is Lopatinski determinant and $L_{j,k}$ is (j,k) cofactor of L. We can show

$$L_{5,1},\ L_{5,2},\ L_{5,3},\ L_{5,4} \sim |\xi'|(|\lambda|^{1/2} + |\xi'|)^4.$$

And also, $\det L$ is given by

$$\begin{aligned}
\det L =& A(B^2 - A^2)[(B^4 + 2A^2B^2 + A^4)(1 + e^{-2Ah})(1 - e^{-2Bh}) \\
&- 4A^3B(1 - e^{-2Ah})(1 + e^{-2Bh})] \\
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\sim& |\xi'|(|\lambda|^{1/2} + |\xi'|)^6
\end{aligned}$$

§3 Outline of proof

Especially, $\widehat{u}_N(\xi', x_N)$ is given by

$$\widehat{u}_N(x_N) = \left[\frac{L_{5,1}e^{-A(h-x_N)}}{\det L} + \frac{L_{5,2}e^{-Ax_N}}{\det L} + \frac{L_{5,3}e^{-B(h-x_N)}}{\det L} + \frac{L_{5,4}e^{-Bx_N}}{\det L}\right]\sum_{j=1}^{N-1} i\xi_j \widehat{g_j^-}(0),$$

where $A = |\xi'|$, $B = \sqrt{\lambda + |\xi'|^2}$, $\det L$ is Lopatinski determinant and $L_{j,k}$ is (j,k) cofactor of L. We can show

$$L_{5,1},\ L_{5,2},\ L_{5,3},\ L_{5,4} \sim |\xi'|(|\lambda|^{1/2} + |\xi'|)^4.$$

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$$\begin{aligned}\det L =& A(B^2 - A^2)[(B^4 + 2A^2B^2 + A^4)(1 + e^{-2Ah})(1 - e^{-2Bh}) \\ &- 4A^3B(1 - e^{-2Ah})(1 + e^{-2Bh})] \\ &+ A[(B^5 + 2A^2B^3 + 5A^4B)(1 + e^{-2Ah})(1 + e^{-2Bh}) \\ &- (16A^2B^3 + 16A^4B)e^{-Ah}e^{-Bh} \\ &- (AB^4 + 6A^3B^2 + A^5)(1 - e^{-2Ah})(1 - e^{-2Bh})] \\ &\sim |\xi'|(|\lambda|^{1/2} + |\xi'|)^6\end{aligned}$$

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where $A = |\xi'|$, $B = \sqrt{\lambda + |\xi'|^2}$, $\det L$ is Lopatinski determinant and $L_{j,k}$ is (j,k) cofactor of L. L is given by

$$L = \begin{bmatrix} (B^2 - A^2) & 0 & 0 & 0 & A & 0 \\ 0 & (B^2 - A^2) & 0 & 0 & 0 & -A \\ 2A^2 & 2A^2 e^{-Ad} & A^2 + B^2 & (A^2 + B^2)e^{-Bd} & 0 & 0 \\ 2A & -2Ae^{-Ad} & 2B & -2Be^{-Bd} & -1 & -e^{-Ad} \\ (A-1)Ae^{-Ad} & (A+1)A & (B-1)Be^{-Bd} & (B+1)B & 0 & 0 \\ e^{-Ad} & 1 & e^{-Bd} & 1 & 0 & 0 \end{bmatrix}.$$

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§3 Outline of proof

Therefore,

$$\lambda\widehat{u_N}(\xi', x_N) \sim \frac{\lambda L_{5,1}}{\det L}, \frac{\lambda L_{5,2}}{\det L}, \frac{\lambda L_{5,3}}{\det L}, \frac{\lambda L_{5,4}}{\det L} \sim \frac{\lambda|\xi'|(|\lambda|^{1/2}+|\xi'|^2)^4}{|\xi'|(|\lambda|^{1/2}+|\xi'|^2)^6} \sim 1.$$

By using this fact, we can show $\{(\tau\partial_\tau)^s \lambda\mathcal{R}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ ($s = 0, 1$, $\lambda = \gamma + i\tau$ and $\mathcal{R}(\lambda)$ is the solution operator from $|\lambda|^{1/2}g^-$ and ∇g^- to u_N) are $\mathcal{R}$-bounded families in $\mathcal{L}(L_q(\Omega)^{N^2-1}, L_q(\Omega))$.
Therefore, by the $\mathcal{R}$-boundedness and the Weis's operator-valued Fourier multiplier theorem, we obtain the following estimates:

$$||\lambda u_N||_{L_q(\Omega)} \leq C_1||(|\lambda|^{1/2}g^-, \nabla g^-)||_{L_q(\Omega)},$$
$$||e^{-\gamma_0 t}\partial_t U_N||_{L_p(\mathbf{R},L_q(\Omega))} \leq C_2||e^{-\gamma_0 t}(\Lambda_{\gamma_0}^{1/2}G^-, \nabla G^-)||_{L_p(\mathbf{R},L_q(\Omega))}$$

for some positive constants C_1 and C_2.

§3 Outline of proof

Therefore,

$$\lambda\widehat{u}_N(\xi', x_N) \sim \frac{\lambda L_{5,1}}{\det L}, \frac{\lambda L_{5,2}}{\det L}, \frac{\lambda L_{5,3}}{\det L}, \frac{\lambda L_{5,4}}{\det L} \sim \frac{\lambda|\xi'|(|\lambda|^{1/2} + |\xi'|^2)^4}{|\xi'|(|\lambda|^{1/2} + |\xi'|^2)^6} \sim 1.$$

By using this fact, we can show $\{(\tau\partial_\tau)^s\lambda\mathcal{R}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}$ ($s = 0, 1$, $\lambda = \gamma + i\tau$ and $\mathcal{R}(\lambda)$ is the solution operator from $|\lambda|^{1/2}g^-$ and ∇g^- to u_N) are $\mathcal{R}$-bounded families in $\mathcal{L}(L_q(\Omega)^{N^2-1}, L_q(\Omega))$.

Therefore, by the $\mathcal{R}$-boundedness and the Weis's operator-valued Fourier multiplier theorem, we obtain the following estimates:

$$\|\lambda u_N\|_{L_q(\Omega)} \leq C_1\|(|\lambda|^{1/2}g^-, \nabla g^-)\|_{L_q(\Omega)},$$

$$\|e^{-\gamma_0 t}\partial_t U_N\|_{L_p(\mathbf{R},L_q(\Omega))} \leq C_2\|e^{-\gamma_0 t}(\Lambda_{\gamma_0}^{1/2}G^-, \nabla G^-)\|_{L_p(\mathbf{R},L_q(\Omega))}$$

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§3 Outline of proof

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Therefore, by the $\mathcal{R}$-boundedness and the Weis's operator-valued Fourier multiplier theorem, we obtain the following estimates:

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for some positive constants C_1 and C_2.

§4 Further results

$$
(1) \qquad \begin{cases} \lambda u - \Delta u + \nabla\theta = f, & \mathrm{div} u = 0 \quad \text{in} \quad \Omega, \\ S(u,\theta)\mathbf{n} = g^+ & \text{on} \quad \Gamma_h, \\ [\alpha u - \beta\partial_N u]_{\tan} = [g^-]_{\tan} & \text{on} \quad \Gamma_0, \\ u_N = 0 & \text{on} \quad \Gamma_0. \end{cases}
$$

Theorem

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $\gamma_0 > 0$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Then, the unique solution (u, θ) to problem (1) satisfies the estimate:

$$
\|(\lambda u, |\lambda|^{1/2}\nabla u, \nabla^2 u)\|_{L_q(\Omega)} + \|\theta\|_{W_q^1(\Omega)}
$$

$$
\leq C\{\|(f, |\lambda|^{1/2} g, \nabla g^+)\|_{L_q(\Omega)} + \beta^{-1}\|(|\lambda|^{1/2} g^-, \nabla g^-)\|_{L_q(\Omega)}\}
$$

for some positive constant C independent of α and β.

This result related to Giga (1982) in bounded domains, Saal (2006) in $\mathbf{R}_+^N$, Shibata and Shimada (2007) in bounded or exterior domains, Shimada (2007) ...