Global solutions for the Navier-Stokes equations in the rotational framework

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(\mathrm{NSC})\begin{cases}\partial_t u - \Delta u + \Omega e_3 \times u + (u\cdot\nabla)u + \nabla p = 0 & t>0, x\in\mathbb{R}^3,\\ \mathbf{div}\ \ u = 0 & t>0, x\in\mathbb{R}^3,\\ u(0) = u_0 & x\in\mathbb{R}^3,\end{cases}
$$

where $\Omega \in \mathbb{R}$ is the speed of the rotation, $e_3 := (0,0,1)$.

Aim. Global solutions for $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with $s \geq 1/2$.

Known Results (Global solutions).

- $\Omega = 0$. **Fujita-Kato (1964), T. Kato (1984), Kozono-Yamazaki (1994), Koch-Tataru (2001).**
  $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ **: small.**

- $\Omega \in \mathbb{R}$

Babin-Mahalov-Nikolaenko (2001)

Chemin-Desjardins-Gallagher-Grenier (2006)

$\forall u_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)^3$, $\exists \omega_0 > 0$ **s.t.** $|\Omega| > \omega_0$.

Giga-Inui-Mahalov-Saal (2008),

$\exists \delta > 0$ **: independent of** $\Omega \in \mathbb{R}$ **s.t.** $\|u_0\|_{FM_0^{-1}} \leq \delta$.

Hieber-Shibata (2010), $\|u_0\|_{H^{\frac{1}{2}}} \leq \delta$.

Konieczny-Yoneda (2011), $1 < p \leq \infty$, $\|u_0\|_{F\dot{B}_{p,\infty}^{2-\frac{3}{p}}} \leq \delta$.

__Problem__. **Characterize the sufficient speed of rotation ω_0 for global existence.**

Theorem 1. Let s, p, θ satisy

$$\frac{1}{2} < s < \frac{3}{4}, \quad \frac{1}{3} + \frac{s}{9} < \frac{1}{p} < \frac{2}{3} - \frac{s}{3},$$
$$\frac{s}{2} - \frac{1}{2p} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}, \quad \frac{3}{4} - \frac{3}{2p} \leq \frac{1}{\theta} < 1 - \frac{2}{p}.$$

Then, $^{\exists}C > 0$ such that if $\|u_0\|_{\dot{H}^s} \leq C|\Omega|^{\frac{s}{2}-\frac{1}{4}}$, there exists a unique solution $u \in C([0,\infty), \dot{H}^s(\mathbb{R}^3))^3 \cap L^\theta(0,\infty; \dot{H}^s_p(\mathbb{R}^3))^3$ to (NSC).

Remark.

(1) $|\Omega| \geq \omega_0 = \left(C^{-1}\|u_0\|_{\dot{H}^s}\right)^{\frac{2}{s-\frac{1}{2}}}$.

(2) $\frac{2}{\theta} + \frac{3}{p} < 1 + s$ **since** $s > \frac{1}{2}$**.**

Solutions in Theorem 1 is more regular than functions in scaling invariant spaces.

(3) Let $u_\lambda := \lambda u(\lambda^2 t, \lambda x)$**,** $\Omega_\lambda := \lambda^2 \Omega$**.**

$$\|u_\lambda(0)\|_{\dot{H}^s} = \lambda^{s-\frac{1}{2}} \|u_0\|_{\dot{H}^s}, \qquad |\Omega_\lambda|^{\frac{s}{2}-\frac{1}{4}} = \lambda^{s-\frac{1}{2}} |\Omega|.$$

Theorem 2. Let $s = \frac{1}{2}$. For any precompact set $K \subset \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$, $^\exists\omega_0 = \omega_0(K) > 0$ s.t. if $u_0 \in K$ and $|\Omega| > \omega_0$, there exists a unique solution $u \in C([0,\infty), \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))^3 \cap L^4(0,\infty; \dot{H}^{\frac{1}{2}}_{3}(\mathbb{R}^3))^3$ to (NSC).

Remark. $s > 1/2$, $|\Omega| \geq c\|u_0\|_{\dot{H}^s}^{\frac{2}{s-\frac{1}{2}}}$.

$s = 1/2$, $|\Omega| \geq \omega(K)$.

Proof.

$$T_\Omega(t)f = \mathcal{F}^{-1}\left[\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)e^{-t|\xi|^2}\widehat{f} + \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)e^{-t|\xi|^2}R(\xi)\widehat{f}\right],$$

where $R(\xi) := \dfrac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}$.

$$T_\Omega(t)f = \mathcal{F}^{-1}\left[\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)e^{-t|\xi|^2}\widehat{f} + \sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)e^{-t|\xi|^2}R(\xi)\widehat{f}\right].$$

Coriolis force : $\cos\left(\frac{\xi_3}{|\xi|}\Omega t\right)$, $\sin\left(\frac{\xi_3}{|\xi|}\Omega t\right)$

Heat equation : $e^{-t|\xi|^2}$

Fourier multiplier : $e^{i\frac{\xi_3}{|\xi|}\Omega t}$

Lemma 1. **Let** $2 \le p \le \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

$$\left\| \mathcal{F}^{-1}\left[e^{i\frac{\xi_3}{|\xi|}\Omega t} \hat{f} \right] \right\|_{\dot{B}^0_{p,2}} \le C \left\{ \frac{\log(e + |\Omega| t)}{1 + |\Omega| t} \right\}^{\frac{1}{2}(1-\frac{2}{p})} \|f\|_{\dot{B}^{3(1-\frac{2}{p})}_{p',2}}$$

Proof. $p = 2$, **isometric estimate.**

$$p = \infty, \left\| \phi_j * \mathcal{F}^{-1}\left[e^{i\frac{\xi_3}{|\xi|}\Omega t} \hat{f} \right] \right\|_{L^\infty} \quad (\phi_j : \text{Littlewood-Paley's decomposition})$$

$$\le \underbrace{\left\| \mathcal{F}^{-1}\left[e^{i\frac{\xi_3}{|\xi|}\Omega t} \chi_{\{2^{j-1} \le |\xi| \le 2^{j+1}\}} \right] \right\|_{L^\infty}} \|\phi_j * f\|_{L^1}.$$

$$\le 2^{3j} \left\{ \frac{\log(e + |\Omega| t)}{1 + |\Omega| t} \right\}^{\frac{1}{2}} (\because \text{ integration by parts.})$$

Interpolation of the two cases. □

Proposition 2. **Let** $2 \le p < \infty$, $1 < q \le \dfrac{p}{p-1}$.

$$\|T_\Omega(t)f\|_{L^p} \le C \left\{ \frac{\log(e + |\Omega| t)}{1 + |\Omega| t} \right\}^{\frac{1}{2}(1-\frac{2}{p})} t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^q}.$$

Proposition 3. **Let** $2 < \theta < \infty$, $2 < p < 6$ **satisfy**

$$\frac{3}{4} - \frac{3}{2p} \le \frac{1}{\theta} < 1 - \frac{2}{p}.$$

Let $\alpha := 1/\theta - (3/4 - 3/2p)$.

$$\|T_\Omega(\cdot)f\|_{L^\theta(0,\infty;L^p(\mathbb{R}^3))} \le C|\Omega|^{-\alpha}\|f\|_{L^2}.$$

Remark. $\alpha \ge 0$. **For** $\alpha = 0$,

$$\lim_{|\Omega| \to \infty} \|T_\Omega(t)f\|_{L^\theta(0,\infty;L^p(\mathbb{R}^3))} = 0 \text{ for } f \in L^2(\mathbb{R}^3)^3.$$

Proof of Theorem 1. For s, p, θ in Theorem 1, let

$$\Psi(u)(t) := T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)d\tau,$$

$$X := \Big\{ u \in L^\theta(0,\infty;\dot{H}^s_p(\mathbb{R}^3)) \,\Big|\, \|u\|_{L^\theta \dot{H}^s_p} \le C|\Omega|^{-\alpha}\|u_0\|_{\dot{H}^s} \Big\},$$

where $\alpha := 1/\theta - (3/4 - 3/2p)$.

Claim. $\|\Psi(u)\|_{L^\theta \dot{H}^s_p} \le C|\Omega|^{-\alpha}\|u_0\|_{\dot{H}^s} + C|\Omega|^{-\alpha}|\Omega|^{-(\frac{s}{2}-\frac{1}{4})}\|u_0\|^2_{\dot{H}^s}$

for any $u \in X$.

Remark. $|\Omega|^{-(\frac{s}{2}-\frac{1}{4})}\|u_0\|_{\dot{H}^s} \le \delta$ **iff** $\|u_0\|_{\dot{H}^s} \le \delta|\Omega|^{-\frac{s}{2}+\frac{1}{4}}$

<u>Proof of Claim</u>. Let $1/q := 2/p - s/3$, $\gamma(\Omega t) := \frac{\log(e+|\Omega|t)}{1+|\Omega|t}$.

$$\left\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)d\tau \right\|_{L^\theta(0,\infty;\dot H^s_p)}$$

$$\le C\left\| \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \gamma\big(\Omega(t-\tau)\big)^{\frac{1}{2}(1-\frac{2}{p})} \|u\otimes u\|_{\dot H^s_q} d\tau \right\|_{L^\theta(0,\infty)}$$

$$\le C\left\| t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \gamma\big(\Omega t\big)^{\frac{1}{2}(1-\frac{2}{p})} \right\|_{L^r(0,\infty)} \|u\otimes u\|_{L^{\frac{\theta}{2}}\dot H^s_q}$$

$$(\because \quad 1/\theta = 1/r + 2/\theta - 1 \ )$$

$$\le C|\Omega|^{-\frac{1}{r}+\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \|u\otimes u\|_{L^{\frac{\theta}{2}}(0,\infty;\dot H^s_q)} \qquad (\because \quad t\mapsto |\Omega|^{-1}\tilde t \ )$$

$$\le C|\Omega|^{-\frac{1}{r}+\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \|u\|^2_{L^\theta(0,\infty;\dot H^s_p)}$$

$$\le C|\Omega|^{-\frac{1}{r}+\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} |\Omega|^{-2\alpha} \|u_0\|^2_{\dot H^s} \qquad (\because \quad u\in X \ )$$

$$= C|\Omega|^{-\alpha}|\Omega|^{-(\frac{s}{2}-\frac{1}{4})}\|u_0\|^2_{\dot H^s}. \qquad \square$$

Proposition 4. Let $2 < \theta < \infty$, $2 < p < 6$, $\frac{3}{4} - \frac{3}{2p} = \frac{1}{\theta}$.
For any $f \in L^2(\mathbb{R}^3)^3$, there exists $C > 0$ such that

$$\limsup_{|\Omega|\to\infty} \sup_{\|g-f\|_{L^2}\le\varepsilon} \|T_\Omega(t)g\|_{L^\theta(0,\infty;L^p(\mathbb{R}^3))} \le C\varepsilon$$

for any $\varepsilon > 0$.

Proof. If $\|g - f\|_{L^2} \le \varepsilon$, we have

$$\|T_\Omega(t)g\|_{L^\theta(0,\infty 0;L^p(\mathbb{R}^3))}$$
$$\le \underbrace{\|T_\Omega(t)(g-f)\|_{L^\theta(0,\infty 0;L^p(\mathbb{R}^3))}}_{(*)} + \underbrace{\|T_\Omega(t)f\|_{L^\theta(0,\infty 0;L^p(\mathbb{R}^3))}}_{(**)},$$

$$(*) \le C\|g-f\|_{L^2} \le C\varepsilon,$$

$(**) \to 0$ as $|\Omega| \to \infty$. $\square$