On global L^p-solutions for a fluid model of Oldroyd kind

Matthias Geissert
(with M. Nesensohn and Y. Shibata)

Techn. Universität Darmstadt
Germany

Jun. 2013

Outline

The problem (NS)

Consider

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= f + \operatorname{div} T(u) - \nabla \pi && \text{in } J \times \Omega, \\ \operatorname{div} u &= 0 && \text{in } J \times \Omega, \\ u &= 0, && \text{on } J \times \partial\Omega, \\ u(0) &= u_0 && \text{in } \Omega, \end{aligned}$$

Here:

- u velocity of the fluid, π pressure of the fluid,
- u_0 initial velocity of the fluid, f extra body force,
- $J = (0, T)$, $\Omega \subset \mathbb{R}^n$ domain.
- $T(u)$ extra stress tensor.

Viscoelastic Fluids (OBF)

We set

$$T(u) := E(u) + \tau,$$

where

$$\begin{aligned} \partial_t \tau + (u \cdot \nabla)\tau + \gamma\tau &= \delta D u + g(\nabla u, \tau) && \text{in } J \times \Omega, \\ \tau(0) &= \tau_0 && \text{in } \Omega. \end{aligned}$$

Here:

- $Eu = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$,
- τ: elastic part of the stress,
- *Oldroyd-B fluids*:

$$g(\nabla u, \tau) = -\tau W u + W u \tau + a(D u \tau + \tau D u)$$

for $\gamma \geq 0$, $\delta > 0$, $-1 \leq a \leq 1$ and $Wu = \frac{1}{2}(\nabla u - \nabla u^T)$,

The Helmholtz projection

- Let $1 < q < \infty$, $\Omega \subset \mathbb{R}^n$ be a domain.
- We say that the *Helmholtz decomposition* exists if

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus G_q(\Omega),$$

where

$$G_q(\Omega) := \{g \in L^q(\Omega)^n : \exists h \in \widehat{H}^{1,q}(\Omega) \text{ such that } g = \nabla h\},$$
$$L^q_\sigma(\Omega) := \{\varphi \in C_c^\infty(\Omega)^n : div\, \varphi = 0\}^{\|\cdot\|_{L^q(\Omega)}}$$

In this case there exists the *Helmholtz projection*

$$P : L^q(\Omega)^n \to L^q_\sigma(\Omega).$$

The Stokes operator

Let $1 < q < \infty$ and $\Omega \subset \mathbb{R}^n$ be a domain such that the Helmholtz projection exists. Set

$$D(A_{St}) = H^{2,q}(\Omega) \cap H_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$$

and define the *Stokes operator*

$$A_{St} : \begin{cases} D(A_{St}) & \to & L_\sigma^q(\Omega), \\ u & \mapsto & P\Delta u. \end{cases}$$

Basic Idea

Rewrite as fixed point problem:

$$\begin{aligned}
\partial_t u - \Delta u + \nabla\pi &= f - (\tilde{u}\cdot\nabla)\tilde{u}) + \operatorname{div}\tilde{\tau} && \text{in } J\times\Omega,\\
\operatorname{div}\ u &= 0 && \text{in } J\times\Omega,\\
\partial_t\tau + (\tilde{u}\cdot\nabla)\tau + \gamma\tau &= \delta Du + g(\nabla\tilde{u},\tilde{\tau}) && \text{in } J\times\Omega,\\
u &= 0, && \text{on } J\times\partial\Omega,\\
u(0) &= u_0 && \text{in } \Omega,\\
\tau(0) &= \tau_0 && \text{in } \Omega.
\end{aligned}$$

and consider

$$\Phi : (\tilde{u},\tilde{\tau}) \mapsto (u,\tau).$$

Global Solutions for small data

- L^2, δ small, $n = 2, 3$, $\gamma > 0$, bounded [1], exterior [2].
- L^2, $\delta > 0$, $n = 3$, $\gamma > 0$, bounded [3], exterior [4].
- L^2, $\delta > 0$, $\gamma > 0$, $n = 3$, $\mathbb{R}^3$ [5].
- L^2, $\delta > 0$, $\gamma = 0$, $n = 2$, $\mathbb{R}^2$, $a = 1$ [6].

C. Guillopé and J.-C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, **15**(1990), 849–869.

M. Hieber, Y. Naito, and Y. Shibata. Global existence results for oldroyd-b fluids in exterior domains. *J. Differential Equations*, **252**(2012), 2617–2629.

L. Molinet and R. Talhouk. On the global and periodic regular flows of viscoelastic fluids with a differential constitutive law. *NoDEA Nonlinear Differential Equations Appl.*, **11**(2004), 349–359.

D. Fang, M. Hieber, and R. Zi. Global existence results for oldroyd-b fluids in exterior domains: The case of non-small coupling parameters, submitted.

D. A. Vorotnikov and V. G. Zvyagin. On the solvability of the initial-value problem for the motion equations of nonlinear viscoelastic medium in the whole space. *Nonlinear Anal.*, **58**(2004), 631–656.

F.-H. Lin, C. Liu, and P. Zhang. On hydrodynamics of viscoelastic fluids. *Comm. Pure Appl. Math.*, **58**(2005), 1437–1471.

Lagrangian coordinates

We set

$$\Theta : \begin{cases} J \times \Omega_0 & \to & J \times \Omega \\ (t, x) & \mapsto & (t, X(t, \xi) = \xi + \int_0^t v(s, X(s, \xi))\, \mathrm{d}s) \end{cases}$$

and

$$v(t,x) := (\Theta^* v)(t,x) := u(\Theta^{-1}(t,x)),$$
$$\theta(t,x) := (\Theta^* \theta)(t,x) := \pi(\Theta^{-1}(t,x)),$$
$$\eta(t,x) := (\Theta^* \eta)(t,x) := \tau(\Theta^{-1}(t,x)).$$

Note: $(\partial_t \tau + u.\nabla \tau)(t, X(t,\xi)) = \partial_t \eta(t,\xi)$

Lagrangian coordinates

We set

$$\Theta : \begin{cases} J \times \Omega_0 & \to & J \times \Omega \\ (t,x) & \mapsto & (t, X(t,\xi) = \xi + \int_0^t v(s, X(s,\xi))\, \mathrm{d}s) \end{cases}$$

and

$$v(t,x) := (\Theta^* v)(t,x) := u(\Theta^{-1}(t,x)),$$
$$\theta(t,x) := (\Theta^* \theta)(t,x) := \pi(\Theta^{-1}(t,x)),$$
$$\eta(t,x) := (\Theta^* \eta)(t,x) := \tau(\Theta^{-1}(t,x)).$$

Note: $(\partial_t \tau + u.\nabla\tau)(t, X(t,\xi)) = \partial_t \eta(t,\xi)$

The Problem in Lagrangian Coordinates

$$
\begin{aligned}
\partial_t v - \Delta v + \nabla \theta - \operatorname{Div} \eta &= F(v, \theta, \eta) && \text{in } (0, \infty) \times \Omega, \\
\operatorname{div} v &= F_d(v) && \text{in } (0, \infty) \times \Omega, \\
\partial_t \eta + \gamma \eta - \delta E v &= G(v, \eta) && \text{in } (0, \infty) \times \Omega, \\
v &= 0 && \text{on } (0, \infty) \times \partial\Omega, \\
v(0) &= u_0 && \text{in } \Omega, \\
\eta(0) &= \tau_0 && \text{in } \Omega,
\end{aligned}
$$

Here:

- $F(v, \theta, \eta) = \operatorname{Div} \left[V_1(I(\nabla v)) \nabla v + V_2(I(\nabla v)) \eta + V_3(I(\nabla v)) \theta \right]$,
- $F_d(v) = V_4(I(\nabla v)) \nabla v = \operatorname{div} V_5(I(\nabla v)) v$,
- $G(v, \eta) = V_6(I(\nabla v)) \nabla v + g(\nabla v, \eta) + g(V_7(I(\nabla v)) \nabla v, \eta)$,
- $I(\nabla v)(t) = \int_0^t \nabla v(s) ds$,
- V_j are smooth functions with $V_j(0) = 0$.

The associated linear problem (ALP)

Consider

$$
\begin{aligned}
\partial_t v - \Delta v + \nabla\theta - \operatorname{Div}\eta &= f && \text{in } (0,\infty)\times\Omega,\\
\operatorname{div} v &= f_d && \text{in } (0,\infty)\times\Omega,\\
\partial_t\eta + \gamma\eta - \delta E v &= g && \text{in } (0,\infty)\times\Omega,\\
v &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\
v(0) &= u_0 && \text{in } \Omega,\\
\eta(0) &= \tau_0 && \text{in } \Omega.
\end{aligned}
$$

for

$$
\begin{aligned}
&f \in L^p_\mu(0,\infty; L_q(\Omega)), \quad g \in L^p_\mu(0,\infty; H^{1,q}(\Omega)),\\
&f_d \in H^{1,p}_\mu(0,\infty; \widehat{H}_0^{-1,q}(\Omega)) \cap L^p_\mu(0,\infty; H^{1,q}(\Omega)),\\
&v_0 \in (D(A_{St}, L^q_\sigma(\Omega))_{1-\frac{1}{p},p}, \tau_0 \in H^1_q(\Omega).
\end{aligned}
$$

Here:

- $f \in L^p_\mu(\mathbb{R}_+; X) \Leftrightarrow e^{-\mu\cdot} f \in L^p(\mathbb{R}_+; X)$

The associated linear problem
Reduction to $u_0, \eta_0, f_d = 0$

$$
\begin{aligned}
\partial_t v - \Delta v + \nabla\theta - \operatorname{Div}\eta &= f && \text{in } (0,\infty)\times\Omega,\\
\operatorname{div} v &= 0 && \text{in } (0,\infty)\times\Omega,\\
\partial_t\eta + \gamma\eta - Ev &= g && \text{in } (0,\infty)\times\Omega,\\
v &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\
v(0) &= 0 && \text{in } \Omega,\\
\eta(0) &= 0 && \text{in } \Omega.
\end{aligned}
$$

Idea:

- Use Stokes semigroup: $e^{A_{St}t}u_0$,
- Use $e^{-\gamma t}\tau_0$,
- Use Bogovksii's operator.

Oldroyd-B Operator $\mathcal{A}_\gamma$

This is equivalent to

$$\begin{pmatrix} v \\ \eta \end{pmatrix}'(t) - \mathcal{A}_\gamma \begin{pmatrix} v \\ \eta \end{pmatrix}(t) = G(t), \quad t > 0,$$
$$\begin{pmatrix} v \\ \eta \end{pmatrix}(0) = 0,$$

where

$$\mathcal{A}_\gamma = \begin{pmatrix} A_{St} & P\,\mathrm{Div} \\ E & -\gamma \end{pmatrix},$$

in $\mathcal{X} = L^q_\sigma(\Omega) \times H^{1,q}(\Omega)$ with $D(\mathcal{A}_\gamma) = D(A_{St}) \times H^{1,q}(\Omega)$.

Maximal L^p_μ Regularity of A_γ

Theorem

Let $p \in (1, \infty)$. Then:

(a) *It holds*

$$s(\mathcal{A}_\gamma) < 0, \quad \gamma > 0,$$
$$s(\mathcal{A}_0) = 0.$$

(b) $\mathcal{A}_\gamma$ *has maximal L^p_μ regularity for $\mu > s(\mathcal{A}_\gamma)$.*

Remark

The theorem above yields maximal regularity estimates for (ALP)

Idea of proof: Maximal regularity

- Note that A_{St} and $-\gamma$ are $\mathcal{R}$-sectorial.
- Show

$$\mathcal{A}_\gamma = \begin{pmatrix} A_{St} & P\,\mathrm{Div} \\ 0 & -\gamma \end{pmatrix}$$

  is $\mathcal{R}$-sectorial.
- Show E is small perturbation.

Spectrum of the Oldroyd Operator

Proposition

Fix $q \in (1, \infty)$. Then, the spectrum of $\mathcal{A}_\gamma$ is independent of q and

$$\sigma(-\mathcal{A}_\gamma) = \{\lambda \in \mathbb{C} : \kappa(\lambda) \in \sigma(A_{St})\} \cup \{-\gamma, -1-\gamma\}$$
$$\sigma_p(-\mathcal{A}_\gamma) = \sigma(-\mathcal{A}_\gamma) \setminus \{-1-\gamma\}.$$

All eigenvalues of $-\mathcal{A}_\gamma$ are isolated and their limit points are $-1-\gamma$ and $-\infty$.

Here:

$$\kappa(\lambda) := \lambda(1 + \frac{\delta}{\lambda + \gamma})^{-1}.$$

Nonlinear Terms

We define

$$N(v,\theta,\eta) := (F(v,\theta,\eta), F_d(v), G(v,\eta)).$$

Recall:

- $F(v,\theta,\eta) = \mathrm{Div}\left[V_1(I(\nabla v))\nabla v + V_2(I(\nabla v))\eta + V_3(I(\nabla v))\theta\right]$,
- $F_d(v) = V_4(I(\nabla v))\nabla v = \mathrm{div}\, V_5(I(\nabla v))v$,
- $G(v,\eta) = V_6(I(\nabla v))\nabla v + g(\nabla v,\eta) + g(V_7(I(\nabla v))\nabla v,\eta)$,

Lemma

Let $p \in (1,\infty)$, $q \in (n,\infty)$, $\mu < 0$. *Then*

$$N \in C^1$$

Moreover, $N(0) = 0$ *as well as* $DN(0) = 0$.

Global Solutions ($\gamma > 0$)

Theorem

Let $\gamma > 0$, $p \in (1, \infty)$, $q \in (n, \infty)$.

(a) *Then there exists a unique, global strong* (u, π, τ) *of the Oldroyd-B*

$$\begin{aligned} u &\in L^p_\mu(\mathbb{R}_+; H^{2,q}(\Omega)) \cap H^{1,p}_\mu(\mathbb{R}_+; L^q(\Omega)) \\ \pi &\in L^p_\mu(\mathbb{R}_+; \widehat{H}^{1,q}(\Omega)) \\ \tau &\in L^p_\mu(\mathbb{R}_+; H^{1,q}(\Omega)) \cap H^{1,p}_\mu(\mathbb{R}_+; L^q\Omega)) \end{aligned}$$

of problem (OBF) for initial data small w.r.t. $(D(\mathcal{A}_\gamma), \mathcal{X})_{1-\frac{1}{p},p}$*-norm and some* $\mu < 0$.

(b) $(u, \tau) \to 0$ *für* $t \to \infty$ *in* $(D(\mathcal{A}_\gamma), \mathcal{X})_{1-\frac{1}{p},p}$.

Idea of Proof

- Fixed point iteration.
- Let $\mathrm{e}^{-\mu\cdot}v \in H^{1,p}(0,\infty;L^q(\Omega)) \cap L^p(0,\infty;D(A_{St}))$. Then,

$$\begin{aligned}\mathrm{e}^{-\mu\cdot}u &\in H^{1,p}(0,\infty;L^q(\Omega)) \cap L^p(0,\infty;D(A_{St}))\\ &\hookrightarrow BUC([0,\infty);(D(A_{St}),L^q_\sigma)_{1-\frac{1}{p},p}).\end{aligned}$$

  u converges exponetially fast to 0 in $(D(A_q),L_{q,\sigma})_{1-\frac{1}{p},p}$.
- Let $\mathrm{e}^{-\mu\cdot}\eta \in H^{1,p}(0,\infty;H^{1,q}(\Omega))$. Then,

$$\begin{aligned}\mathrm{e}^{-\mu\cdot}\tau &\in H^{1,p}(0,\infty;L^q(\Omega)) \cap L^p(0,\infty;H^{1,q}(\Omega))\\ \mathrm{e}^{-\mu\cdot}\tau &\in BUC([0,\infty);H^{1,q}(\Omega)).\end{aligned}$$

  τ converges exponetially fast to 0 in $H^1_q(\Omega)$.

Spectrum of $\mathcal{A}_0$

Recall:

$$\mathcal{A}_\gamma = \begin{pmatrix} A_{St} & P\,\mathrm{Div} \\ E & 0 \end{pmatrix}.$$

Theorem

(a) 0 *is isolated eigenvalue* $\mathcal{A}_0$.

(b) $\mathcal{X} = N(\mathcal{A}_0) \oplus R(\mathcal{A}_0)$.

Kernel and Range of the Oldroyd Operator

Lemma

Let $\gamma = 0$. *Kernel* $N(\mathcal{A}_0)$ *and range* $R(\mathcal{A}_0)$ *of* $\mathcal{A}_0$ *are*

$$N(\mathcal{A}_0) = \{0\} \times \{\eta \in H^{1,q}(\Omega) \colon P \operatorname{Div} \eta = 0\},$$
$$R(\mathcal{A}_0) = L^q_\sigma(\Omega) \times \{\eta = Ew \colon w \in D(A_{St})\}.$$

The corresponding projections

$$P^s \colon \mathcal{X} \to R(\mathcal{A}_0) \quad \textit{and} \quad P^c \colon \mathcal{X} \to N(\mathcal{A}_0)$$

are given by

$$P^s(f,g) = (f, -EA_{St}^{-1} P \operatorname{Div} g), \quad (f,g) \in \mathcal{X},$$
$$P^c(f,g) = (0, g + EA_{St}^{-1} P \operatorname{Div} g), \quad (f,g) \in \mathcal{X}.$$

The Problem in Lagrangian Coordinates ($\gamma = 0$)

Recall:

$$
\begin{aligned}
\rho\partial_t v - \Delta v + \nabla\theta - \operatorname{Div}\eta &= F(v,\theta,\eta) && \text{in } (0,\infty)\times\Omega,\\
\operatorname{div} v &= F_d(v) && \text{in } (0,\infty)\times\Omega,\\
\partial_t\eta - Ev &= G(v,\eta) && \text{in } (0,\infty)\times\Omega,\\
v &= 0 && \text{on } (0,\infty)\times\partial\Omega,\\
v(0) &= u_0 && \text{in } \Omega,\\
\eta(0) &= \tau_0 && \text{in } \Omega,
\end{aligned}
$$

Here:

- $F(v,\theta,\eta) = \operatorname{Div}\left[V_1(I(\nabla v))\nabla v + V_2(I(\nabla v))\eta + V_3(I(\nabla v))\theta\right]$,
- $F_d(v) = V_4(I(\nabla v))\nabla v = \operatorname{div} V_5(I(\nabla v))v$,
- $G(v,\eta) = V_6(I(\nabla v))\nabla v + g(\nabla v,\eta) + g(V_7(I(\nabla v))\nabla v,\eta)$,

Maximal L^p_μ Regularity: $\gamma = 0$ $(LOBF_0)$

Consider

$$\begin{aligned} \partial_t u - \Delta u + \nabla \pi + \operatorname{Div} \tau = f + \operatorname{Div} F \quad & \text{in } (0,\infty) \times \Omega, \\ \operatorname{div} u = 0 \quad & \text{in } (0,\infty) \times \Omega, \\ \partial_t \tau - Eu = g \quad & \text{in } (0,\infty) \times \Omega, \\ u = 0 \quad & \text{auf } (0,\infty) \times \partial\Omega, \\ u(0,\cdot) = 0 \quad & \text{in } \Omega, \\ \tau(0,\cdot) = 0 \quad & \text{in } \Omega. \end{aligned}$$

for

$$\begin{aligned} & f \in \mathbb{F}_1 := L^p_\mu(\mathbb{R}_+; L^q(\Omega)), \\ & F \in \mathbb{F}_2 := \{F \in BUC([0,\infty); H^{1,q}(\Omega)) \colon \partial_t F \in L^p_\mu(\mathbb{R}_+; H^{1,q}(\Omega))\} \\ & g \in \mathbb{G} := L^p_\mu(\mathbb{R}_+; H^{1,q}(\Omega)). \end{aligned}$$

Maximal L^p_μ Regularity of $(LOBF_0)$

Theorem

Let $p, q \in (1, \infty)$, $\gamma = 0$, Then, there exists $\mu < 0$, s.t. for $f \in \mathbb{F}_1$, $F \in \mathbb{F}_2$ and $g \in \mathbb{G}$ there exists a unique solution (u, π, τ) of $(LOBF_0)$ satisfying

$$\begin{aligned}
u \in & L^p_\mu(\mathbb{R}_+; H^{2,q}(\Omega)) \cap H^{1,p}_\mu(\mathbb{R}_+; L^q(\Omega)) \\
\pi \in & L^p_\mu(\mathbb{R}_+; \widehat{H}^{1,q}(\Omega)) \\
& \qquad + \left\{\theta \in BUC([0, \infty); \widehat{H}^{1,q}(\Omega)) : \partial_t\theta \in L^p_\mu(\mathbb{R}_+ \widehat{H}^{1,q}(\Omega))\right\} \\
\tau \in & \left\{\eta \in BUC([0, \infty); H^{1,q}(\Omega)) : \partial_t\eta \in L^p_\mu(\mathbb{R}_+; H^{1,q}(\Omega))\right\}
\end{aligned}$$

Global Solutions ($\gamma = 0$)

Theorem

Let $\gamma = 0$. $p \in (1, \infty)$, $q \in (n, \infty)$.

(a) *Then, there exists a global solution* (u, π, τ)

$$
\begin{aligned}
u \in & L^p_\mu(\mathbb{R}_+; H^{2,q}(\Omega)) \cap H^{1,p}_\mu(\mathbb{R}_+; L^q(\Omega)) \\
\pi \in & L^p_\mu(\mathbb{R}_+; \widehat{H}^{1,q}(\Omega)) \\
& \quad + \left\{\tau \in BUC([0,\infty); \widehat{H}^{1,q}(\Omega)) \colon \partial_t \pi \in L^p_\mu(\mathbb{R}_+ L^q(\Omega))\right\} \\
\tau \in & \left\{\tau \in BUC([0,\infty); H^{1,q}(\Omega)) \colon \partial_t \tau \in L^p_\mu(\mathbb{R}_+; L^q(\Omega))\right\}
\end{aligned}
$$

of (OBF) for small data w.r.t. $(D(\mathcal{A}_0), \mathcal{X})_{1-\frac{1}{p},p}$*-norm and some* $\mu < 0$.

(b) $(u, \tau) \to (0, \tau_0)$ *for* $t \to \infty$ *in* $(D(\mathcal{A}_0), \mathcal{X})_{1-\frac{1}{p},p}$.

Idea of Proof

- Fixed point iteration.
- Let $\mathrm{e}^{-\mu\cdot} v \in H^{1,p}(0,\infty; L^q(\Omega)) \cap L^p(0,\infty; D(A_{St}))$. Then,

$$\mathrm{e}^{-\mu\cdot} u \in H^{1,p}(0,\infty; L^q(\Omega)) \cap L^p(0,\infty; D(A_{St})) \hookrightarrow BUC([0,\infty); (D(A_{St}), L^q_\sigma)_{1-\frac{1}{p},p}).$$

  u converges exponetially fast to 0 in $(D(A_{St}), L^q_\sigma)_{1-\frac{1}{p},p}$.
- Let $\eta \in BUC([0,\infty); H^{1,q}(\Omega))$, $\mathrm{e}^{-\mu\cdot}\partial_t \eta \in L^p(0,\infty; H^{1,q}(\Omega))$. Then,

$$\tau \in BUC([0,\infty); H^{1,q}(\Omega)), \quad \mathrm{e}^{-\mu\cdot} \frac{\mathrm{d}}{\mathrm{dt}} \tau \in L^q(0,\infty; L^q(\Omega)).$$

  Moreover, there exists $\tau^\infty \in H^{1,q}(\Omega)$ such that τ converges to τ^∞ in $H^{1,q}(\Omega)$.

Equilibria (τ symmetric)

Proposition

Assume that $\gamma = 0$. *Then* (u, π, τ) *is an equilibrium of (OBF) if and only if*

$$u = 0 \quad \textit{and} \quad \operatorname{Div} \tau = \nabla \pi.$$

Furthermore, the energy

$$\mathsf{E} = \begin{cases} \frac{\rho}{2} \int_\Omega |u|^2 + \frac{1}{2\delta} \int_\Omega |\tau|^2 & \textit{if } a = 0, \\ \frac{\rho}{2} \int_\Omega |u|^2 + \frac{1}{2a} \int_\Omega \operatorname{tr} \tau & \textit{if } a \neq 0 \end{cases}$$

is a strict Ljapunov functional, i.e. for every solution (u, π, τ) *of (OBF) the energy satisfies* $\frac{d}{dt}\mathsf{E} \leq 0$ *and* $\frac{d}{dt}\mathsf{E} = 0$ *if only if* (u, π, τ) *is an equilibrium.*