Besov space regularity conditions for weak solutions of the Navier-Stokes equations

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The Problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded C^2-domain, $u_0 \in L^2_\sigma(\Omega)$ and

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$$

be a weak solution of the instationary Navier-Stokes equations

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= 0 \quad \text{in } \Omega \times (0,T) \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,T) \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T) \\ u(0) &= u_0 \quad \text{at } t = 0 \end{aligned}$$

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Find criteria so that u is a strong solution in Serrin's sense

$$u \in L^s(0,T; L^q(\Omega)), \quad \frac{2}{s} + \frac{3}{q} = 1, \; s > 2, \; q > 3.$$

First Idea: Optimal Initial Values (I)

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The optimal necessary and sufficient condition is described in

R. Farwig, H. Sohr, W. Varnhorn Ann. Univ. Ferrara 55, 89-110 (2009)

R. Farwig, H. Sohr Math. Ann. 345, 631-642 (2009)

R. Farwig, C. Komo (exterior domains) Analysis (Munich) (2013)

First Idea: Optimal Initial Values (II)

Theorem 1 Let the weak solution u satisfy the energy inequality.

- The condition

$$(*) \qquad \int_0^\infty \|e^{-tA}u_0\|_q^s \, dt < \infty$$

is **necessary and sufficient** to guarantee the existence of a unique local strong solution $u \in L^s(0,T';L^q(\Omega))$, $T' > 0$: the solution $e^{-tA}u_0$ of the corresponding Stokes system satisfies

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- Quantify T': $\exists\, \varepsilon_* = \varepsilon_*(\Omega,q) > 0$ such that if

$$\Big(\int_0^T \|e^{-tA}u_0\|_q^s \, dt\Big)^{1/s} \le \varepsilon_* \,, \quad 0 < T \le \infty,$$

  then $\exists\,!$ strong solution $u \in L^s(0,T;L^q(\Omega))$.

First Idea: Optimal Initial Values (III)

- Since $\|e^{-tA}u_0\|_q \le c\,t^{-\alpha}\,\|u_0\|_2$, $\alpha = \frac{1}{2}\left(\frac{3}{2} - \frac{3}{q}\right)$, only the integrability of $\|e^{-tA}u_0\|_q^s$ on some interval $(0,\delta)$ in (*) is important!

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- (*) is equivalent to the condition

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- We do need only that $u_0 \in \hat{\mathcal{B}}^{s,q}_\delta(\Omega) := \mathcal{B}^{s,q}_\delta(\Omega) \cap L^2_\sigma(\Omega)$ where

$$\mathcal{B}^{s,q}_\delta(\Omega) := \{u_0 : \|u_0\|^s_{\mathcal{B}^{s,q}_\delta} = \int_0^\delta \|e^{-tA}u_0\|_q^s\, dt < \infty\}$$

and that

$$\|u_0\|_{\mathcal{B}^{s,q}_\delta} < \varepsilon_*$$

to see that the weak solution u is strong on $[0,\delta)$.

Ideas

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- For the identification of the weak solution with the local strong solution starting at t_0 we need the *strong energy inequality (SEI)*, i.e.,

$$(SEI) \qquad \frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^{t} \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u(t_0)\|_2^2$$

for a.a. $0 \le t_0 \le T$ (including $t_0 = 0$) and all $t \in (t_0, T)$.

Notes

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- For $3 < q \leq s < \infty$ we have the embeddings

$$D(A_2^{1/4}) \subset L^3_\sigma(\Omega) \subset L^{3,s}_\sigma(\Omega) \subset \hat{\mathcal{B}}_\delta^{s,q}(\Omega) = \hat{\mathcal{B}}^{s,q}(\Omega)$$

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- For $3 < q \leq s < \infty$

$$C^0([0,T]; D(A_2^{1/4}) \subset C^0([0,T]; L^3_\sigma(\Omega))$$
$$\subset C^0([0,T]; L^{3,s}_\sigma(\Omega)) \subset C^0([0,T]; \hat{\mathcal{B}}^{s,q}_\delta(\Omega))$$

Theorem 2 (Regularity)

Let the weak solution u satisfy $u \in L^{\infty}_{\mathrm{loc}}(0,T;\mathcal{B}^{s,q}(\Omega))$. Then

u is regular if one of the following conditions is satisfied:

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3. $u \in C([0,T);\mathcal{B}^{s,q}(\Omega))$
4. In (3) the space $\mathcal{B}^{s,q}(\Omega)$ may be replaced by $D(A_2^{1/4})$, $L^3_\sigma(\Omega)$ and $L^{3,s}_\sigma(\Omega)$.

Proof of Theorem 2

1. $u \in L^\infty_{\text{loc}}(0,T';\mathcal{B}^{s,q}(\Omega)) \cap C^0_w([0,T'];L^2_\sigma(\Omega))$ and $\hat{\mathcal{B}}^{s,q}(\Omega) \subset L^2_\sigma(\Omega) \Rightarrow u \in C^0_w([0,T'];\mathcal{B}^{s,q}(\Omega))$ and hence

$$(**) \qquad \|u\|_{L^\infty(0,T';\mathcal{B}^{s,q})} = \sup_{t\in[0,T']} \|u(t)\|_{\mathcal{B}^{s,q}}$$

The same result holds for $\mathcal{B}^{s,q}_\delta(\Omega))$.

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Choose $\delta > 0$ such that $\|u(t_0)\|_{\mathcal{B}^{s,q}_\delta} < \varepsilon_*$ for all $t_0 \in [0,T']$, but consider only those t_0 where (SEI) holds.
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2. Let $\mathcal{N} \subset (0,T)$ be the exceptional set in (2). By $(**)$

$$\sup_{t\in[0,T']} \|u(t)\|_{\mathcal{B}^{s,q}_\delta} = \|u\|_{L^\infty(0,T';\mathcal{B}^{s,q}_\delta)} \le \sup_{t\in[0,T']\setminus\mathcal{N}} \|u(t)\|_{\mathcal{B}^{s,q}_\delta}$$

$\to 0$ as $\delta \to 0$. Use (1)

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3. Use uniform continuity on $[0,T']$ and the pointwise limit $\|u(t_j)\|_{\mathcal{B}^{s,q}_\delta}$ in equidistant epochs t_j.

Theorem 3 (Uniqueness)

Let the weak solution u satisfy $u \in L^\infty_{\mathrm{loc}}(0,T;\mathcal{B}^{s,q}(\Omega))$. Then

$$u \text{ is uniquely determined by } u_0 \in L^2_\sigma(\Omega)$$

if one of the following conditions is satisfied:

1. The energy inequality holds for each $t_0 \in [0,T)$
2. The kinetic energy $\|u(t)\|_2^2$ is continuous on $[0,T)$
3. $u \in L^\infty_{\mathrm{loc}}([0,T);L^{3,\infty}_\sigma(\Omega))$

Proof of Theorem 3

1. Let $u, \tilde{u}$ be two weak solutions with $u(0) = \tilde{u}(0)$ and assume $u = \tilde{u}$ on a maximal interval $[0, T_1)$ where $T_1 < T$
$u, \tilde{u} \in C_w^0([0,T); L^2) \Rightarrow u(T_1) = \tilde{u}(T_1)$.

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   $u(T_1) \in \mathcal{B}^{s,q}(\Omega) \Rightarrow \exists\, \delta > 0 \ \|u(T_1)\|_{\mathcal{B}^{s,q}_\delta} < \varepsilon^*$
   u satisfies energy inequality at $T_1 \Rightarrow$
   $u \in L^s([T_1, T_1 + \delta); L^q(\Omega))$
   $\tilde{u}$ satisfies energy inequality at $T_1 \Rightarrow u = \tilde{u}$ on $[T_1, T_1 + \delta)$,
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   $\tilde{u}$ satisfies energy inequality at $T_1 \Rightarrow u = \tilde{u}$ on $[T_1, T_1 + \delta)$, i.e., T_1 was not maximal
2. (2) $\Rightarrow$ (1)
3. $\|v \otimes w\|_{L^{2,2}} \le c\|v\|_{L^{3,\infty}}\|w\|_{L^{6,2}} \le c\|v\|_{L^{3,\infty}}\|\nabla w\|_{L^2}$
   $\Rightarrow u \otimes u \in L^2(0,T; L^2(\Omega))$, $u \in L^4(0,T; L^4(\Omega))$
   $\Rightarrow u$ even satisfies the energy identity

$\kappa^{s,q}$−space regularity conditions for weak solutions of the Navier-Stokes equations