Some remarks on generalized Korn's inequalities

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Introduction (Korn's Inequality)

$\Omega \subset \mathbb{R}^n$: bounded Lipschitz domain, $\quad \|\cdot\| := \|\cdot\|_{L^2(\Omega)}$

$\varepsilon(\boldsymbol{u}) := \big(\varepsilon_{ij}(\boldsymbol{u})\big),\ \varepsilon_{ij}(\boldsymbol{u}) := \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ for $\boldsymbol{u} = (u_i) \in \boldsymbol{H}^1(\Omega)$

Korn's Inequality

$$\|\varepsilon(\boldsymbol{u})\|^2 + \|\boldsymbol{u}\|^2 \geqslant {}^{\exists}C\,\|\nabla \boldsymbol{u}\|^2 \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) := H^1(\Omega; \mathbb{C}^n)$$

For $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ (even for $\boldsymbol{u} \in \boldsymbol{\mathcal{D}}'(\Omega)$), it holds

$$\varepsilon_{ij}(\boldsymbol{u}) \in L^2(\Omega)\ \ (1 \leqslant i \leqslant j \leqslant n) \quad \Rightarrow \quad \partial_j u_i \in L^2(\Omega)\ \ (1 \leqslant i, j \leqslant n)$$

only $\frac{1}{2}n(n+1)$ linear combinations of partial derivatives of $\boldsymbol{u}$ $\qquad$ n^2 partial derivatives of $\boldsymbol{u}$

"This implication is *definitely unexpected*" by Ciarlet (1986)

Long History

Korn (1906,...), Friedrichs (1947), Gobert (1962), Nečas (1967), Duvaut-Lions (1972), Nitsche (1981), Kondrat'ev-Oleinik (1988), Horgan (1995), Ciarlet (2005,...), ...

Energy Functional

$$E[\boldsymbol{u}] = \int_\Omega W(\nabla \boldsymbol{u})\,dx \quad \text{for } \boldsymbol{u} = (u_r) \in \boldsymbol{H}^1(\Omega) := H^1(\Omega;\mathbb{C}^m)$$

$$\text{with } \; W(\nabla \boldsymbol{u}) = \sum_{i,j=1}^{n} (A^{ij}\partial_j \boldsymbol{u}, \partial_i \boldsymbol{u})_{\mathbb{C}^m}, \quad (A^{ij})^{\mathsf{T}} = A^{ji} \in M_m(\mathbb{R}) \quad (1 \leqslant i,j \leqslant n)$$

Case of linear elasticity $(m = n)$

$$W(\nabla \boldsymbol{u}) = \tilde{W}(\varepsilon(\boldsymbol{u})) = \sum_{i,j,k,h} a^{ijkh} \varepsilon_{kh}(\boldsymbol{u})\overline{\varepsilon_{ij}(\boldsymbol{u})} \quad (a^{ijkh} = a^{khij} = a^{jikh})$$

$$= \sum_{i,j} (A^{ij}\partial_j \boldsymbol{u}, \partial_i \boldsymbol{u}) \quad \text{with } \; A^{ij} = \left(a^{ikjh}\right)_{\substack{k=1\downarrow n \\ h=1\to n}}$$

strong convexity :

$$\sum_{i,j,k,h} a^{ijkh} s_{kh} s_{ij} \geqslant {}^{\exists} c_0 \sum_{i,j} {s_{ij}}^2 \quad \forall (s_{ij}) : \text{real sym.} \quad \Rightarrow \quad E[\boldsymbol{u}] \geqslant c_0 \|\varepsilon(\boldsymbol{u})\|^2$$

isotropic elasticity :

$$A^{ij} = \lambda\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j + \mu(\boldsymbol{e}_j \otimes \boldsymbol{e}_i + I_n), \quad E[\boldsymbol{u}] = \lambda \|\mathrm{tr}\,\varepsilon(\boldsymbol{u})\|^2 + 2\mu \|\varepsilon(\boldsymbol{u})\|^2$$

Variety of Coerciveness

$$E[\boldsymbol{u}] = \int_\Omega \Big(\overbrace{-\sum_{i,j} A^{ij}\partial_i\partial_j \boldsymbol{u}}^{\mathcal{L}\boldsymbol{u}},\, \boldsymbol{u} \Big) dx + \int_{\partial\Omega} \Big(\overbrace{\sum_{i,j} \nu_i(x) A^{ij}\partial_j \boldsymbol{u}}^{\mathcal{N}\boldsymbol{u}},\, \boldsymbol{u} \Big) dS$$

$E[\boldsymbol{u}] \quad \longleftrightarrow \quad$ Neumann-type B.V.P. $\{\mathcal{L}, \mathcal{N}\}$

①	*strongly elliptic*	$E[\boldsymbol{u}] \geqslant {}^\exists c_1 \|\nabla \boldsymbol{u}\|^2$	$\forall \boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$
②	*coercive*	$E[\boldsymbol{u}] + {}^\exists c_0 \|\boldsymbol{u}\|^2 \geqslant {}^\exists c_1 \|\nabla \boldsymbol{u}\|^2$	$\forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$
③	*Korn-type ineq.*	$E[\boldsymbol{u}] + {}^\forall c_0 \|\boldsymbol{u}\|^2 \geqslant {}^\exists c_1 \|\nabla \boldsymbol{u}\|^2$	$\forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$
④	*strongly coercive*	$E[\boldsymbol{u}] \geqslant {}^\exists c_1 \|\nabla \boldsymbol{u}\|^2$	$\forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$

Remark

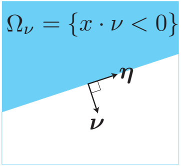
- Generally, ④ ⇒ ③ ⇒ ② ⇒ ①. If $m = 1$ (or $n = 1$), all are **equivalent**.
- Condition for ①, ② ($\partial\Omega \in C^1$) to hold is well known.
- ③ ⇔ $E[\boldsymbol{u}] \geqslant 0$ & $E[\boldsymbol{u}] + \|\boldsymbol{u}\|^2 \geqslant c_1 \|\nabla \boldsymbol{u}\|^2 \quad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$

  $E[\boldsymbol{u}] \geqslant 0 \ (\forall \boldsymbol{u}) \ \Leftrightarrow \ E[\boldsymbol{u}] = \|P(\partial)\boldsymbol{u}\|^2 \quad ({}^\exists P(\xi)$: homog. in ξ of deg. 1)

Condition for ①, ② to Hold

$$L(\xi) := \sum_{i,j} A^{ij}\xi_i\xi_j, \quad N_\nu(\xi) := \sqrt{-1}\sum_{i,j} A^{ij}\nu_i\xi_j, \quad \xi = (\xi_i),\ \nu = (\nu_i) \in \mathbb{R}^n.$$

- ① $\Leftrightarrow$ $L(\xi) > O \ \ \forall\xi \in \mathbb{S}^{n-1}$ ($L(D)$ is **strongly elliptic**)
- When $L(D)$ is strongly elliptic, let $Z_\nu(\eta)$ $(\eta \perp \nu)$ be the symbol of **Dirichlet-Neumann map** for B.V.P. $\{L(D), N_\nu(D)\}$ on $\Omega_\nu = \{x \mid x\cdot\nu < 0\}$



If $\underline{\partial\Omega \in C^1}$,

② $\Leftrightarrow$ $L(\xi) > O \ \ \forall\xi \in \mathbb{S}^{n-1}$, $Z_\nu(\eta) > O \ \ \forall\nu,\eta \in \mathbb{S}^{n-1}$ with $\eta \perp \nu$

$L(D)$: **strongly elliptic** $\{L(D), N_\nu(D)\}$: **strongly complementing**

$$\Big(\Leftrightarrow \quad E[\boldsymbol{u}] \geqslant {}^{\exists}C_\nu\|\nabla\boldsymbol{u}\|^2 \quad \forall\boldsymbol{u} \in \boldsymbol{H}^1(\Omega_\nu),\ \forall\nu \in \mathbb{S}^{n-1}\Big)$$

Example (isotropic elasticity)

$$A^{ij} = \lambda\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j + \mu\,(\delta_{ij} I + \boldsymbol{e}_j \otimes \boldsymbol{e}_i) \quad 1 \leqslant i, j \leqslant n\,(= m \geqslant 2),$$

$$E[\boldsymbol{u}] = \lambda\,\|\mathrm{div}\,\boldsymbol{u}\|^2 + 2\mu\,\|\varepsilon(\boldsymbol{u})\|^2 \qquad (\lambda,\ \mu : \text{Lamé constants}).$$

$L(\xi) = \mu|\xi|^2 I + (\lambda+\mu)\boldsymbol{\xi}\otimes\boldsymbol{\xi}$ with eigenvalues $\underbrace{\mu|\xi|^2, \ldots, \mu|\xi|^2}_{n-1}, \ (\lambda+2\mu)|\xi|^2$

$$N_\nu(\xi) = \sqrt{-1}\,\{\lambda\,\boldsymbol{\nu}\otimes\boldsymbol{\xi} + \mu((\boldsymbol{\xi}\cdot\boldsymbol{\nu})I + \boldsymbol{\xi}\otimes\boldsymbol{\nu})\}$$

Thus, ① $\Leftrightarrow$ $\mu > 0$ & $\lambda + 2\mu > 0$

Under this condition,

$$Z_\nu(\eta) = \mu\left\{I + \frac{\lambda+\mu}{\lambda+3\mu}\left(\frac{\boldsymbol{\eta}}{|\eta|}\otimes\frac{\boldsymbol{\eta}}{|\eta|} + \frac{\boldsymbol{\nu}}{|\nu|}\otimes\frac{\boldsymbol{\nu}}{|\nu|}\right) + \sqrt{-1}\,\frac{2\mu}{\lambda+3\mu}\,\frac{\boldsymbol{\eta}}{|\eta|}\wedge\frac{\boldsymbol{\nu}}{|\nu|}\right\}|\eta||\nu|$$

with eigenvalues $\underbrace{\mu|\eta||\nu|, \ldots, \mu|\eta||\nu|}_{n-2}, \ 2\mu|\eta||\nu|, \ \dfrac{2\mu(\lambda+\mu)}{\lambda+3\mu}|\eta||\nu|$

Hence, ② $\Leftrightarrow$ $\mu > 0$ & $\lambda + \mu > 0$

Nonnegative Energy

$$\boxed{\begin{array}{c} E[\boldsymbol{u}] \geqslant 0 \\ (W(\nabla \boldsymbol{u}) \geqslant 0) \\ \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \end{array}} \Leftrightarrow \overbrace{\sum_{i,j=1}^{n} A^{ij} \boldsymbol{\zeta}_j \cdot \boldsymbol{\zeta}_i}^{\text{quad. form of } Z} \geqslant 0 \quad \begin{array}{l} (\zeta_{ri} \leftrightarrow \partial_i u_r) \\ \forall Z = (\boldsymbol{\zeta}_1 \cdots \boldsymbol{\zeta}_n) \in M_{m,n}(\mathbb{R}) \\ \boldsymbol{\zeta}_i = (\zeta_{ri}) \in \mathbb{R}^m, 1 \leqslant i \leqslant n. \end{array}$$

$$\| \quad = \sum_{\ell=1}^{L_1} \left(\sum_{r=1}^{m} \sum_{i=1}^{n} {}^{\exists} a^i_{\ell r} \, \zeta_{ri} \right)^2 \quad (L_1 : \textit{positive} \text{ index of the form})$$

$$A^{ij} = \left(\sum_{\ell=1}^{L_1} a^i_{\ell r} a^j_{\ell s} \right)_{\substack{r=1 \downarrow m \\ s=1 \rightarrow m}}$$

Hence,

$$E[\boldsymbol{u}] = \sum_{\ell=1}^{L_1} \left\| \sum_{r=1}^{m} \sum_{i=1}^{n} a^i_{\ell r} \, \partial_i u_r \right\|^2 = \sum_{\ell=1}^{L_1} \left\| \sum_{r=1}^{m} p_{\ell r}(\partial) \, u_r \right\|^2 = \| P(\partial) \boldsymbol{u} \|^2,$$

$$p_{\ell r}(\xi) := \sum_{i=1}^{n} a^i_{\ell r} \, \xi_i, \quad P(\xi) := \big(p_{\ell r}(\xi) \big)_{\substack{\ell=1 \downarrow L_1 \\ r=1 \rightarrow m}}$$

determined uniquely
up to its *row space*

and $\boxed{④ \Leftrightarrow L_1 = mn}$

PROBLEM (Generalized Korn's Inequality)

We consider inequality ③ for

$$E[\boldsymbol{u}] \equiv \|P(\partial)\boldsymbol{u}\|^2 = \sum_{\ell=1}^{L} \left\| \sum_{r=1}^{m} p_{\ell r}(\partial)\, u_r \right\|^2 \qquad (\textit{Remark: } L_1 \leqslant L)$$

$$P(\xi) = \begin{pmatrix} p_{11}(\xi) & p_{12}(\xi) & \cdots & p_{1m}(\xi) \\ p_{21}(\xi) & p_{22}(\xi) & \cdots & p_{2m}(\xi) \\ \vdots & \vdots & & \vdots \\ p_{L1}(\xi) & p_{L2}(\xi) & \cdots & p_{Lm}(\xi) \end{pmatrix}, \quad p_{\ell r}(\xi) = \sum_{i=1}^{n} a^i_{\ell r}\, \xi_i .$$

Generalized Korn's inequality

(K) $\|P(\partial)\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2 \geqslant {}^{\exists}C\, \|\nabla \boldsymbol{u}\|^2 \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) = H^1(\Omega; \mathbb{C}^m)$

Our Problems

easy to check

1. Find a necessary and sufficient condition of $P(\xi)$ for **(K)** to hold.
2. Given m and n, find the *minimum* of L over all $P(\xi)$ for which **(K)** holds.

Main Results

Theorem

Each of the following is necessary and sufficient for (K) to hold.

1. $\operatorname{rank} P(\xi) = m \quad \forall \xi \in \mathbb{C}^n \setminus \{0\}$.

2. $\exists k \in \mathbb{N} : \ \forall |\alpha| = k \ (\alpha \in \mathbb{N}_0{}^n) \ \exists Q_\alpha(\xi)$ (homog. of deg. $k-1$) s.t.

$$\underbrace{Q_\alpha(\xi)}_{m\times L} \underbrace{P(\xi)}_{L\times m} = \xi^\alpha I_m.$$

3. $\mathcal{N}_P := \{\boldsymbol{u} \in \boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}^n) \mid P(\partial)\boldsymbol{u} = \boldsymbol{0}\}$ is finite dimensional.

4. All $m \times m$ minors of $P(\xi)$ spans the space of homogeneous polynomials of ξ of degree m.

Remark

- 1, 2 (3) are already known to be necessary and sufficient (Nečas (1967))
- 4 is a new condition, which has some advantages over the others.

Outline of Proof

1 $\operatorname{rank} P(\xi) = m \ \ \forall \xi \neq 0$

$\Downarrow$ Hilbert's Nullstellensatz $\cdots$ less information about k and $Q_\alpha(\xi)$

2 $\exists k \in \mathbb{N} : \ (\ \forall |\alpha| = k \ \exists Q_\alpha(\xi) : \ Q_\alpha(\xi) P(\xi) = \xi^\alpha I\ )$

$\Downarrow$ Lions's lemma (or Sobolev's representation formula & sing. integral)

(K) $\|P(\partial)\boldsymbol{u}\|^2 + \|\boldsymbol{u}\|^2 \geqslant {}^\exists C \|\nabla \boldsymbol{u}\|^2 \quad \forall \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$

- **Lions's Lemma**

  "$u \in H^{-1}(\Omega), \nabla u \in \boldsymbol{H}^{-1}(\Omega)$" implies $u \in L^2(\Omega)$ and

$$\|u\|_{L^2} \leqslant C(\|u\|_{H^{-1}} + \|\nabla u\|_{H^{-1}}).$$

  Accordingly, for any $p \in \mathbb{N}_0$ and $q \in \mathbb{N}$,

$$\|u\|_{H^{-p}} \leqslant C \sum_{|\alpha| \leqslant q} \|\partial^\alpha u\|_{H^{-p-q}}.$$

$\neg\boxed{1}$ $\exists \xi_0 : \operatorname{rank} P(\xi_0) < m$

$\Downarrow$ *constructive*

$\neg\boxed{3}$ $\dim \mathcal{N}_P = \infty$

$\Downarrow$ *fundamental*

$\neg$(K)

- If $\boldsymbol{a} \neq \boldsymbol{0}$ solves $P(\xi_0)\boldsymbol{a} = \boldsymbol{0}$,

$$\boldsymbol{u}(x) = \mathrm{Re}(e^{\lambda \xi_0 \cdot x} \boldsymbol{a})$$

belongs to $\mathcal{N}_P$ for all $\lambda \in \mathbb{R}$.

- If (K) holds, then

$$\|\boldsymbol{u}\|^2 \geqslant C \|\nabla \boldsymbol{u}\|^2 \quad \forall \boldsymbol{u} \in \mathcal{N}_P.$$

$\boxed{1}$

easy $\Uparrow\Downarrow$ **Eagon-Northcott complex**

$\boxed{4}$

Advantage of New Condition

- In general situation, $\boxed{4}$ is the easiest to check. (algorithmically checked)
- If $\boxed{4}$ holds, we can explicitly construct $Q_\alpha(\xi)$ in $\boxed{2}$ for $k = m$.
- $\boxed{4}$ leads us to the following interesting corollary.

COROLLARY

Let any one of the conditions in THEOREM be satisfied.

ⓐ The number L of rows of $P(\xi)$ must be $\geqslant m + n - 1$.

ⓑ We can take k appearing in $\boxed{2}$ above so that $k \leqslant m$.
Hence, we have $\dim \mathcal{N}_P \leqslant \binom{m+n-1}{n}$ a priori.

Example 1 (optimal case)

$$E[\boldsymbol{u}] = \sum_{\ell=2}^{m+n} \Bigg\| \sum_{\substack{i+r=\ell \\ 1\leqslant i\leqslant n \\ 1\leqslant r\leqslant m}} \partial_i u_r \Bigg\|^2 \overset{\text{if } m=n}{=} \sum_{\ell=2}^{2n} \Bigg\| \sum_{\substack{i+j=\ell \\ 1\leqslant i,j\leqslant n}} \varepsilon_{ij}(\boldsymbol{u}) \Bigg\|^2 \quad \longleftrightarrow \quad P(\xi) = \begin{pmatrix} \xi_1 & & & \\ \xi_2 & \xi_1 & & \\ \vdots & \xi_2 & \ddots & \\ \xi_{n-1} & \ddots & \ddots & \xi_1 \\ \xi_n & \xi_{n-1} & \ddots & \xi_2 \\ & \xi_n & \ddots & \vdots \\ & & \ddots & \xi_{n-1} \\ & & & \xi_n \end{pmatrix}$$

Then,

$$\operatorname{rank} P(\xi) = m \quad \forall \xi \in \mathbb{C}^n \setminus \{0\};$$

$$\dim \mathcal{N}_P = \binom{m+n-1}{n}, \quad L = m+n-1.$$

If $n = m \geqslant 3$, this gives an inequality strictly stronger than Korn's ineq.

Example 2 (isotropic elasticity)

$$E[\boldsymbol{u}] = \lambda \left\|\operatorname{div} \boldsymbol{u}\right\|^2 + 2\mu \left\|\varepsilon(\boldsymbol{u})\right\|^2 \left(\geqslant \min\{n\lambda + 2\mu,\ 2\mu\} \left\|\varepsilon(\boldsymbol{u})\right\|^2\right)$$

$$= \frac{n\lambda + 2\mu}{n} \left\|\sum_{i=1}^{n} \varepsilon_{ii}(\boldsymbol{u})\right\|^2 + \sum_{j=1}^{n-1} \frac{2\mu}{j(j+1)} \left\|\sum_{i=1}^{j} \varepsilon_{ii}(\boldsymbol{u}) - \varepsilon_{j+1,j+1}(\boldsymbol{u})\right\|^2 + 4\mu \sum_{i<j} \left\|\varepsilon_{ij}(\boldsymbol{u})\right\|^2$$

$$= \frac{n\lambda + 2\mu}{n} \left\|\sum_{i=1}^{n} \varepsilon_{ii}(\boldsymbol{u})\right\|^2 + \frac{2\mu}{n} \sum_{i<j} \left\|\varepsilon_{ii}(\boldsymbol{u}) - \varepsilon_{jj}(\boldsymbol{u})\right\|^2 + 4\mu \sum_{i<j} \left\|\varepsilon_{ij}(\boldsymbol{u})\right\|^2$$

$$= \frac{n\lambda + 2\mu}{n} \left\|\operatorname{tr} \varepsilon(\boldsymbol{u})\right\|^2 + 2\mu \left\|\varepsilon(\boldsymbol{u}) - \frac{1}{n}(\operatorname{tr} \varepsilon(\boldsymbol{u}))I\right\|^2.$$

③ *Korn-type ineq.*

$$\Leftrightarrow \quad \mu > 0 \ \& \ \begin{cases} n\lambda + 2\mu > 0 \ \ (n \geqslant 2) & L_1 = \frac{1}{2}n(n+1) \\ n\lambda + 2\mu = 0 \ \ (n \geqslant 3) & L_1 = \frac{1}{2}n(n+1) - 1 \end{cases}$$

Applications

$n\lambda + 2\mu > 0$: isotropic elasticity, Navier-Stokes eq., Killing eq. $(\lambda = 0)$

$n\lambda + 2\mu = 0 \ (n \geqslant 3)$: conformal Killing eq. (Dain 2006)

- $E[\boldsymbol{u}] \cong \|\varepsilon(\boldsymbol{u})\|^2,$

  $L_1 = \frac{1}{2}n(n+1)$ (optimal if $n = 2$)

- $\partial_j \partial_k u_i = \partial_j \varepsilon_{ki}(\boldsymbol{u}) + \partial_k \varepsilon_{ij}(\boldsymbol{u}) + \partial_i \varepsilon_{jk}(\boldsymbol{u}) \quad (\forall i, j, k)$

- $$\mathcal{N} = \{\boldsymbol{u}(x) \mid \overbrace{\varepsilon(\boldsymbol{u}) = O}^{\boldsymbol{u}\text{ : Kiling vector field}}\}$$

  $$= \{S\boldsymbol{x} + \boldsymbol{c} \mid S \in \mathfrak{so}(n),\ \boldsymbol{c} \in \mathbb{R}^n\}, \quad \dim \mathcal{N} = \tfrac{n(n+1)}{2}.$$

- $$E[\boldsymbol{u}] = 2\mu \left\| \varepsilon(\boldsymbol{u}) - \tfrac{1}{n}(\operatorname{tr}\varepsilon(\boldsymbol{u}))I \right\|^2 \cong \sum_{1 \leqslant i < j \leqslant n} \left(\|\zeta_{ij}(\boldsymbol{u})\|^2 + \|\varepsilon_{ij}(\boldsymbol{u})\|^2 \right) \quad (\zeta_{ij}(\boldsymbol{u}) := \varepsilon_{ii}(\boldsymbol{u}) - \varepsilon_{jj}(\boldsymbol{u})),$$

  $L_1 = \frac{1}{2}n(n+1) - 1$ (optimal if $n = 3$)

- $$\begin{cases} \partial_j\partial_k u^i = \partial_j\varepsilon_{ki}(\boldsymbol{u}) - \partial_i\varepsilon_{jk}(\boldsymbol{u}) + \partial_k\varepsilon_{ij}(\boldsymbol{u}), \\ \partial_j^3 u^i = 2\partial_j^2\varepsilon_{ij}(\boldsymbol{u}) - \partial_i\partial_j\zeta_{jk}(\boldsymbol{u}) - \partial_k(\partial_i\partial_j u^k), \\ \partial_i^2\partial_j u^i = 2\partial_i^2\varepsilon_{ij}(\boldsymbol{u}) - \partial_i^3 u^j, \\ \partial_i^4 u^i = \partial_i^3\zeta_{ij}(\boldsymbol{u}) + \partial_j(\partial_i^3 u^j) \qquad (\forall i, j, k : \text{distinct}) \end{cases}$$

- $$\mathcal{N} = \{\boldsymbol{u}(x) \mid \overbrace{\varepsilon(\boldsymbol{u}) - \tfrac{1}{n}(\operatorname{tr}\varepsilon(\boldsymbol{u}))I = O}^{\boldsymbol{u}\ :\ \text{conformal Killing vector field}}\ \} = \left\{ 2(\boldsymbol{a}, \boldsymbol{x})\boldsymbol{x} - |\boldsymbol{x}|^2\boldsymbol{a} + (bI + S)\boldsymbol{x} + \boldsymbol{c} \;\middle|\; \boldsymbol{a}, \boldsymbol{c} \in \mathbb{R}^n, b \in \mathbb{R}, S \in \mathfrak{so}(n) \right\},$$

  $\dim \mathcal{N} = \frac{(n+1)(n+2)}{2}$.