On some numerical tests of viscoelastic flows

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Outline

What are viscoelastic fluids?

viscous elastic

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Viscoelastic flow is a kind of non-Newtonian flow, which has "memory" (the state-of-stress depends on flow history).

Introduction

Like all fluids, we have Navier-Stokes equations.

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$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \nabla \cdot \mathbf{T}$$

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$$\mathbf{T} = 2\mu_s \mathbf{D} + \boldsymbol{\sigma},\ with\ \boldsymbol{\sigma} = \frac{\mu_0 - \mu_s}{\Lambda}(\boldsymbol{\tau} - \mathbf{I})$$

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Constitutive law for elasticity(Oldroyd-B model):

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\tau} - \nabla \mathbf{u} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot (\nabla \mathbf{u})^T = \frac{1}{\Lambda}(\mathbf{I} - \boldsymbol{\tau}) \qquad (1)$$

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Retardation time(Λ) over characteristic flow time(L/U).

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Numerical phenomenen?

Validity of the constitutive law?

A cartoon-model to show the instability[1].

$$\frac{\partial \phi}{\partial t} + a(x)\frac{\partial \phi}{\partial x} - b(x)\phi = -\frac{1}{We}\phi \qquad (2)$$

$\phi(x,0) = 0$. $a(x), b(x) > 0$ play the role as $\mathbf{u}, \nabla \mathbf{u}$.

[1] R. Fattal, R.Kupfermann

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$$\phi_j^{k+1} = [1 - \frac{a_j \Delta t}{\Delta x} + \Delta t(b_j - \frac{1}{We})]\phi_j^k + \frac{a_j \Delta t}{\Delta x}\phi_{j-1}^k$$

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$$0 \leq \frac{a_j \Delta t}{\Delta x} \leq 1$$

$$0 \leq 1 - \frac{a_j \Delta t}{\Delta x} + \Delta t(b_j - \frac{1}{We}) \leq 1 \quad !$$

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Else, $b_j > 1/We \implies$ restriction on mesh size!

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This situation remains unchanged if using higher-order method.

The use of implicit schemes does not help either.

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Again, first-order upwind scheme,

$$\psi_j^{k+1} = (1 - \frac{a_j \Delta t}{\Delta x})\psi_j^k + \frac{a_j \Delta t}{\Delta x}\psi_{j-1}^k + \Delta t(b_j - \frac{1}{We})$$

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There's no longer restriction on mesh size Δx!

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There are also other transformation, such as square root($\psi = \phi^{1/2}$), but it does no contribution to the HWNP.

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Reformulate the constitutive law from $\boldsymbol{\tau}$ for $\boldsymbol{\psi} = \log \boldsymbol{\tau}$

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we do it from the following decomposition

$$\nabla \mathbf{u} = \mathbf{B} + \boldsymbol{\Omega} + \mathbf{N}\boldsymbol{\tau}^{-1}.$$

where $\mathbf{N}, \boldsymbol{\Omega}$ are anti-symmetric, $\mathbf{B}$ is symmetric and commutes with $\boldsymbol{\tau}$.

$$\nabla \mathbf{u} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot (\nabla \mathbf{u})^T = \boldsymbol{\Omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\Omega} + 2\mathbf{B}\boldsymbol{\tau}$$

(3)Rotation:

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \boldsymbol{\Omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\Omega} \implies \boldsymbol{\tau}(t) = e^{\boldsymbol{\Omega} t}\boldsymbol{\tau}_0 e^{-\boldsymbol{\Omega} t}$$

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(4)Extension:

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = 2\mathbf{B}\boldsymbol{\tau} \implies \boldsymbol{\tau}(t) = e^{2\mathbf{B} t}\boldsymbol{\tau}_0$$

$$\boldsymbol{\psi}(t) = \log \boldsymbol{\tau}(t) = 2\mathbf{B}t + \boldsymbol{\psi}_0 \implies \frac{\partial \boldsymbol{\psi}}{\partial t} = 2\mathbf{B}$$

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Note:
If $\mathbf{Y}$ is skew-symmetric, $e^{\mathbf{Y}}$ is orthogonal,
If Y is invertible, then $e^{\mathbf{YXY}^{-1}} = \mathbf{Y}e^{\mathbf{X}}\mathbf{Y}^{-1}$,
If $\mathbf{XY} = \mathbf{YX}$, then $e^{\mathbf{X}}e^{\mathbf{Y}} = e^{\mathbf{X}+\mathbf{Y}}$.

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Dimensionless model

$$\begin{cases} \frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla)\psi - (\mathbf{\Omega}\psi - \psi\mathbf{\Omega}) - 2\mathbf{B} = \frac{1}{We}(e^{-\psi} - \mathbf{I}) \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{Re}\nabla p + \frac{\alpha}{Re}\Delta \mathbf{u} + \frac{1}{We}\frac{1-\alpha}{Re}\nabla \cdot \boldsymbol{\tau} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

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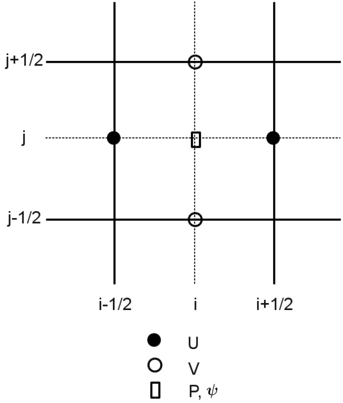
Staggered mesh



Combined FV-FD scheme

FV for constitutive law, FD[2] for NS equations

[2] B. Seibold

FV for constitutive law, FD[2] for NS equations

Part 1: Finite volume for elasticity

$$\boldsymbol{\psi}^{k+1} = \boldsymbol{\psi}^{k} - \frac{\Delta t}{|S|}\iint_S \mathbf{u}^k \cdot \nabla \boldsymbol{\psi}^k + \Delta t(\boldsymbol{\Omega}^k \boldsymbol{\psi}^k - \psi^k \boldsymbol{\Omega}^k + 2\mathbf{B}^k) + \frac{\Delta t}{We}(e^{-\boldsymbol{\psi}^k} - \mathbf{I})$$

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upwind approach for flux

$$FLUX = \iint_S \mathbf{u}^k \cdot \nabla \boldsymbol{\psi}^k = \oint_{\partial S} \mathbf{u}^k \mathbf{n} \boldsymbol{\psi}^k = \sum \mathbf{u}^k \mathbf{n} \widehat{\psi}^k|_l |l|$$

Let u_{ij} be the velocity vector in the middle point of l_{ij}, then

$$\widehat{\psi}_{ij} = \begin{cases} \widehat{\psi}_i & \text{if} \quad \mathbf{u}_{ij}\mathbf{n}_{ij} > 0 \\ \widehat{\psi}_j & \text{if} \quad \mathbf{u}_{ij}\mathbf{n}_{ij} \leq 0 \end{cases}$$

[2]B. Seibold

Part 2: Navier-Stokes equations.

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In this part, we decompose the NS equations into three steps.

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1-\alpha}{Re} \frac{1}{We} \nabla \cdot \boldsymbol{\tau}$$

$$\mathbf{u}_t = \frac{\alpha}{Re} \Delta \mathbf{u}$$

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2.1 Explicit nonlinear terms and viscoelasticity

$$\frac{U^* - U^k}{\Delta t} = -((U^k)^2)_x - (U^k V^k)_y + \frac{1-\alpha}{Re} \frac{1}{We} (\tau_{11x}^{k+1} + \tau_{12y}^{k+1})$$
$$\frac{V^* - V^k}{\Delta t} = -(U^k V^k)_x - ((V^k)^2)_y + \frac{1-\alpha}{Re} \frac{1}{We} (\tau_{21x}^{k+1} + \tau_{22y}^{k+1})$$

2.2 Implicit viscosity

$$
\begin{aligned}
\frac{U^{**}-U^{*}}{\Delta t} &= \frac{\alpha}{Re}(U^{**}_{xx} + U^{**}_{yy}) \\
\frac{V^{**}-V^{*}}{\Delta t} &= \frac{\alpha}{Re}(V^{**}_{xx} + V^{**}_{yy})
\end{aligned}
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implicit; centeral difference

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implicit; centeral difference

2.3 Chorin's projection method

$$\frac{1}{\Delta t}(\mathbf{u}^{k+1} - \mathbf{u}^{**}) = -\nabla P^{k+1}$$

The equation is solved as following:

a). $F = \nabla \cdot \mathbf{u}^{**}$

b). $-\Delta P^{k+1} = -\frac{1}{\Delta t} F$

c). $\mathbf{G} = \nabla P^{k+1}$

d). $\mathbf{u}^{k+1} = \mathbf{u}^{**} - \Delta t \mathbf{G}$.

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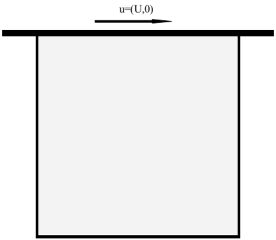
$$\nabla \cdot \mathbf{u}^{k+1} = \nabla \cdot (\mathbf{u}^{**} - \Delta t \mathbf{G}) = F - \Delta t \nabla \cdot (\nabla P^{k+1}) = 0.$$

Geometry: (0,1)x(0,1)
Mesh points Nx=Ny=64,128,256
Initial values: u=0.
Boundary condition: solid wall; for left, right and bottom, $\mathbf{u} = 0$, for the top $\mathbf{u} = (8(1 + tanh(8t - 4))x^2(1 - x)^2, 0)$
In the first few steps $dt = 0.02T_{total}$,
then $dt = CFL \min(\Delta x, \Delta y)/|u|_{\max}$ with CFL=0.6.
Re=1, We=1,3,4,5. $\alpha = 0.5$



Numerical Test

streamline–different Weissenberg number

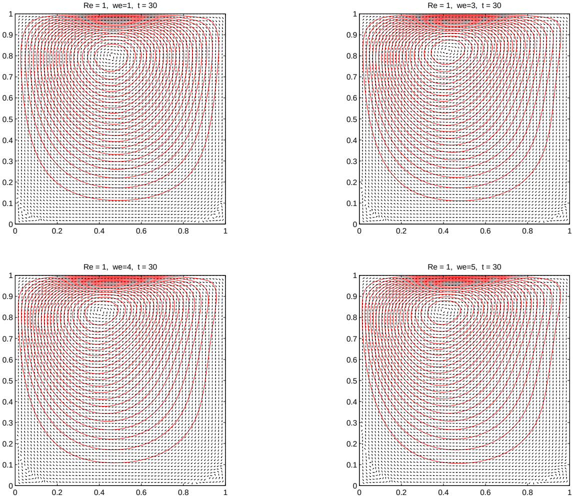


Figure: streamline of different Weissenberg numbers(n=64)

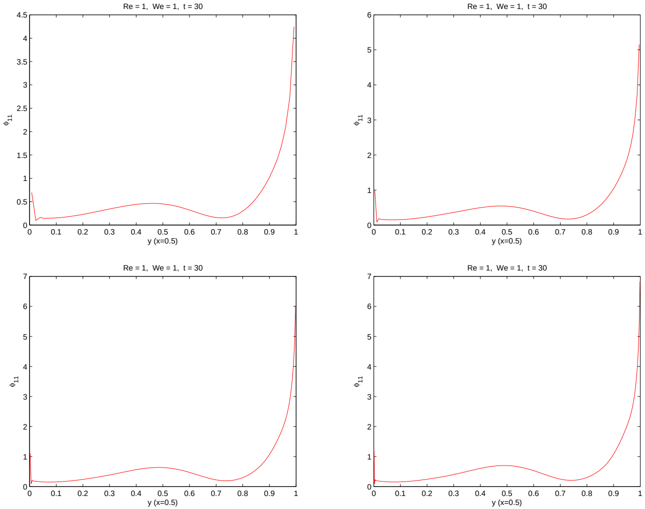


Figure: ψ_{11}–results of different mesh size.

Table: error of ψ_{11}

n	$\|\|\psi_h - \psi_{h/2}\|\|_{L_2}$	EOC
32/64	0.0684	
64/128	0.0818	-0.2597
128/256	0.0730	0.1655
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Nobody gets the convergence! What's the source for this?

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128/256	5.3517	-0.6914
256/512	8.0288	-0.5852

Nobody gets the convergence! What's the source for this?

To be continued...

Thank you for your attention!