Hardy type inequalities on balls*

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Hardy inequality for $n \geq 3$

$$\left\| \frac{f}{|x|} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{n-2} ||\nabla f||_{L^2(\mathbb{R}^n)}, \ f \in H^1(\mathbb{R}^n)$$

- A special case of Pitt's inequality (Beckner, Proc. AMS, 2008)
- Uncertainty principle lemma (Reed & Simon, Methods of MMP Ⅱ, 1975)
- Dilational characterization (Sasaki & T. O, Commun. Contemp. Math., 2009)

Hardy inequality for $n = 2$ (Edmunds & Triebel, Math. Nachr., 1999) :

$$\left\| \frac{f}{|x|(1+|\log|x||)} \right\|_{L^2(\mathbb{R}^2)} \leq C\|f\|_{H^1(\mathbb{R}^2)}, \ f \in H^1(\mathbb{R}^2)$$

an equivalent form :

$$\left\| \frac{f}{|x|(1+|\log|x||)} \right\|_{L^2(B_1)} \leq C\|f\|_{H^1(\mathbb{R}^2)}, \ f \in H^1(\mathbb{R}^2)$$

where $B_R \equiv \{x \in \mathbb{R}^n; |x| < R\}, \ R > 0.$

Hardy inequality on $B_1 = \{x \in \mathbb{R}^2; |x| < 1\}$

$$\left\| \frac{f}{|x|\,|\log|x||} \right\|_{L^2(B_1)} \leq 2\|\nabla f\|_{L^2(B_1)}, \;\; f \in C_0^\infty(B_1)$$

Leray, J. Math. Pures Appl., 1933.
Ladyzhenskaya, “The mathematical theory of viscous incompressible flow,” 1969.

By density, the inequality holds for all $f \in H_0^1(B_1)$.

Question

1. $H^1(B_R)$ vs $H_0^1(B_R)$? $\cdots$ Boundary behavior of functions
2. $\|f\|_{H^1}$ vs $\|\nabla f\|_{L^2}$? $\cdots$ Homogeneous norm control

Theorem 1. $n \geq 3,\ R > 0.$

(1)

$$\Big(\int_{B_R} \frac{1}{|x|^2}\Big|f(x) - f\Big(R\frac{x}{|x|}\Big)\Big|^2 dx\Big)^{1/2} \leq \frac{2}{n-2}\Big(\int_{B_R}\Big|\frac{x}{|x|}\cdot\nabla f(x)\Big|^2 dx\Big)^{1/2}$$

holds for all $f \in H^1(\mathbb{R}^n)$.

(2)

$$\Big(\int_{B_R} \frac{1}{|x|^2}|f(x)|^2 dx\Big)^{1/2} \leq \frac{2}{n-2}\Big(\int_{B_R}\Big|\frac{x}{|x|}\cdot\nabla f(x)\Big|^2 dx\Big)^{1/2}$$

holds for all $f \in H_0^1(B_R)$ and fails for some $f \in H^1(B_R)$.

Theorem 2. $n = 2,\ R > 0.$

(1)

$$\Big(\int_{B_R} \frac{1}{|x|^2 \big|\log \frac{R}{|x|}\big|^2} \Big| f(x) - f\Big(R\frac{x}{|x|}\Big)\Big|^2 dx\Big) \leq 2\Big(\int_{B_R} \Big|\frac{x}{|x|} \cdot \nabla f(x)\Big|^2 dx\Big)^{1/2}$$

holds for all $f \in H^1(\mathbb{R}^2)$.

(2)

$$\Big(\int_{B_R} \frac{1}{|x|^2 \big|\log \frac{R}{|x|}\big|^2} |f(x)|^2 dx)\Big)^{1/2} \leq 2\Big(\int_{B_R} \Big|\frac{x}{|x|} \cdot \nabla f(x)\Big|^2 dx\Big)^{1/2}$$

holds for all $f \in H^1_0(B_R)$ and fails for some $f \in H^1(B_R)$.

Theorem 3. $n = 2,\ R > 0,\ f \in H^1(B_R).$ Then

$$\frac{f}{|x|\left|\log\frac{R}{|x|}\right|} \in L^2(B_R) \iff f \in H_0^1(B_R).$$

Theorem 4. $n = 2$

$$\left(\int_{B_1} \frac{|f(x)|^2}{(1+|x|)^2(1+|\log|x||)^2}dx\right)^{1/2} \leq C\|\nabla f\|_{L^2(\mathbb{R}^2)}$$

fails for some $f \in H^1(\mathbb{R}^2)$.

Proof of Theorem 1 (1)

$$\int_{B_R} \frac{1}{|x|^2}\Big|f(x) - f\Big(R\frac{x}{|x|}\Big)\Big|^2 dx = \int_0^R r^{n-3}\int_{S^{n-1}} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega) dr$$

$$= \Big[\frac{1}{n-2} r^{n-2}\int_{S^{n-1}} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega)\Big]_{r=0}^{r=R}$$

$$\quad - \frac{1}{n-2}\int_0^R r^{n-2}\Big(\frac{d}{dr}\int_{S^{n-1}} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega)\Big) dr$$

$$= -\ \frac{2}{n-2}\int_0^R r^{n-2}\ \mathrm{Re}\int_{S^{n-1}} (f(r\omega) - f(R\omega))\omega\cdot\overline{\nabla f(r\omega)} d\sigma(\omega) dr$$

$$\le \frac{2}{n-2}\Big(\int_0^R r^{n-3}\int_{S^{n-1}} |f(r\omega) - f(R\omega)|^2 d\sigma(\omega) dr\Big)^{1/2}$$

$$\quad \cdot \Big(\int_0^R r^{n-1}\int_{S^{n-1}} |\omega\cdot\nabla f(r\omega)|^2 d\sigma(\omega) dr\Big)^{1/2}$$

$$= \frac{2}{n-2}\Big(\int_{B_R} \frac{1}{|x|^2}\Big|f(x) - f\Big(R\frac{x}{|x|}\Big)\Big|^2 dx\Big)^{1/2}\Big(\int_{B_R} \Big|\frac{x}{|x|}\cdot\nabla f\Big|^2 dx\Big)^{1/2}.$$

Proof of Theorem 2 (1)

$$\int_{B_R} \frac{1}{|x|^2|\log(R/|x|)|^2}\Big|f(x)-f\Big(R\frac{x}{|x|}\Big)\Big|^2dx = \int_0^R \frac{1}{r\big(\log(R/r)\big)^2}\int_{S^1}|f(r\omega)-f(R\omega)|^2d\sigma(\omega)dr$$

$$= \Big[\frac{1}{\log(R/r)}\int_{S^1}|f(r\omega)-f(R\omega)|^2d\sigma(\omega)\Big]_{r=0}^{r=R}$$

$$- \int_0^R \frac{1}{\log(R/r)}\Big(\frac{d}{dr}\int_{S^1}|f(r\omega)-f(R\omega)|^2d\sigma(\omega)\Big)dr$$

$$= -2\int_0^R \frac{1}{\log(R/r)}\ \mathrm{Re}\int_{S^1}(f(r\omega)-f(R\omega))\omega\cdot\overline{\nabla f(r\omega)}d\sigma(\omega)dr$$

where the boundary value at $r=R$ vanishes since

$$0 \le \log\frac{R}{r} = \log\Big(1+\Big(\frac{R}{r}-1\Big)\Big) \le \frac{R}{r}-1 = \frac{R-r}{r},$$

$$|f(r\omega)-f(R\omega)|^2 \le \|\nabla f\|_{L^\infty}^2|R-r|^2.$$

$$\le 2\Big(\int_0^R \frac{1}{r(\log(R/r))^2}\int_{S^1}|f(r\omega)-f(R\omega)|^2d\sigma(\omega)dr\Big)^{1/2}\Big(\int_0^R r\int_{S^1}|\omega\cdot\nabla f(r\omega)|^2d\sigma(\omega)dr\Big)^{1/2}$$

$$= 2\Big(\int_{B_R}\frac{1}{|x|^2|\log(R/|x|)|^2}\Big|f(x)-f\Big(R\frac{x}{|x|}\Big)\Big|^2dx\Big)^{1/2}\Big(\int_{B_R}\Big|\frac{x}{|x|}\cdot\nabla f(x)\Big|^2dx\Big)^{1/2}.$$

Proof of Theorem 3. Let $f \in H^1(B_R)$ satisfy $f/(|x|\log(R/|x|)) \in L^2(B_R)$.

Then $\dfrac{f}{|x|-R} \in L^2(B_R)$. Let $\zeta \in C^\infty(\mathbb{R})$ satisfy $0 \le \zeta \le 1$,

$\zeta = 0$ on $(-\infty,\ 1/2]$, $\zeta = 1$ on $[1, \infty)$. Define $\rho_j(x) = \zeta(j(1-|x|/R))$.

Then $\rho_j = 1$ on $\overline{B_{R(1-1/j)}}$ and $\rho_j = 0$ on $\mathbb{R}^2 \setminus B_{R(1-1/2j)}$,

$$|(\nabla\rho_j)(x)| \le \frac{\|r\zeta'\|_\infty}{R-|x|}\chi_{B_{R(1-\frac{1}{2j})}\setminus\overline{B_{R(1-\frac{1}{j})}}}(x).$$

Therefore, supp$(\rho_j f)$ is compact in B_R, $\rho_j f \to f$, $\rho_j \nabla f \to \nabla f$,

$(\nabla\rho_j)f \to 0$ in $L^2(B_R)$. By mollyfing $\rho_j f$, we see that $f \in H^1_0(B_R)$.

Proof of Theorem 4. Define $f_j(x) = \varphi_j(|x|)$, where

$$\varphi_j(r) = \begin{cases} 1 & \text{if} \quad |\log r| \le j, \\ 2 - |\log r|/j & \text{if} \quad j < |\log r| < 2j, \\ 0 & \text{if} \quad |\log r| \ge 2j. \end{cases}$$

$$\int_{B_1} \frac{1}{(1+|x|)^2(1+|\log|x||)^2} |f_j(x)|^2 dx = 2\pi \int_0^1 \frac{1}{(1+r)^2(1+|\log r|)^2} |\varphi_j(r)|^2 r dr$$

$$= 2\pi \int_0^\infty \frac{1}{e^{2t}(1+e^{-t})^2(1+t)^2} |\varphi_j(e^{-t})|^2 dt$$

$$\ge 2\pi \int_0^1 \frac{1}{(e^t+1)^2(1+t)^2} |\varphi_j(e^{-t})|^2 dt \ge \frac{2\pi}{(e+1)^2} \int_0^1 \frac{1}{(1+t)^2} dt = \frac{2\pi}{(e+1)^2},$$

while, with $\psi_j(t) = \varphi_j(e^{-t})$,

$$\|\nabla f_j\|^2_{L^2(\mathbb{R}^2)} = 2\pi \int_0^\infty |\varphi_j'(r)|^2 r dr = 2\pi \int_{-\infty}^\infty |\varphi_j'(e^{-t})|^2 e^{-2t} dt$$

$$= 2\pi \int_{-\infty}^\infty |\psi_j'(t)|^2 dt = 4\pi \int_j^{2j} \frac{1}{j^2} dt = \frac{4\pi}{j} \to 0 \quad \text{as} \quad j \to \infty.$$