Very Weak Solutions of the Stationary Stokes Equations in Unbounded Domains of Half Space Type

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Unbounded Domains of Half Space Type

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Unbounded Domains of Half Space Type

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- a *bent half space* is a domain of the form

$$H_\omega = \{x = (x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\},$$

where $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz continuous function in $W^{2,1}_{loc}(\mathbb{R}^{n-1})$, such that the gradient $\nabla'\omega = (\partial_1, \dots, \partial_{n-1})\omega$ is bounded in $\mathbb{R}^{n-1}$

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- a *perturbed half space* is domain of class $C^{1,1}$ such that $\Omega \backslash B = \mathbb{R}^n_+ \backslash B$ for some open ball B
- an *aperture domain* is a domain of class $C^{1,1}$ such that $\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B$ for some open ball $B = B_R(0) \subset \mathbb{R}^n$ of radius R and center 0, where

$$\mathbb{R}^n_- := \{x \in \mathbb{R}^n : x_n < -d\}$$

  for some $d \geq 0$

The Concept of Very Weak Solutions

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Definition

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be an unbounded domain of half space type. Moreover, let $1 < r < q < \infty$ such that $\frac{1}{n} + \frac{1}{q} = \frac{1}{r}$. Then for given data

$$F \in L^r(\Omega), k \in L^r(\Omega), g \in W^{-\frac{1}{q},q}(\partial\Omega), \tag{1}$$

we call a vector field $u \in L^q(\Omega)$ a *very weak solution* to the Stokes system if it satisfies the following conditions:

$$\begin{array}{rcll} -(u, \Delta w) & = & -(F, \nabla w) - \langle g, \mathbf{n} \cdot \nabla w \rangle_{\partial\Omega}, & w \in C^2_{0,\sigma}(\overline{\Omega}), \\ -(u, \nabla\psi) & = & (k, \psi) - \langle g, \psi\mathbf{n} \rangle_{\partial\Omega}, & \psi \in C^1_0(\overline{\Omega}), \end{array} \tag{2}$$

where $C^2_{0,\sigma}(\overline{\Omega}) = \{w \in C^2(\overline{\Omega}) : \operatorname{div} w = 0, \operatorname{supp} w \text{ compact in } \overline{\Omega}, w|_{\partial\Omega} = 0\}$.

Remark

For $n' < q < \infty$ and given data F, k and g as in the foregoing definition, $\mathcal{F}$ and $\mathcal{K}$ defined via

$$\begin{array}{rcll} \langle \mathcal{F}, w \rangle & := & -(F, \nabla w) - \langle g, N \cdot \nabla w \rangle_{\partial\Omega}, & w \in \hat{Y}_\sigma^{2,q'}(\Omega), \\ \langle \mathcal{K}, \psi \rangle & := & (k, \psi) - \langle g, \psi N \rangle_{\partial\Omega}, & \psi \in \hat{W}^{1,q'}(\Omega) \end{array} \tag{3}$$

yield elements in $\hat{Y}_\sigma^{-2,q}(\Omega)$ and $\hat{W}^{-1,q}(\Omega)$, respectively. Moreover, it holds

$$\|\mathcal{F}\|_{\hat{Y}_\sigma^{-2,q}(\Omega)} + \|\mathcal{K}\|_{\hat{W}^{-1,q}(\Omega)} \leq c(\|F\|_r + \|k\|_r + \|g\|_{\hat{W}^{-1/q,q}(\partial\Omega)}). \tag{4}$$

Definition

Let $n \geq 2$, $1 < q < \infty$, Ω an unbounded domain of half space type and let $\mathcal{F} \in \hat{Y}_\sigma^{-2,q}(\Omega), \mathcal{K} \in \hat{W}^{-1,q}(\Omega)$ be given. Then $u \in L^q(\Omega)$ is called a *very weak solution of the Stokes problem with data* $\mathcal{F}, \mathcal{K}$, if

$$\begin{aligned} -(u, \Delta w) &= \langle \mathcal{F}, w \rangle\,, \quad w \in \hat{Y}_\sigma^{2,q'}(\Omega), \\ -(u, \nabla \psi) &= \langle \mathcal{K}, \psi \rangle\,, \quad \psi \in \hat{W}^{1,q'}(\Omega), \end{aligned} \tag{5}$$

Generalized Definition

Definition

Let $n \geq 2$, $1 < q < \infty$, Ω an unbounded domain of half space type and let $\mathcal{F} \in \hat{Y}_\sigma^{-2,q}(\Omega), \mathcal{K} \in \hat{W}^{-1,q}(\Omega)$ be given. Then $u \in L^q(\Omega)$ is called a *very weak solution of the Stokes problem with data* $\mathcal{F}, \mathcal{K}$, if

$$\begin{aligned} -(u, \Delta w) &= \langle \mathcal{F}, w \rangle, \quad w \in \hat{Y}_\sigma^{2,q'}(\Omega), \\ -(u, \nabla \psi) &= \langle \mathcal{K}, \psi \rangle, \quad \psi \in \hat{W}^{1,q'}(\Omega), \end{aligned} \tag{5}$$

or, equivalently,

$$(u, -\Delta w - \nabla \psi) = \langle \mathcal{F}, w \rangle + \langle \mathcal{K}, \psi \rangle . \tag{6}$$

Observation

If for every $v \in L^{q'}(\Omega)$ there exists a unique solution $w \in \hat{Y}^{2,q'}_{\sigma}(\Omega)$, $\psi \in \hat{W}^{1,q'}(\Omega)$ to the problem

$$-\Delta w - \nabla\psi = v, \quad \operatorname{div} w = 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega \tag{7}$$

depending linearly on v and satisfying the estimate

$$\|\nabla^2 w\|_{q'} + \|\nabla\psi\|_{q'} \leq c\,\|v\|_{q'}\,,$$

then $u \in L^q(\Omega)$ defined via the relation

$$(u, v) = \langle\mathcal{F}, w\rangle + \langle\mathcal{K}, \psi\rangle \qquad \forall v \in L^{q'}(\Omega),$$

is the unique very weak solution of the Stokes system with data $\mathcal{F}$ and $\mathcal{K}$.

System (*)

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Definition

For a domain $\Omega \subset \mathbb{R}^n$, we call

$$\begin{aligned} -\Delta w - \nabla\psi &= v && \text{in } \Omega \\ \nabla \operatorname{div} w &= \nabla\gamma && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{*}$$

the system (*) corresponding to Ω.

Theorem (Farwig, Sohr 1994)

Let $n \geq 2$, $1 < q < \infty$ and let $\Omega = \mathbb{R}^n_+$. Then to each

$$v \in L^q(\Omega), \qquad \gamma \in W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega),$$

there is a solution $(w, \psi) \in \hat{Y}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ of the system (*) and it holds

$$\|\nabla^2 w\|_q + \|\nabla\psi\|_q \leq c \left(\|v\|_q + \|\nabla\gamma\|_q\right) \tag{8}$$

with a constant $c = c(n, q) > 0$.

Strong Solutions in the Half Space

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Theorem (Farwig, Sohr 1994; Farwig, S. 2012)

Let $n \geq 2$, $1 < q < \infty$ and let $\Omega = \mathbb{R}^n_+$. Then to each

$$v \in L^q(\Omega), \qquad \gamma \in \hat{W}^{1,q}(\Omega),$$

there is a solution $(w, \psi) \in \hat{Y}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ of the system (*) and it holds

$$\|\nabla^2 w\|_q + \|\nabla\psi\|_q \leq c\left(\|v\|_q + \|\nabla\gamma\|_q\right) \tag{8}$$

with a constant $c = c(n, q) > 0$.

Theorem

Let $n \geq 3$, $1 < q < n-1$ and let H_ω be a bent half space. Then there is a constant $K = K(n,q) > 0$ such that if $\|\nabla'\omega\|_\infty \leq K$ and if $\|\nabla'^2\omega\|_{L^{n-1}(\mathbb{R}^{n-1})} \leq K$ or $\||\cdot|\nabla'^2\omega\|_\infty \leq K$, then for all $v \in L^q(H_\omega)$ and $\gamma \in \hat{W}^{1,q}(H_\omega)$ there exists a solution $(w,\psi) \in \hat{Y}^{2,q}(H_\omega) \times \hat{W}^{1,q}(H_\omega)$ of the system (*) satisfying the estimate

$$\|\nabla^2 w\|_q + \|\nabla\psi\|_q \leq c\left(\|v\|_q + \|\nabla\gamma\|_q\right) \tag{9}$$

with a constant $c = c(\omega, n, q) > 0$.

Strong Solutions in the Perturbed Half Space

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Theorem

Let $n \geq 3$, $1 < q < n/2$. then for all $v \in L^q(\Omega)$ and $\gamma \in \hat{W}^{1,q}(\Omega)$ there exists a solution $(w, \psi) \in \hat{Y}^{2,q}(\Omega) \times \hat{W}^{1,q}(\Omega)$ of the system (*) with $\|\nabla^2 w\|_q < \infty$ satisfying the estimate

$$\|\nabla^2 w\|_q + \|\nabla\psi\|_q \leq c\left(\|v\|_q + \|\nabla\gamma\|_q\right) \tag{10}$$

with a constant $c = c(\Omega, n, q) > 0$.

Properties of the Aperture Domain

Lemma

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an aperture domain. Suppose $1 < q < n$ and let r be defined via $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$. Then for every $\psi \in \dot{W}^{1,q}(\Omega)$ there are constants $\psi_\pm \in \mathbb{C}$ such that

$$\|\psi - \psi_+\|_{L^r(\Omega_+)} + \|\psi - \psi_-\|_{L^r(\Omega_-)} + |\psi_+ - \psi_-| \leq c\|\nabla\psi\|_q.$$

Furthermore, the sum

$$\dot{W}^{1,q}(\Omega) = \hat{W}^{1,q}(\Omega) \oplus \{K\varphi^0 : K \in \mathbb{C}\}$$

is direct. Here, $\varphi^0 \in C^\infty(\Omega)$ is a function satisfying

$$\varphi^0(x) = \begin{cases} 1 & \text{for} \quad x \in \Omega_+ \\ 0 & \text{for} \quad x \in \Omega_- \setminus B \end{cases} \qquad \text{and} \qquad \int_{B\cap\Omega_-} \varphi^0 \, dx = 0.$$

Theorem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an aperture domain and let $v \in L^q(\Omega)$, $\gamma = 0$, $1 < q < \frac{n}{2}$. Furthermore let r, ρ be defined via $\frac{1}{n} + \frac{1}{r} = \frac{1}{q}$ and $\frac{2}{n} + \frac{1}{\rho} = \frac{1}{q}$, respectively. Then for every $\alpha \in \mathbb{C}$ there is a unique solution $(w, \psi) \in \hat{Y}^{2,q}(\Omega) \times \dot{W}^{1,q}(\Omega)$ of the system (*) such that $\hat{\phi}(w) = \alpha$. It holds

$$\|\nabla^2 w\|_q + \|\nabla \psi\|_q \leq c(\|v\|_q + |\alpha|)$$

for some $c = c(n, q, \Omega)$.

Thank you very much for your attention!

Questions?