Jeffery-Hamel's flow in the plane

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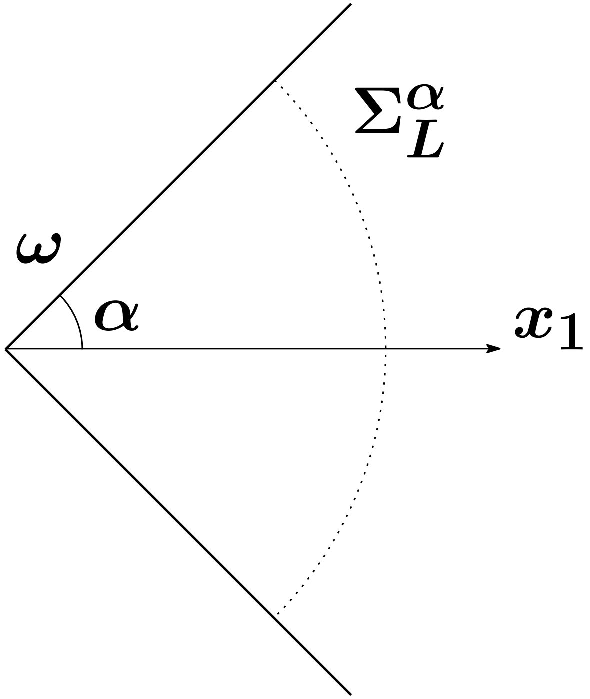
1. Jeffery-Hamel's flow

$$\alpha \in (0, \pi).$$
$$\omega = \{(r, \theta); r > 0, -\alpha < \theta < \alpha\} \subset \mathbb{R}^2.$$
$$\Sigma_L^\alpha = \{(r, \theta); r = L, -\alpha < \theta < \alpha\} \quad (\forall L > 0).$$



$$\alpha = \frac{1}{2}\pi \quad \Rightarrow \quad \omega = \text{the half space.}$$

Navier-Stokes equations

$$-\nu\Delta u + (u\cdot\nabla)u + \nabla p = 0 \quad \text{in} \quad \omega, \tag{1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \omega, \tag{2}$$

$$u = 0 \quad \text{on} \quad \partial\omega, \tag{3}$$

$$u \to 0 \quad \text{as} \quad r \to \infty \tag{4}$$

with the flux condition

$$\int_{\Sigma_L^\alpha} u \cdot e_r d\sigma = \gamma. \tag{5}$$

u : velocity

p : pressure

ν : kinematic viscosity

γ : flux constant

$e_r = (\cos\theta, \sin\theta)$

Jeffery-Hamel's flow is found in the form

$$\boldsymbol{u} = \frac{\gamma g(\theta)}{r} \boldsymbol{e_r}, \;\; \boldsymbol{e_r} = (\cos\theta, \sin\theta). \tag{6}$$

$g = g(\theta)$ **: unknown scalar function**

ODE.

$$g'' + 4g + \frac{\gamma}{\nu} g^2 = \frac{A}{\nu\gamma} \quad \text{on} \quad (-\alpha, \alpha), \tag{7}$$

$$g(\pm\alpha) = 0, \tag{8}$$

$$\int_{-\alpha}^{\alpha} g(\theta) d\theta = 1, \tag{9}$$

$$g(\theta) = g(-\theta). \tag{10}$$

A **: arbitrary constant**

2. Known result (L. Rosenhead, 1940)

$\boldsymbol{R} = \dfrac{|\gamma|}{\boldsymbol{\nu}}$: Reynolds number

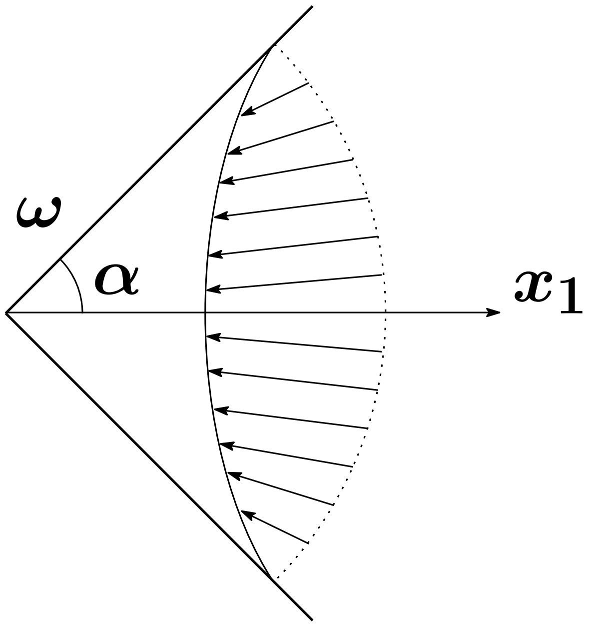
$\gamma < 0,\ \boldsymbol{\alpha} \in (0, \frac{\pi}{2})$.

For $\forall \boldsymbol{R} > 0$,

$\Downarrow$

$\gamma \boldsymbol{g}(\boldsymbol{\theta}) < 0$

$\gamma \boldsymbol{g}(\boldsymbol{\theta})$ is symmetric.



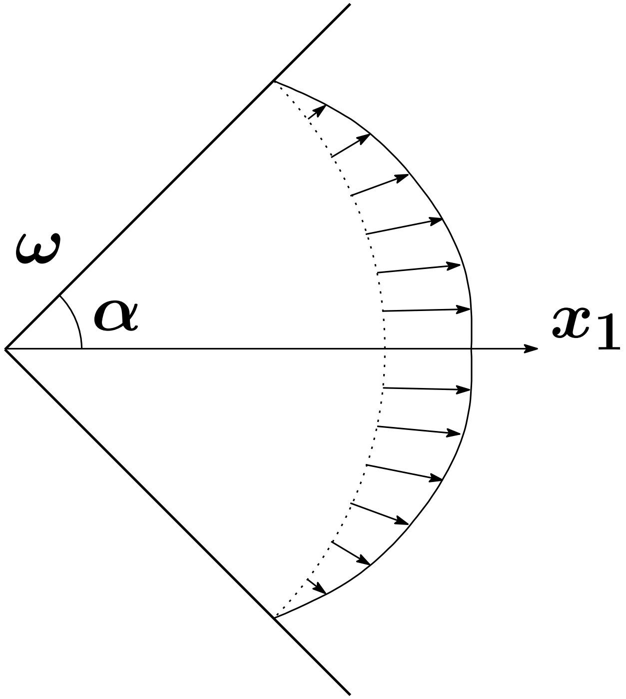
$\gamma > 0,\ \alpha \in (0, \frac{\pi}{2})$.

$\exists R_{\max}$ such that $R < R_{\max}$

$\Downarrow$

$\gamma g(\theta) > 0$

$\gamma g(\theta)$ is symmetric.



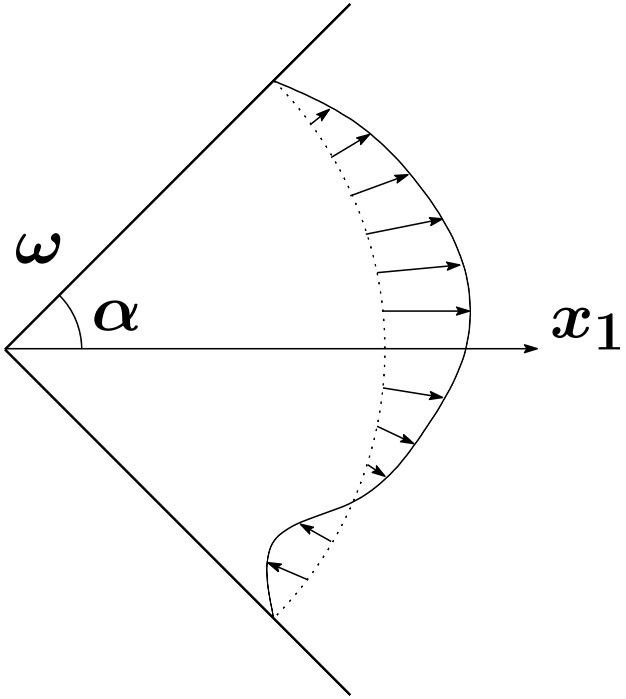
$\gamma > 0, \ \alpha \in (0, \frac{\pi}{2})$.

$R > R_{\max}$: unsteady flow

$\Downarrow$

$\gamma g(\theta)$ is not always positive.

$\gamma g(\theta)$ is not symmetric.



$\gamma > 0,\ \alpha \in (0, \frac{\pi}{2})$.

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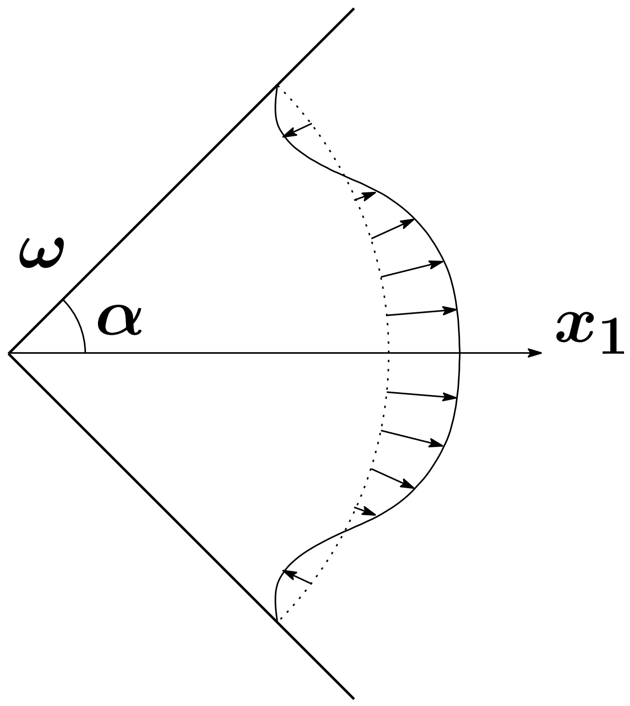
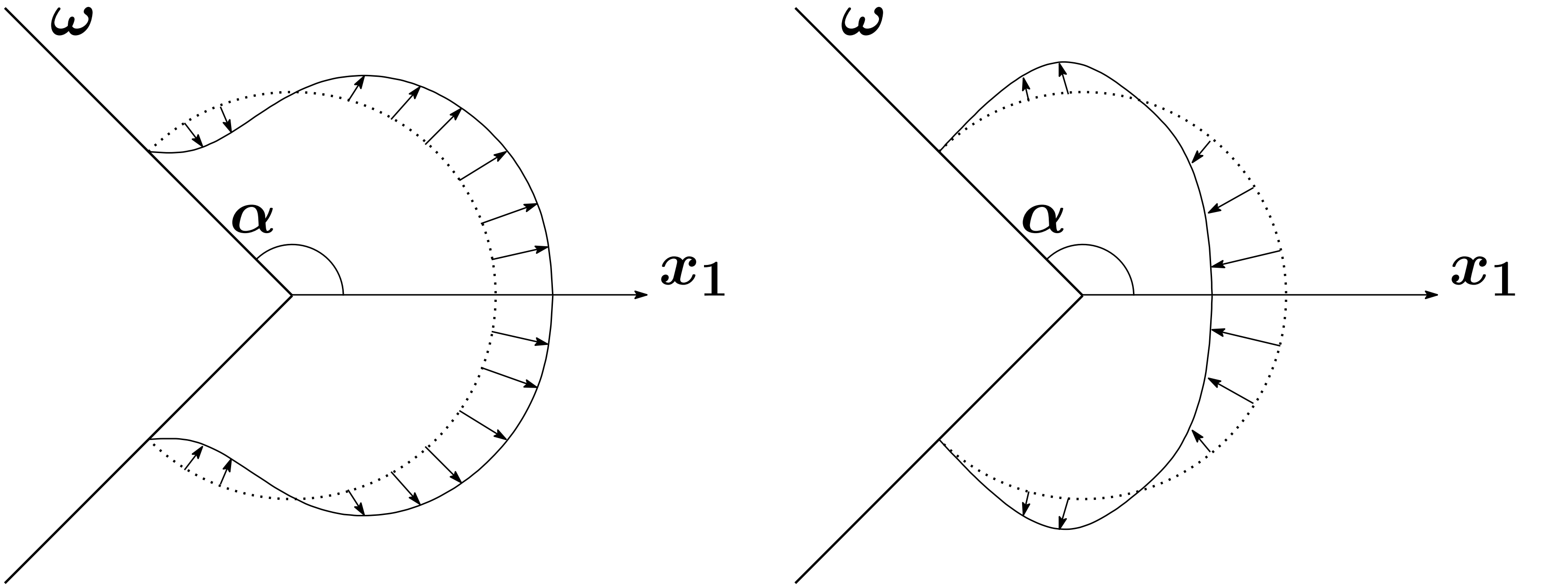


$\alpha \in [\frac{\pi}{2}, \pi)$.

For $\forall R > 0$, $\theta \in (-\alpha, \alpha)$,

$\gamma > 0$ $\gamma < 0$



Proposition (G. P. Galdi, M. Padula, V. A. Solonnikov(1996))

Let $\alpha = \frac{1}{2}\pi$ **and** $|\gamma| < \frac{\nu}{36}$.

$$\Downarrow$$

$\exists$ **solution of (N-S) in the half space of the form**

$$u = \frac{\gamma g(\theta)}{r} e_r,\ p = \frac{2\nu\gamma g(\theta) + C_1}{r^2} + C_2 \quad (C_1, C_2 \in \mathbb{R})$$

with

$$g \in C_0^S[-\frac{1}{2}\pi, \frac{1}{2}\pi] \cap C^\infty(-\frac{1}{2}\pi, \frac{1}{2}\pi),\ \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} g(\theta) d\theta = 1,$$

$$|g(\theta)| \leq 6,$$

$$C_0^S[-\frac{1}{2}\pi, \frac{1}{2}\pi] = \{g \in C[-\frac{1}{2}\pi, \frac{1}{2}\pi] | g(\theta) = g(-\theta), g(\pm\frac{1}{2}\pi) = 0\}.$$

Moreover the function g **is unique in the above class.**

3. Today's result

Theorem

Let $\alpha \in (0,\pi)\backslash\{\frac{1}{4}\pi, \alpha_0(2\alpha_0 = \tan 2\alpha_0), \frac{3}{4}\pi\}$ and $|\gamma| < \gamma_0$.

$\Downarrow$

$\exists$ solution of (N-S) in ω of the form

$$u = \frac{\gamma g(\theta)}{r} e_r, \; p = \frac{2\nu\gamma g(\theta) + C_1}{r^2} + C_2 \quad (C_1, C_2 \in \mathbb{R})$$

with

$$g \in C_0^S[-\alpha,\alpha] \cap C^\infty(-\alpha,\alpha), \quad \int_{-\alpha}^{\alpha} g(\theta)d\theta = 1,$$

$$|g(\theta)| \le \frac{2(1-\cos 2\alpha)}{|\tan 2\alpha - 2\alpha||\cos 2\alpha|} (:= M).$$

Moreover the function g is unique in the above class.

$$\gamma_0 = \frac{\nu\pi}{24} \quad (\alpha = \frac{1}{2}\pi).$$

$$\gamma_0 = \begin{cases} \gamma_1 \ (\alpha \in (0, \frac{1}{4}\pi)), \\ \gamma_2 \ (\alpha \in (\frac{1}{4}\pi, \frac{3}{4}\pi)), \\ \gamma_3 \ (\alpha \in (\frac{3}{4}\pi, \pi)). \end{cases}$$

$$\gamma_1 := \frac{\nu(\tan 2\alpha - 2\alpha)\cos 2\alpha}{(1 - \cos 2\alpha + \sin 2\alpha \tan 2\alpha)\left(\frac{2\alpha(1-\cos 2\alpha)}{(\tan 2\alpha - 2\alpha)\cos 2\alpha} + 1\right)(1 - \cos 2\alpha)},$$

$$\gamma_2 := \frac{\nu|2\alpha - \tan 2\alpha||\cos 2\alpha|}{\left(1 - \cos 2\alpha + \frac{2 - \sin 2\alpha}{|\cos 2\alpha|}\right)\left(\frac{2\alpha(1-\cos 2\alpha)}{|2\alpha - \tan 2\alpha||\cos 2\alpha|} + 1\right)(1 - \cos 2\alpha)},$$

$$\gamma_3 := \frac{\nu(2\alpha - \tan 2\alpha)\cos 2\alpha}{\left(3 + \cos 2\alpha + \frac{4 + \sin 2\alpha}{\cos 2\alpha}\right)\left(\frac{2\alpha(1-\cos 2\alpha)}{(2\alpha - \tan 2\alpha)\cos 2\alpha} + 1\right)(1 - \cos 2\alpha)}.$$

5. The linear problem

For any $\alpha \in (0,\pi)\backslash\{\frac{1}{4}\pi, \frac{3}{4}\pi\}$,

$$h'' + 4h = b(\theta) \quad \text{on} \quad (-\alpha,\alpha) \tag{11}$$

with

$$h(\theta) = h(-\theta), \tag{12}$$

$$h(\pm\alpha) = 0, \tag{13}$$

where $b \in C^S[-\alpha,\alpha] = \{g \in C[-\frac{1}{2}\pi, \frac{1}{2}\pi] | g(\theta) = g(-\theta)\}$.

The function

$$h(\theta) = \frac{1}{2}\int_\theta^\alpha \sin 2(s-\theta) b(s) ds - \frac{1}{2\cos 2\alpha}\int_0^\alpha \cos 2s b(s) ds \cdot \sin 2(\alpha-\theta)$$

is a unique solution of (11) with (12) and (13).

$\alpha = \frac{1}{4}\pi,$

$$h'' + 4h = b(\theta) \quad \text{on} \quad (0, \tfrac{1}{4}\pi) \tag{14}$$

with

$$h(\tfrac{1}{4}\pi) = 0, \tag{15}$$

$$h'(0) = 0, \tag{16}$$

where $b \in C^S[-\alpha, \alpha]$.

The function

$$h(\theta) = \frac{1}{2}\int_\theta^\alpha \sin 2(s-\theta)b(s)ds + C_1 \sin 2\theta + C_2 \cos 2\theta$$

is a general solution of (14).

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ * \end{pmatrix}. \quad \text{rank}(A) = 1, \quad \text{rank}(A, f) = 2.$$

5. The formulation

We define an operator $\mathcal{L}$ such as

$$\mathcal{L}[b] = h \quad (b \in C^S[-\alpha, \alpha]).$$

$$g'' + 4g + \frac{\gamma}{\nu} g^2 = \frac{A}{\nu\gamma}$$

$$\mathcal{L}[g'' + 4g + \frac{\gamma}{\nu} g^2 = \frac{A}{\nu\gamma}]$$

$$g = \frac{A}{\nu\gamma}\mathcal{L}[1] - \frac{\gamma}{\nu}\mathcal{L}[g^2]$$

$$\int_{-\alpha}^{\alpha} g d\theta = \frac{A}{\nu\gamma} \int_{-\alpha}^{\alpha} \mathcal{L}[1] d\theta - \int_{-\alpha}^{\alpha} \frac{\gamma}{\nu} \mathcal{L}[g^2] d\theta$$

$$1 = \frac{A}{\nu\gamma} \left(\frac{1}{2}\alpha - \frac{1}{4}\tan 2\alpha \right) - \int_{-\alpha}^{\alpha} \frac{\gamma}{\nu} \mathcal{L}[g^2] d\theta$$

$$A = \frac{\nu\gamma}{\frac{1}{2}\alpha - \frac{1}{4}\tan 2\alpha} \left(1 + \frac{\gamma}{\nu} \int_{-\alpha}^{\alpha} \mathcal{L}[g^2](\theta) d\theta \right)$$

$$g = \frac{1}{\frac{1}{2}\alpha - \frac{1}{4}\tan 2\alpha} \left(1 + \frac{\gamma}{\nu} \int_{-\alpha}^{\alpha} \mathcal{L}[g^2](\theta) d\theta \right) \mathcal{L}[1] - \frac{\gamma}{\nu} \mathcal{L}[g^2] \qquad (g \in C_0^S[-\alpha, \alpha]).$$

For any $\alpha \in (0,\pi)\backslash\{\frac{1}{4}\pi, \alpha_0(2\alpha_0 = \tan 2\alpha_0), \frac{3}{4}\pi\}$, we set

$$\mathcal{J}[g] := \frac{1}{\frac{1}{2}\alpha - \frac{1}{4}\tan 2\alpha}\left(1 + \frac{\gamma}{\nu}\int_{-\alpha}^{\alpha} \mathcal{L}[g^2](\theta)d\theta\right)\mathcal{L}[1] - \frac{\gamma}{\nu}\mathcal{L}[g^2]$$

$$(g \in C_0^S[-\alpha,\alpha]).$$

Lemma

Let $|\gamma| < \gamma_0$.

$\Downarrow$

Then $\mathcal{J}$ is a contraction from $B(0,M)$ to $B(0,M)$,

where $B(0,M) = \{\varphi \in C_0^S[-\alpha,\alpha]; \|\varphi\|_C \leq M\}$,

$$M = \frac{2(1-\cos 2\alpha)}{|\tan 2\alpha - 2\alpha||\cos 2\alpha|}.$$