Regularity question of the incompressible Navier-Stokes equations I

by Hi Jun Choe

Department of Mathematics and MSRI, Yonsei University
Seoul, Korea

2013. 6. 17

Contents

§1. Helmholtz decomposition and weak solutions

Helmholtz observed that smooth vector field can be decomposed of gradient field and solenoidal field. His observation has influenced a great impact on science and mathematics. In fluid mechanics Helmholtz decomposition appears in the formulation of incompressibility due to volume preserving deformation.

To be more specific, we consider a smooth vector field

$$u : R^3 \to R^3$$

satisfying

$$u(x) = o(\frac{1}{|x|}).$$

Define Newtonian potential $N(f)$ for $f = o(\frac{1}{|x|^2})$ as $|x| \to \infty$ by

$$N(f)(x) = -\Delta^{-1} f(x) = \frac{1}{4\pi} \int_{R^3} \frac{f(y)}{|x-y|} dy.$$

From the integration by parts and vector identity

$$\Delta u = -\nabla \times \nabla \times u + \nabla(\text{div}(u))$$

we have

$$\begin{aligned} u &= -\nabla \phi + \nabla \times v \\ \phi &= N(\text{div}(u)) \\ v &= N(\nabla \times u) \end{aligned}$$

and the decomposition is unique.

In view of analysis, we need more general framework to consider various function spaces. For notational simplicity we do not separate vectors and scalars and omit domains in symbols.

We let $\Omega \subset R^3$ be an open set and introduce the Lebesgue measurable function spaces $L^p(\Omega) = \{\int_\Omega |f|^p dx < \infty\}, 1 \leq p \leq \infty$ and $C_0^\infty(\Omega)$ that is the set of smooth function with compact support in Ω.

$C_{0,\sigma}^\infty$ is a subset of solenoidal vectors in C_0^∞ and L_σ^p is the closure of $C_{0,\sigma}^\infty$ in L^p.

Since L^2 is Hilbert space, we have Helmholtz decomposition so that

$$\begin{aligned} &L^2(\Omega) = L_\sigma^2(\Omega) \oplus G^2(\Omega), \\ &G^2(\Omega) = \{\nabla p \in L^2(\Omega); p \in L_{loc}^1(\Omega)\}. \end{aligned}$$

Sometimes, there needs deeper understanding about domains. For example, we let Ω be Lipscitz.

Definition 1.1. *We call* $D \subset \mathbf{R}^n$, $n \geq 2$ *is Lipschitz if for every* $Q \in S(= \partial D)$, *there is a ball* $B(r, Q) = \{P \in \mathbf{R}^n \,|\, |P - Q| < r\}$ *and a coordinate system such that*

$$B(r, Q) \cap D = B(r, Q) \cap \{(x', x_n) \mid x_n > \phi(x'),\ ||\nabla \phi||_{L^\infty} \leq M\},$$

where M *and* r *are independent of* Q.

For $1 \leq p \leq \infty$ and $-\infty < s < \infty$ Sobolev space $L^p_s(R^n)$ is defined by

$$L^p_s = \{(I - \Delta)^{-s/2}u : u \in L^p(R^n)\}$$

with the norm

$$||u||_{L^p_s} = ||(I - \Delta)^{s/2}u||_{L^p(R^n)}$$

and define for $s \geq 0$ $L^p_s(\Omega)$ by the space of the restrictions of functions in $L^p_s(R^n)$ to Ω. $L^p_{s,0}$ is the subspace of functions with support in $\overline{\Omega}$ of $L^p_s(R^n)$. We denote $H^k = L^2_k$.

Define Besov space $\mathcal{B}^p_s(\partial\Omega)$ as the collection of all measurable function f on $\partial\Omega$ such that

$$||f||_{\mathcal{B}^p_s(\partial\Omega)} =: ||f||_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(P) - f(Q)|^p}{|P - Q|^{n-1+sp}} d\sigma(P) d\sigma(Q)\right)^{\frac{1}{p}}.$$

The case $p = \infty$ corresponds to the non-homogeneous version of the space of Holder continuous functions on $\partial\Omega$. We also define $\mathcal{B}^p_{-s}(\partial\Omega)$ as the dual of $\mathcal{B}^p_s(\partial\Omega)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1, 0 < s < 1, 1 < p \leq \infty$.

Theorem 1.2. *Suppose that* $u \in L^p(\Omega)$ *and* $div\, u \in L^p_{-1}(\Omega)$. *Then* $u \cdot N \in \mathcal{B}^p_{-\frac{1}{p}}(\partial\Omega)$, *where* N *is the outward unit normal vector.*

Proof

(1) We begin by noting that the pairing of $L^q_{-s+\frac{1}{q}}(\Omega)$ and $L^q_{s-1+\frac{1}{p}}(\Omega)$ is well defined for any $0 < s < 1, 1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(2) In fact, since $C_0^\infty(\Omega)$ is dense in $L^q_\alpha(\Omega)$ for $0 \le \alpha < \frac{1}{q}$, it is not difficult to see that $L^q_{\alpha,0}(\Omega) = L^q_\alpha(\Omega)$ for $0 \le \alpha < \frac{1}{q}$.

(3) In particular,

$$L^q_{-s+\frac{1}{q}}(\Omega) = \left(L^p_{s-1+\frac{1}{p}}(\Omega)\right)^* \quad \text{if} \quad \frac{1}{q} \le s$$

and

$$L^p_{s-1+\frac{1}{p}}(\Omega) = \left(L^q_{-s+\frac{1}{q}}(\Omega)\right)^* \quad \text{if} \quad s \le \frac{1}{q}.$$

(4) For

$$f \in \left(L^p_{s+\frac{1}{p}}(\Omega)\right)^* = L^q_{-s-\frac{1}{p},0}(\Omega),$$

the distribution $\mathrm{div}\, f \in (C_0^\infty(\Omega))'$, we denote by $f \cdot N$ the normal component of f and define it by the linear functional in $\mathcal{B}^q_{-s}(\partial\Omega)$ by

$$< f \cdot N, \phi > := < \mathrm{div} f, \widetilde{\phi} > + < f, \nabla\widetilde{\phi} >$$

for all $\phi \in \mathcal{B}^p_s(\partial\Omega)$, where $\widetilde{\phi} \in L^p_{s+\frac{1}{p}}(\Omega)$ is an extension of ϕ in the trace sense.

(5) In particular, when $s = \frac{1}{q}, \widetilde{\phi} \in L^p_1(\Omega)$.

(6) It remains to show that $\nabla\widetilde{\phi} \in L^p_{s-\frac{1}{q}}$. But this follows from extension lemma by [25] and duality lemma by [18].

Remark 1

From the proof, we can deduce

$$||u \cdot N||_{\mathcal{B}^p_{-\frac{1}{p}}(\partial\Omega)} \leq c(\Omega, p) \left(||u||_{L^p(\Omega)} + ||\operatorname{div} u||_{L^p(\Omega)} \right).$$

Remark 2

If $u \in L^p(\Omega)$ has $\operatorname{div} u = 0$, then $u \cdot N$ as a functional in $\left(\mathcal{B}^p_{1-\frac{1}{q}}(\partial\Omega) \right)^*$, annihilates all functions of the form $\chi_{\partial\Omega'}$ with Ω' connected component of Ω. We denote the collection of all such functionals by $\widetilde{\mathcal{B}}^p_{-\frac{1}{p}}(\partial\Omega)$. We have

$$L^p_\sigma(\Omega) := \{u \in L^p(\Omega) : \operatorname{div} u = 0 \text{ and } u \cdot N = 0\}$$

$$grad\, L^p_1(\Omega) := \{\nabla u : u \in L^p_1(\Omega)\}.$$

They are easily seen to be closed subspaces of $L^p(\Omega)$ and , for $p = 2$, we denote P, D the corresponding orthogonal projections from $L^2(\Omega)$ onto $L^2_\sigma(\Omega)$ and $G^2(\Omega)$ respectively. P is called Helmholtz projection.

Theorem 1.3. *For each Lipschitz domain Ω in $\mathbf{R^n}$, with arbitrary topology, there exists a positive number ϵ depending on Ω such that P, D extend to bounded operators from $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$ and onto $grad\, L^p_1(\Omega)$, respectively, for each $\frac{3}{2} - \epsilon < p < 3 + \epsilon$. Hence in this range*

$$L^p(\Omega) = grad L^p_1(\Omega) \oplus L^p_\sigma(\Omega)$$

where the sum is topological. In the class of Lipschitz domain the result is sharp. If however $\partial\Omega \in C^1$ *then we may take* $1 < p < \infty$.

For unbounded domain like infinite cylinder and various function spaces with weight, there are results by Farwig, Specovius-Neugebauer, Fujiwara-Morimoto, Miyakawa, Sohr, Simader, Wiegner. For Lipschitz domain, there results by Kenig-Jerison, Fabes-Mitrea-Mendez.

The Helmhotz decomposition depends essentially on Neumann problem:

$$\Delta\phi = 0 \quad \text{in} \quad \Omega$$
$$\frac{\partial\phi}{\partial N} = (u - \nabla(\mathrm{div}(N(u)))) \cdot N \quad \text{on} \quad \partial\Omega$$

In the whole domain R^3 or periodic domain $T^3 = R^3/(2\pi Z)^3$ Riesz operator is useful to apply Navier-Stokes equations. In R^3, we define for $u \in L^2(R^3)$

$$R^i(u) = F^{-1}(\frac{\xi^i}{|\xi|}\hat{u}), i = 1, 2, 3$$

and for $u(x) = \sum_{k \in Z^3} u_k e^{ik\cdot x}$ in $L^2(T^3)$

$$R^i(u) = \sum_{k \in Z^3} \frac{k^i}{|k|} u_k e^{ik\cdot x}, i = 1, 2, 3.$$

The Helmholtz projection will be

$$P(u) = (I - R \otimes R)u$$

in $L^2(R^3)$ and

$$P(u) = \sum_{k \in Z^3} (I - \frac{k}{|k|} \otimes \frac{k}{|k|}) u_k e^{ik\cdot x}$$

in $L^2(T^3)$.

First, we consider a Cauchy problem in $\Omega_T = R^3 \times (0, T)$ for a fixed time T of the Navier-Stokes equations

$$\frac{\partial}{\partial t}u - \nu\Delta u + \mathrm{div}(u \otimes u) + \nabla p = 0, \tag{1.1}$$

$$\mathrm{div} u = 0$$

with an initial data

$$u(x, 0) = u_0(x),$$

where the velocity fields u and u_0 are three dimensional solenoidal vector fields and the pressure p is a scalar field. We let the viscosity $\nu = 1$.

Definition 1.4. *We say $u \in L^2_{loc}(\Omega_T)$ is a weak solution if for Q an open subset of Ω_T and $\phi \in C_0^\infty(Q)$*

$$\int u \cdot \nabla\phi dz = 0$$

and for $\psi \in C^\infty_{0,\sigma}(Q)$

$$\int u \cdot (\psi_t + \Delta\psi) + u \otimes u : \nabla \otimes \psi dz = 0,$$

where $z = (x, t)$.

We construct Galerkin solution when the initial data u_0 is in $L^2_\sigma(T^3)$ for the periodic domain. Certainly our construction can be modified for more general domain like Lipschitz domain. We follow the idea of Hopf. For each n we let P_n the projection to the finite dimensional space $(span\{e^{ik\cdot x}, |k| \leq n\})_\sigma$ such that

$$P_n(\sum_{k\in Z^3} u^k e^{ik\cdot x}) = \sum_{|k|\leq n}(I - \frac{k}{|k|}\otimes\frac{k}{|k|})u^k e^{ik\cdot x}.$$

The n-th Galerkin approximation solution u_n is the solution to

$$\frac{\partial}{\partial t}u_n = \Delta u_n + P_n(\mathrm{div}(u_n \otimes u_n))$$

$$u_n(0) = P_n(u_0)$$

with $P_n u_n = u_n$. Applying u_n as a test function to Galerkin equation, we have an approximation energy estimate.

Theorem 1.5. *Suppose $u_0 \in L^2_\sigma(T^3)$, then the Galerkin approximation solution satisfies that for all $T > 0$*

$$\int |u_n(T)|^2 dx + 2\int_0^T \int |\nabla u_n(t)|^2 dxdt = \int |P_n u_0|^2 dx$$

and

$$\lim_{T\to 0}\int |u_n(T)|^2 dx = \int |P_n u_0|^2 dx.$$

By taking $\phi \in C_\sigma^\infty(T)$ we have

$$\int u_n(T) \cdot \phi dx + \int_0^T \int \nabla u_n : \nabla \phi dx dt$$

$$+ \int_0^T \int u_n \otimes u_n : P_n \nabla \phi dx dt = \int P_n u_0 \cdot \phi dx$$

and from the a priori estimate of u_n and ∇u_n, we have

$$\int_0^T \int \nabla u_n : \nabla \phi dx dt \to 0 \quad \text{as} \quad T \to 0$$

$$\int_0^T \int u_n \otimes u_n : P_n \nabla \phi dx dt \to 0 \quad \text{as} \quad T \to 0.$$

Since ϕ is arbitrary periodic smooth function, $u_n(T)$ converges u_0 weakly in L^2 as T goes to zero.Therefore the norm convergence and weak convergence in L^2 imply the strong convergence.

Theorem 1.6.

$$\int |u_n(T) - P_n u_0|^2 dx \to 0 \quad as \quad T \to 0.$$

The norm convergence of u_n to $P_n u_0$ as t goes to zero By Sobolev embedding and compactness if necessary choosing a subsequence there is $u \in L^\infty(0,T : L^2_\sigma(T^3)) \cap L^2(0,T : H^1(T^3))$ such that

$$\begin{aligned} u_n &\to u \quad \text{weakly* in} \quad L^\infty(0,T : L^2(T^3)) \\ u_n &\to u \quad \text{weakly in} \quad L^2(0,T : H^1(T^3)) \\ u_n &\to u \quad \text{strongly in} \quad L^p(\Omega_T), 1 < p < \frac{10}{3}. \end{aligned}$$

We denote the existence space by

$$V = L^\infty(0,T : L^2_\sigma(T^3)) \cap L^2(0,T : H^1(T^3)).$$

From the completeness of L^2_σ we have $P_n u_0$ converges to $Pu_0 = u_0$ strongly in L^2 and we have the following theorem.

Theorem 1.7. *$u \in V$ is a weak solution and satisfies*

$$\int |u(T)|^2 dx + \int_0^T \int |\nabla u|^2 dx dt \leq \int |u_0|^2 dx$$

and

$$\int |u(T) - u_0|^2 dx \to 0 \quad as \quad T \to 0.$$

Theorem 1.8. *u is in $C(0, T : L^2_{weak})$, namely, for all $v \in L^2(T^3)$*

$$\int u(t) \cdot v dx \to \int u(s) \cdot v dx$$

as t goes to s.

Here is a question of energy inequality. If we the weak solution satisfies strict inequality, we call it turbulent solution referring to Kolmogorov energy spectrum structure. It means a certain portion of energy dissipates through heat conduction from viscous friction.

But, as far as I know, there has never been constructed turbulent solution in Navier-Stokes flow.

To get equality, it is enough to construct solutions in $C^\alpha(0,T : L^2)$ for some $\alpha > 0$.

In case of inviscid Euler flow, Onsager conjectured that if the flow is $C^\alpha, \alpha > 1/3$ then the energy identity holds and if $\alpha < 1/3$, the equality fails. The positive answer for $\alpha > 1/3$ has already been answered by Constantin-Titi [8] in 1994. Remarkably enough, when $\alpha < 1/5$, De Lellis-Szekelyhidi [13] constructed solutiions satisfies strict inequality recently.

The Galerkin approximation solution doesn't involved with the pressure. However by defining the approximation pressure as the solution to the Poisson equation

$$\Delta p_n = \frac{\partial^2}{\partial x^i \partial x^j} u_n^i u_n^j$$

we can recover n-th Galerkin pressure, where the double indices mean summation up to 3. Introducing the residual projection $Q_n = P - P_n$, we write the Galerkin equation as

$$\frac{\partial}{\partial t} u_n - \Delta u_n + \mathrm{div}(u_n \otimes u_n) + \nabla p_n - Q_n(\mathrm{div}(u_n \otimes u_n)) = 0.$$

Thus if the projection residual converges in appropriate sense, we may say the Galerkin solution is associated with pressure.

Question:

$$\lim_n \int Q_n u_n \phi dz \to 0 \quad ?$$

Definition 1.9. *We say* $(u, p) \in V \times L^{3/2}(\Omega_T)$ *is suitable weak solution to the initial value problem if for all* $\phi \in C_0^\infty$

$$\int u\cdot\phi_t dz + \int \nabla u : \nabla\phi dz + \int u\otimes u : \nabla\phi dz - \int p div\phi dz = 0 \tag{1.2}$$

and u *is weakly divergence free for almost all time, satisfies the localized energy inequality for almost all* t

$$\int |u(x,t)|^2\phi dx + 2\int_0^t \int |\nabla u|^2 \phi dx ds \tag{1.3}$$
$$\leq \int_0^t \int |u|^2(\phi_t + \Delta\phi) dx ds + \int_0^t \int (|u|^2 + 2p) u \cdot \phi dx ds$$

for all nonnegative $\phi \in C_0^\infty(R^3 \times R_+)$ *and*

$$\int |u(x,t) - u_0(x)|^2 dx \to 0 \quad as \quad t \to 0,$$

where the initial data u_0 *is weakly divergence free in* $L^2(R^3_+)$.

Here is a lemma due to Bryuk-Craig-Ibrahim:

Lemma 1.10. *If for any constant* $c > 0$

$$\lim_{n\to\infty} \int_0^T \sum_{n-c<|k|\le n} |k|^2 |u_n^k(t)|^2 dt = 0,$$

then the Galerkin solution satisfies local energy inequality in suitable sense.

Remark. If the Galerkin approximation has bi-Laplacian, then it is suitable. Also recently, Guermond [23] showed that Finite Element Solution (u^n, p^n) in discrete space satisfying LBB condition converges to a suitable weak solution.

The existence theorem of suitable weak solutioon is due to Scheffer and Caffarelli-Kohn-Nirenberg([36] and [3]).

We introduce suitable approximation scheme: There exists a sequence $\{(u_n, p_n)\}$ of smooth functions and a sequence of positive numbers ϵ_n such that

$$\begin{aligned}
&u_n \to u \quad \text{weakly* in} \quad L^\infty(0, T : L^2(T^3)) \\
&u_n \to u \quad \text{weakly in} \quad L^2(0, T : H^1(T^3)) \\
&u_n \to u \quad \text{strongly in} \quad L^3(\Omega_T), \\
&p_n \to p \quad \text{strongly in} \quad L^{3/2}(\Omega_T),
\end{aligned}$$

with (u_n, p_n) satisfying

$$\begin{aligned}
&\frac{\partial}{\partial t} u_n - \Delta u_n + (\eta_n * u_n \cdot \nabla) u_n + \nabla p_n = 0 \\
&\text{div} u_n = 0 \\
&u_n(0) = \eta_n * u_0.
\end{aligned}$$

Here η is a smooth nonnegative cutoff function and $\eta_n(x) = \frac{1}{\epsilon_n^3} \eta(\frac{x}{\eta_n})$.

Theorem 1.11. *Let (u, p) be a limit of suitable approximation solutions of the Navier-Stokes system on the periodic domain Ω_T , then (u, p) is a suitable weak solution in Ω_T.*

Proof. Denote by $\{(u_n, p_n)\}$ a sequence of solutions of approximate equation which converges in the suitable sense to (u, p), a suitable weak solution. Using Sobolev embedding $H^1(T^3) \to L^6(T^3)$ we deduce that $\{u_n\}$ is a bounded sequence in $L^2(0, T : L^6)$ (where we are using the norm convention that $v \in L^q(0, T : L^p)$ means that $\int_0^T (\int |v(x,t)|^p dx)^{q/p} dt < 1$). From Hölder inequality we have $u_n \in L^{10/3}$ with uniform bound. By the compact embedding we get(if necessary subsequence)

$$(u_n, p_n) \to (u, p) \quad \text{strongly in} \quad L^3 \times L^{3/2}.$$

Now we want to prove that the limit (u, p) satisfies the local energy inequality in Ω_T. Let ϕ be a non-negative smooth cutoff function with support in Ω_T (in particular ϕ vanishes near t = 0). Multiplying the equation by ϕu_n and integrating by parts, and using the integral identity

$$\int (\eta_n * u_n \cdot \nabla) u_n \cdot u_n \phi dx = \frac{1}{2} \int \eta_n * u_n |u_n|^2 \cdot \nabla \phi dx$$

we have

$$2 \int |\nabla u_n|^2 \phi dz = \int |u_n|^2 (\phi_t + \Delta \phi) dz$$

$$+ \int |u_n|^2 \eta_n * u_n \cdot \nabla \phi + 2 p_n u_n \cdot \nabla \phi dz$$

and the strong convergence (u_n, p_n) in $L^3 \times L^{3/2}$ implies the local energy inequality (3.4)

$$\int |u(x,t)|^2 \phi dx + 2 \int_0^t \int |\nabla u|^2 \phi dx ds$$
$$\leq \int_0^t \int |u|^2 (\phi_t + \Delta \phi) dx ds + \int_0^t \int (|u|^2 + 2p) u \cdot \phi dx ds.$$

Here is a brief argument of $L^{3/2}$ integrability of pressure:

i) $p_n = R^i R^j((\eta_n * u_n)^i u_n^j)$ for the Riesz operator R^i.

ii) R^i is bounded operator in $L^p, p \in (1, \infty)$.

iii) For almost all t, $\eta_n * u_n \otimes u_n(t)$ converges strongly to $u \otimes u(t)$ in $L^{3/2}(T^3)$.

iv) We let $p = R^i R^j(u^i u^j)$, then $p_n(t)$ converges to $p(t)$ in $L^{3/2}(T^3)$ for almost all t.

v) From the $L^{3/2}(0, T : L^{3/2}(T^3))$ convergence of $u_n \otimes u_n$ to $u \otimes u$, we get $p \in L^{3/2}(\Omega_T)$.

Kato and Giga-Miyakawa proved that for the initial data $u_0 \in L^3$ there exist $T > 0$ and at least one solution u in

$$C([0,T);L^3) \cap C((0,T);L^\infty)$$

such that u solves Navier-Stokes equations in the sense of the following integral equation:

$$u(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A} P\mathrm{div}(u \otimes u)(s)ds,$$

where e^{tA} denotes the heat semigroup and P denotes the Helmholtz projection. Such u is called a mild solution.

Weak-Strong uniqueness implies uniqueness.

§2. Partial regularity

§2..1 Introduction

In this section, we study partial regularity of the incompressible Navier-Stokes equations for the weak solution in the sense of Leray [31] and Hopf [24]. After Scheffer [36] introduced the idea of suitable weak solution, Caffarelli, Kohn and Nirenberg [3] established a criterion of ϵ regularity and Lin [32] simplified the proof greatly. Choe and Lewis [7] improved the parabolic Hausdorff dimension by logarithmic factor.

All the previous results rely on the localized energy inequality and higher integrability of pressure like $L^{3/2}$ of the suitable weak solutions. From the definition of the suitable weak solution, the pressure satisfies Poisson equation and is represented by Newtonian potential with density that is a quadratic function of velocity gradient.

Meanwhile, Leray-Hopf solution is constructed by Galerkin process with orthonormal basis of the parabolic solution space and from the weak convergence, we have global energy inequality. Here, it is found a localized energy inequality from the vorticity equation and Biot Savart law. In fact, for the Stokes equations, a localized energy inequality is proved by Jin [26].

We say $u \in L^\infty(0,T : L^2(R^3)) \cap L^2(0,T : H^1(R^3))$ is Leray-Hopf solution to the Cauchy problem if for a fixed $v \in L^2(R^3)$ $\int u \cdot v dx(t)$ is continuous in t, for all $\phi \in C_0^\infty(R^3 \times (0,T))$ with $\mathrm{div}\phi = 0$

$$\int u \cdot \phi_t dz - \int \nabla u : \nabla \phi dz + \int u \otimes u : \nabla \phi dz = 0, \quad (2.1)$$

for almost all t_0

$$\frac{1}{2}\int |u(x,t_0)|^2 dx + \int_0^{t_0}\int |\nabla u|^2 dxdt \le \frac{1}{2}\int |u_0(x)|^2 dx$$

and

$$\int |u(x,t) - u_0(x)|^2 dx \to 0 \quad as \quad t \to 0,$$

where the initial data u_0 is weakly divergence free in $L^2(R^3)$. The following existence theorem is due to Leray [31].

Theorem 2.1. *There is a Leray-Hopf solution to the Cauchy problem.*

We define a ball $B_r(x_0) = \{x : |x - x_0| < r\}$ and a parabolic cylinder $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r_0^2)$, and we always assume that the cylinders are in Ω_T. Our main theorem is the following:

Theorem 2.2. *There is an absolute constant ϵ_0 such that*

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q_r} |\nabla u|^2 dz \leq \epsilon_0$$

implies that for an r_0

$$\sup_{z \in Q_{\frac{r_0}{2}}} |u(z)| \leq \frac{c}{r_0}$$

for an absolute constant c

Let h be an increasing function on $(0,1]$ with $\lim_{r\to 0} h(r) = 0$. For fixed $\delta > 0$ and $E \subset R^3 \times R$ we let $L(\delta)$ be the family of all coverings $\{Q_{r_i}(z_i), r_i < \delta\}$ of E. We define

$$\Psi_\delta(E,h) = \inf_{L(\delta)} \sum_i h(r_i) \tag{2.2}$$

and set the corresponding Hausdorff measure

$$\Lambda(E,h) = \lim_{\delta\to 0} \Psi_\delta(E,h).$$

In particular we denote $\Lambda_k(E) = \Lambda(E, r^k)$.

The singular set S is the set of point z in Ω_T such that in any neighborhood of z u is unbounded.

Theorem 2.3. *There is $\sigma > 0$ such that*

$$\Lambda(S, t(\log(1/t))^\sigma) = 0.$$

§2..2 Partial regularity

In this section we assume z_0 is origin, omit z_0 in various expressions if obvious. The compactness of solution space is due to Leray [31] and Lin [32]. Suppose $v_n \in L^\infty(-1,1 : L^2(B_1)) \cap L^2(-1,1 : H^1(B_1))$ is a localized solution to (2.1) in Q_1 with the uniform bound

$$\text{ess} \sup_{t\in(-1,1)} \int_{B_1} |v_n|^2 dx + \int_{-1}^{1}\int_{B_1} |\nabla v_n|^2 dxdt < c$$

for all n. The localized solution in Q_1 means that the integral identity (2.1) holds for all test function $\phi \in C_0^\infty(Q_1)$ with $\text{div}\phi = 0$. Then there is a subsequence v_n by rewriting index and $v \in L^\infty(-1,1 : L^2(B_1))) \cap L^2(-1,1 : H^1(B_1))$ such that

$$v_n \to v$$

weakly in $L^2(-1,1 : H^1(B_1))$ and weakly-* in $L^\infty(-1,1 : L^2(B_1)))$, and the limit v is a solution to (2.1). Furthermore, from Sobolev embedding,

$$v_n \to v$$

strongly in $L^3(Q_1)$.

One of the important step for ϵ-regularity is the localized energy inequality which is essentially due to Jin [26]. We define a bilinear form

$$< f, g >_N = \int_{R^3} \int_{R^3} \frac{f(x)g(y)}{|x-y|} dxdy$$

and the corresponding norm

$$||f||_N = < f, f >_N^{1/2} .$$

Define the vorticity ω by

$$\omega = \nabla \times v.$$

Since we do not know $\omega \in L^\infty(-1, 1 : L^2(B_1))$, we need an indirect definition of $< \omega, \omega >_N (t)$ from $u \in L^\infty(-1, 1 : L^2(B_1)$. To define $< \omega, \omega >_N (t)$ for almost all t, we let $\phi \in C_0^\infty$ and we note that

$$\omega\phi = \nabla \times (u\phi) + u \times \nabla\phi$$

and from the definition

$$< \omega\phi, \omega\phi >_N (t) = \int_{R^3} \int_{R^3} \frac{1}{|x-y|} (\nabla \times (u\phi) + u \times \nabla\phi)(x, t) \cdot$$
$$(\nabla \times (u\phi) + u \times \nabla\phi)(y, t) dxdy$$

and the righthand side is well defined after integration by parts almost all time t.

Theorem 2.4. *Suppose* $v \in L^\infty(-1,1 : L^2(B_1))) \cap L^2(-1,1 : H^1(B_1))$ *is a solution to (2.1). Then, there is a constant* c *such that for almost all* $t_0 \in (-1,1)$

$$||\omega\phi||_N^2(t_0) + ||\nabla v\phi||^2_{L^2(Q_1)} \le c||v||^2_{L^2(Q_1)} + c||v||^3_{L^3(Q_1)} \quad (2.3)$$

for a cutoff function $\phi \in C_0^\infty(Q_1)$, *and if we take* $\phi \in C_0^\infty(B_1)$ *and* $\phi = 1$ *in* $B_{r_0}, r_0 \in (0, \frac{1}{2})$, *then we have*

$$||v||^2_{L^2(B_{r_0})} \le 2||\omega\phi||^2_N + 2||v \times \nabla\phi||^2_N. \quad (2.4)$$

Proof. The vorticity $\omega = \nabla \times v$ satisfies

$$\frac{\partial}{\partial t}\omega - \Delta\omega + (u \cdot \nabla)\omega + (\omega \cdot \nabla)v = 0. \quad (2.5)$$

We let $\phi \in C^\infty(Q_1)$ and $\phi = 0$ on the boundary ∂Q_1, $|\nabla\phi| < c$ and $|\phi_t| < c$. We recall Newtonian potential $N(f)(x) = \frac{1}{4\pi}\int_{R^3} \frac{f(y)}{|x-y|}dy$.

At least formally, we take $\phi N(\omega\phi)$ as a test function to (2.5) and after integration by parts and observing the inverse relation between Newtonian potential and Laplace operator, we have for almost all t_0

$$\frac{1}{2} < \omega\phi, \omega\phi >_N (t_0) + \int_{-1}^{t_0}\int_{B_1} |\omega\phi|^2 dxdt \quad (2.6)$$
$$\le \int_{-1}^{t_0}\int_{B_1} \omega\phi_t N(\omega\phi)dxdt + c\int_{-1}^{t_0}\int_{B_1}|v|^2 dxdt$$
$$+ \int_{-1}^{t_0}\int_{B_1}(v\cdot\nabla)v\nabla \times (\phi N(\nabla \times v\phi))dxdt.$$

We can justify the computations by taking the test function $\nabla \times (N(\omega\phi)\phi)$ to (2.1). Indeed, it is solenoidal and in $L^\infty(0,T : L^2(R^3)) \cap L^2(0,T : H_0^1(R^3))$.

Note that, for all $1 < p < \infty$, $||N(f\chi_{B_1})||_{L^p(B_1)} \leq c||f||_{L^p(B_1)}$, $||\nabla N(f\chi_{B_1})||_{L^p(B_1)} \leq c||f||_{L^p(B_1)}$ and $||\nabla^2 N(f\chi_{B_1})||_{L^p(B_1)} \leq c||f||_{L^p(B_1)}$, where χ_A is the characteristic function of A.

On the other hand, since v is solenoidal, namely, $\nabla \times \omega = -\Delta v$, we have for almost all t_0

$$\int_{-1}^{t_0}\int_{B_1} |\nabla v\phi|^2 dxdt \leq \int_{-1}^{t_0}\int_{B_1} |\omega\phi|^2 dxdt + c\int_{-1}^{t_0}\int_{B_1} |v|^2 dxdt.$$

Combining these estimates, we have

$$\begin{aligned}
&< \omega\phi,\omega\phi >_N^2 (t_0) + \int_{-1}^{t_0}\int_{B_1} |\nabla v\phi|^2 dxdt \\
&\qquad \leq c\int_{-1}^{t_0}\int_{B_1} \omega\phi_t N(\omega\phi) dxdt + c\int_{-1}^{t_0}\int_{B_1} |v|^2 dxdt \\
&\qquad + \int_{-1}^{t_0}\int_{B_1} (v\cdot\nabla)v\nabla \times (\phi N(\nabla \times v\phi)) dxdt.
\end{aligned}$$

From the integration by parts and the potential estimates, we have

$$\int_{-1}^{t_0}\int_{B_1} \omega\phi_t N(\omega\phi) dxdt \leq c\int_{-1}^{t_0}\int_{B_1} |v|^2 dxdt.$$

In the potential expression, we have

$$\nabla \times (\phi N(\nabla \times v\phi)) = \nabla\phi \times N(\nabla \times (v\phi)) - \nabla\phi \times N(\nabla\phi \times v)$$
$$+\phi\nabla \times N(\nabla \times v\phi)$$

and

$$||\nabla(\nabla\phi \times N(\nabla \times (v\phi)) - \nabla\phi \times N(\nabla\phi \times v))||_{L^3} \leq c||v||_{L^3}.$$

The last term can be treated as

$$\phi\nabla \times N(\nabla \times v\phi) = -\phi N(\nabla\phi \times (\nabla \times v)) - \phi N(\nabla \times (\nabla \times v)\phi)$$
$$= \phi N(\nabla\phi \times (\nabla \times v)) - \phi N(\nabla \times (\nabla \times v)\phi)$$

and after integration by parts we have $||\nabla(\phi N(\nabla\phi \times (\nabla \times v)))||_{L^3} \leq c||v||_{L^3}$. Since $\text{div} v = 0$, we have

$$\nabla \times (\nabla \times v)\phi = -\Delta v\phi = -2\nabla \cdot (v\nabla\phi) + v\Delta\phi + \Delta(v\phi)$$

and

$$\phi N(\nabla \times (\nabla \times v)\phi) = -2\phi N(\nabla \cdot (v\nabla\phi)) + \phi N(v\Delta\phi) + v\phi^2.$$

We also have $||\nabla(-2\phi N(\nabla\cdot(v\nabla\phi))+\phi N(v\Delta\phi))||_{L^3} \leq c||v||_{L^3(B_1)}$. Therefore we have

$$\left|\int_{-1}^{t_0}\int_{B_1}(v \cdot \nabla)v\nabla \times (\phi N(\nabla \times v\phi))dxdt\right|$$
$$\leq c\left|\int_{-1}^{t_0}\int_{B_1} v \otimes v : \nabla F dxdt\right| + c\int_{-1}^{t_0}\int_{B_1}|v|^3 dxdt$$

for some F with $||F||_{L^3} \leq c||v||_{L^3(B_1)}$. Applying Holder inequality on F with L^3 norm, we obtain the localized energy estimate (2.3).

From integration by parts, the previous potential estimate and the relation $\omega\phi = \nabla \times (v\phi) - v \times \nabla\phi$, we have

$$< \nabla\times(v\phi), \nabla\times(v\phi) >_N \leq 2 < \omega\phi, \omega\phi >_N +2 < v\times\nabla\phi, v\times\nabla\phi >_N$$

and, from vector calculus, we have that

$$< \nabla\times(v\phi), \nabla\times(v\phi) >_N = < \mathrm{div}(v\phi), \mathrm{div}(v\phi) >_N + \int |v\phi|^2 dx.$$

This proves (2.4).

Theorem 2.5. *Suppose that v is a Leray-Hopf solution to (2.1) in Q_1. Then, there is a constant ϵ such that*

$$\left(\sup_{t\in(-1,1)} ||v\chi_{B_1}||_N(t)\right)^3 + \int_{Q_1} |v|^3 dz \leq \epsilon \qquad (2.7)$$

implies

$$||v||_{C^\alpha(Q_{\frac{1}{2}})} \leq c$$

for $\alpha \leq \frac{1}{3}$ and some c.

For the proof, since the pressure estimate is unnecessary, our argument is much simple. Define the averages in cylinder and ball by $(v)_{Q_r} = \frac{1}{|Q_r|}\int_{Q_r} vdz$ and $(v)_{B_r}(t) = \frac{1}{|B_r|}\int_{B_r} v(x,t)dx$.

Lemma 2.6. *There is an ϵ such that*

$$\sup_{t\in(-1,1)} ||v\chi_{B_1}||_N^3(t) + \int_{Q_1} |v|^3 dz \leq \epsilon$$

implies that for a constant $r \in (0, \frac{1}{2})$,

$$\begin{aligned}\sup_{t\in(-r,r)} \left(\frac{r^{1/2}}{|B_r|}||(v-(v)_{B_r}(t))\chi_{B_r}||_N(t)\right)^3 + \frac{1}{|Q_r|}\int_{Q_r} |v-(v)_{Q_r}|^3 dz \\ \leq \frac{1}{2}\left(\sup_{t\in(-1,1)} ||v\chi_{B_1}||_N^3(t) + \int_{Q_1} |v|^3 dz\right).\end{aligned} \qquad (2.8)$$

Proof. If the lemma is false, we would have a sequence of Leray-Hopf solution v_i with

$$\epsilon_i = \sup_{t\in(-1,1)} ||v_i\chi_{B_1}||_N^3(t) + \int_{Q_1} |v_i|^3 dz \to 0$$

but (2.8) is not valid. We let $u_i = \frac{v_i}{\epsilon^{1/3}}$ and $q_i = \frac{p_i}{\epsilon^{1/3}}$, they satisfy

$$\frac{\partial}{\partial t}u_i - \Delta u_i + \epsilon^{1/3}(u_i \cdot \nabla)u_i + q_i = 0.$$

Then u_i is bounded in $L^\infty(-1, 1 : L^2(B_1)) \cap L^2(-1, 1 : H^1(B_1))$ and a subsequence converges to w in L^3 strongly. By lower semicontinuity of norms, we have

$$\sup_{t\in(-1,1)} ||w\chi_{B_1}||_N^3(t) + \int_{Q_1} |w|^3 dz \leq 1.$$

Furthermore w is a solution to the Stokes equations in Q_1 and satisfies

$$\sup_{t\in(-\frac{1}{2},\frac{1}{2})} \frac{1}{|B_r|} \int_{B_r} \frac{|w(x,t) - (w)_{B_r}(t)|^2}{r^2} dx < c$$

$$\frac{1}{|Q_r|} \int_{Q_r} \frac{|w - (w)_{Q_r}|^3}{r} dz < c$$

for all $r \in (0, \frac{1}{2})$ and a constant c. Therefore we have for all $t \in (-\frac{1}{2}, \frac{1}{2})$

$$||(w - (w)_{B_r}(t))\chi_{B_r}||_N(t) \leq cr^{1/2}|B_r|.$$

There is r_0 such that for $r < r_0$

$$\sup_{t\in(-r,r)} \left(\frac{r^{1/2}}{|B_r|} ||(w - (w)_{B_r}(t))\chi_{B_r}||_N(t) \right)^3$$

$$+\frac{1}{|Q_r|} \int_{Q_r} |w - (w)_{Q_r}|^3 dz < r^{2/3}.$$

Since u_i, after rearranging indices, converges to w strongly in L^3, we have for sufficiently large i

$$\sup_{t\in(-r,r)}\left(\frac{r^{1/2}}{|B_r|}||(u_i-(u_i)_{B_r}(t))\chi_{B_r}||_N(t)\right)^3 + \frac{1}{|Q_r|}\int_{Q_r}|u_i-(u_i)_{Q_r}|^3dz < r^{1/2}$$

and this contradicts to our assumption.

We prove Theorem 2.2.

Proof. Suppose v is Leray-Hopf solution satisfying

$$\sup_{t\in(-1,1)} ||v\chi_{B_1}||_N^3(t) + \int_{Q_1} |v|^3 dz \le \epsilon$$

for ϵ appearing in Lemma 2.3. We let

$$v_1(x,t) = \frac{v-(v)_{Q_r}}{r^{1/6}}(rx, r^2t), \quad p_1(x,t) = r^{5/6}p(rx, r^2t)$$

where r is the scale in Lemma 2.6. So v_1 is a Leray-Hopf solution to (2.1) in Q_1. Moreover, from Lemma 2.6, v_1 satisfies the energy estimate

$$\sup_{t\in(-1,1)} ||v_1\chi_{B_1}||_N^3(t) + \int_{Q_1} |v_1|^3 dz \le \frac{\epsilon}{2}.$$

We follow the argument of Lemma 2.3 except the constant convection term

$$\frac{\partial}{\partial t}v_1 - \Delta v_1 + (b\cdot\nabla)v_1 + p_1 = 0,$$

$b = \lim_i r(v_i)_{Q_r}$ is constant vector with $|b| \le 1$.

Therefore, after iteration, we prove that for all $Q_\rho \subset Q_{\frac{1}{2}}$

$$\frac{1}{|Q_\rho|}\int_{Q_\rho} |v-(v)_{Q_\rho}|^3 dz \le c\rho$$

for all $\rho < \frac{1}{2}$ and $v \in C^{1/3}(Q_{\frac{1}{2}})$. This ends the proof.

First we note that $||\omega\chi_{B_r}||_N^2(t) \le ||\omega\phi||_N^2(t)$ if $\phi = 1$ in B_r. We define several scale invariant quantities:

$$A_0(r) = \sup_{-r^2 \le t \le r^2} \frac{1}{r}||\omega\chi_{B_r}||_N^2(t), \quad A_1(r) = \frac{1}{r}\int_{Q_r} |\nabla u|^2 dz,$$
$$A_2(r) = \sup_{-r^2 \le t \le r^2} \frac{1}{r}\int_{B_r} |u|^2 dx, \quad A_3(r) = \frac{1}{r^2}\int_{Q_r} |u|^3 dz.$$

We have a lemma from Choe and Lewis [7].

Lemma 2.7. *Suppose $Q_\rho \subset \Omega_T$ and $r < \rho$ and $\frac{9}{5} \le b \le 2$. we have that*

$$A_3(r) \le c(\rho/r)^2 A_1(\rho)A_2(\rho)^{1/2} + c\min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\}$$
$$A_3(r) \le c(\rho/r)^2 A_1(\rho)^{3/4}A_2(\rho)^{3/4} + c\min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\}$$
$$A_3(r) \le cr^{-2}\rho^{2b-3}A_2(\rho)^{(3-b)/2}\int_{Q_\rho} |\nabla u|^b dz$$
$$+ c\min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\}.$$

Theorem 2.8. *There is an absolute constant* ϵ_0 *such that*

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r} |\nabla u|^2 dz \leq \epsilon_0$$

implies that for an $r_0 > 0$

$$\sup_{z \in Q_{r_0}} |u(z)| \leq \frac{c}{r_0}$$

for an absolute constant c.

Proof.

We note that

$$\sup_{s\in(t-r^2,t+r^2)} \frac{1}{r^3} ||u\chi_{B_r}||_N^2(s) \leq cA_2(r)$$

and from Lemma 2.4, we have

$$A_3(r) \leq c\left[\left(\frac{r}{\rho}\right)^3 A_2(\rho)^{3/2} + \left(\frac{\rho}{r}\right)^2 A_1(\rho)^{3/4} A_2(\rho)^{3/4}\right] \quad (2.9)$$

for $0 < r < \frac{\rho}{2}$. From the localized energy inequality (2.3) and (2.4) with scaling, it follows that

$$A_0(r) + A_1(r) \leq c\left[\frac{\rho}{r} A_3^{2/3}(\rho) + \left(\frac{\rho}{r}\right)^2 A_3(\rho)\right] \quad (2.10)$$

for a c. Note that, for the cutoff function ϕ supported in B_ρ,

$$u = \nabla \times N(\omega\phi) + \nabla \times N(u \times \nabla\phi) + \nabla N(u \cdot \nabla\phi)$$

and thus

$$\int_{B_r} |u|^2 dx \leq 3\int_{B_r} |\nabla\times N(\omega\phi)|^2 dx + 3\int_{B_r} |\nabla\times N(u\times\nabla\phi)|^2 dx$$

$$+3\int_{B_r} |\nabla N(u\cdot\nabla\phi)|^2 dx.$$

From the definition of bilinear form, we have for $r \le \frac{\rho}{4}$

$$\int_{B_r} |\nabla \times N(\omega\phi)|^2 dx \le c||\omega\chi_{B_\rho}||_N^2.$$

On the other hand

$$\int_{B_r} |\nabla\times N(u\times\nabla\phi)|^2 dx + \int_{B_r} |\nabla N(u\cdot\nabla\phi)|^2 dx$$
$$\le c\int_{B_r}\left(\frac{1}{\rho}\int_{B_\rho\backslash B_{\rho/2}} \frac{1}{|x-y|^2}|u(y)|dy\right)^2 dx$$
$$\le c\left(\frac{r}{\rho}\right)^3 ||u||_{B_\rho}^2.$$

and for $r \le \frac{\rho}{4}$

$$A_2(r) \le c\frac{\rho}{r}A_0(\rho) + c\left(\frac{r}{\rho}\right)^2 A_2(\rho). \qquad (2.11)$$

Combining (3.9), (2.10) and (2.11), we find that there is a large integer n_0 depending only on ϵ such that

$$A_0(r/n_0) + A_2(r/n_0) \le \frac{1}{2}(A_0(r) + A_2(r)) + \epsilon$$

and after iteration with Theorem 2.2 under scaling we complete the proof.

§2..3 Logarithmic dimension of singular set

In this section, we improve one dimensional Hausdorff measure estimate of singular set by logarithmic factor. Choe and Lewis [7] proved the same theorem for the suitable weak solution and we follow a similar path to improve Hausdorff measure. Nonetheless, we do not need to consider the pressure and the argument becomes much simpler. We adopt the same definition of $A_0(r;x,t), A_1(r;x,t), A_2(r;x,t)$ and $A_3(r;x,t)$ of Lemma 2.7 and we define

$$F_i(M) = \{z \in \Omega_T : \limsup_{r\to 0} A_i(r;z) \le M\}, \quad i = 0,1,2,3.$$

To treat A_2, we need an intrinsic behavior of lim sup of functions(see (2.16) in [7]). If g is a real valued function on $(0,r_0]$ which is bounded on any closed subinterval of $(0,r_0]$ and if $\limsup_{r\to 0} g(r) = \infty$, then there is a decreasing sequence (s_k) converging to zero in $(0,r_0]$ with

$$\begin{aligned} g(s_k) &\to \infty, \quad \text{as} \quad k \to \infty \\ 2g(s_k) &\ge \sup_{s\in[s_k,r_0]} g(s). \end{aligned} \tag{2.12}$$

Theorem 2.8 implies the following theorem:

Theorem 2.9. *The Hausdorff dimension of the singular set S is less than or equal to one and*

$$\Lambda_1(S) = 0.$$

We need an equivalence lemma for $F_i(M)$.

Lemma 2.10. *Suppose $z \in F_i(M)$ for $i = 0, ..., 3$, then there is an absolute constant c such that for all $j = 0, ..., 3$,*

$$z \in F_j(c(M^3 + 1)).$$

Proof. We start with $i = 3$. We assume $M \geq 1$ and r_0 so small that

$$A_i(r) \leq 2M$$

for $r \in (0, r_0)$. From (2.10), we have that with $\rho = kr < r_0$ for a large k

$$A_0(r) + A_1(r) \leq c\left[A_3^{2/3}(kr) + A_3(kr)\right] \leq c(M + 1)$$

whenever $Q_{2r} \subset \Omega_T$ and this proves the cases $i = 0, 1$. Suppose $\limsup A_2 = \infty$. From (2.11) and (2.12), if k is sufficiently large, then

$$A_2(s_n) \leq ck(M + 1) + \frac{1}{2}A_2(s_n)$$

so that $A_2(s_n) \leq 2ck(M + 1)$ and letting $n \to \infty$ we get contradiction. Once $\limsup A_2$ is bounded, it follows that

$$\limsup_{r \to 0} A_2(r) \leq c(M + 1)$$

and $z \in F_2(c(M + 1))$.

Next we prove the case for $i = 2$. From (3.9), we have for $r < r_0$

$$A_3(r) \leq c\left[M^{3/2} + A_1(2r)^{3/4}M^{3/4}\right]$$

and (2.10) implies

$$A_1(r) \leq c\left[M^{3/2} + A_1(2r)^{3/4}M^{3/4}\right].$$

Then, as in the proof of the case $i = 3$, we have $\limsup A_1(r)$ is bounded and

$$\limsup A_1(r) \leq cM^3$$

and thus A_3 estimate follows too. The estimate (2.10) implies

$$\limsup A_0(r) \leq cM^3.$$

We consider the case $i = 1$. First we prove that

$$\limsup_{r\to 0} A_2(r) < \infty.$$

Let s_k be in (2.12) for $g = A_2$ and from Sobolev embedding like (3.9)(also see (2.6a) in [7]) we have that if $\frac{r}{\rho}$ is small and $s_k < r < \rho$, then

$$A_3(r) \leq cMA_2^{1/2}(s_k) + \frac{1}{2}A_3(\rho).$$

Iterating this inequality, for sufficiently large enough k, we have that for $s_k < r$

$$A_3(r) \leq cMA_2^{1/2}(s_k)$$

and (2.10) implies that $s_k < r$

$$A_0(r) \leq cMA_2^{1/2}(s_k).$$

From (2.11), it follows that for $s_k < r < \rho$

$$A_2(r) \le c\frac{\rho}{r} M A_2^{1/2}(s_k) + c\left(\frac{r}{\rho}\right)^2 A_2(\rho)$$

and replacing $r = 2s_k$ and $\rho = 4cs_k$ and iterating we prove that

$$A_2(2s_k) \le cMA_2^{1/2}(s_k)$$

and this contradicts to (2.12). Therefore, setting $\eta_i = \limsup_{r\to 0} A_i(r)$, we have

$$\eta_3 \le cM\eta_2^{1/2} + \frac{1}{2}\eta_3$$
$$\eta_2 \le c\eta_0 + \frac{1}{2}\eta_2$$
$$\eta_0 \le c\eta_3^{2/3} + c\eta_3$$

and we find

$$\eta_0, \eta_2, \eta_3 \le cM^3.$$

Finally we assume that $\limsup_{r\to 0} A_0(r) = M$. Like the other cases, from (2.11) we can show that $\limsup A_2$ is bounded and if $\lim A_0(s_k) = M$ and $\lim A_2(r_k) = \eta_2$, then for $4cr_n < s_k <$ with

$$A_2(r_n) \le 4c^2 A_0(4cr_n) + \frac{1}{2}A_2(4cr_n)$$

and taking limit we have

$$\eta_2 \le cM$$

and η_1 and η_3 are estimated in the same way. This ends proof.

For a given b satisfying $\frac{9}{5} \le b < 2$, we put a scale invariant quantity

$$J_b(r) = J(r, z, b) = r^{2b-5} \int_{Q_r(z)} |\nabla u|^b dz.$$

Lemma 2.11. *Suppose that* $z \in F_i(M) \cap S, i = 1, 2, 3$. *Then there is* $\delta > 0$ *depending on* M *such that*

$$\liminf_{r \to 0} J_b(r) \ge \delta.$$

Proof. Let $N = c(M^3+1)$ and assume all cylinders considered is contained in Ω_T. We suppose that r_1 so small that $A_2(r) \le 2N$ for $r < r_1$ and Lemma 2.7 implies that for $r < \rho < r_1$

$$A_3(r) \le c(\rho/r)^2 J_b(\rho) + c \min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho) A_3(\rho)\}.$$

Hence if $J_b(\rho) < \delta$ for a small δ, then

$$A_3(r) \le c \left(\frac{\rho}{r}\right)^2 \delta + c(r/\rho)^3 N^{3/2}$$

and hence for given $\epsilon_2 > 0$ there are δ and r_1 small enough so that for $r < r_1$

$$A_3(r) \le \epsilon_2.$$

Indeed, we choose small r initially and then choose δ to make $c\left(\frac{\rho}{r}\right)^2 \delta$ small. Also from (2.10), for $r < r_1$

$$A_0(r) + A_1(r) \le c \left[\frac{r_1}{r} \epsilon_2^{2/3} + \left(\frac{r_1}{r}\right)^2 \epsilon_2\right]$$

for a c and from (2.11), we have $r \le \frac{\rho}{4}$

$$A_2(r) \le c\frac{\rho}{r} A_0(\rho) + c \left(\frac{r}{\rho}\right)^2 A_2(\rho).$$

Combining these two estimates, we find that there is a large integer n_0 depending only on ϵ such that

$$A_0(r/n_0) + A_2(r/n_0) \leq \frac{1}{2}(A_0(r) + A_2(r)) + c\epsilon_2^{2/3}$$

and after iteration we have for sufficiently small r

$$\sup_t \frac{1}{r^3} < v\chi_{B_r}, v\chi_{B_r} >_N \leq c\epsilon_2^{2/3}.$$

Therefore if ϵ_2 is small enough, then the condition of Theorem 2.2 is satisfied and z is regular point. This contradicts $z \in S$ if δ is small. This ends the proof.

Define the parabolic distance *dist* by

$$dist(z, A) = \inf\{|x-y| + \sqrt{|t-s|} : z = (x,t), (y,s) \in A\}$$

and denote $S_i(M) = F_i(M) \cap S$.

Proposition 2.12. *There is $\alpha > 0$ depending only on M such that*

$$\int_{\Omega_T} dist(z, S_i(M))^{-\alpha} |\nabla u|^2 dz < \infty$$

for $i = 1, 2, 3$.

Proof. For a fixed i, we let $r(z)$ be the generic radius of z in $S_i(M)$ such that for $0 < r < r(z)$

$$J_b(r) \geq \delta/2 \quad \text{and} \quad A_1(r) \leq 2c(M^3 + 1) \tag{2.13}$$

and since we can assume that $S_i(M)$ is compact, we have a finite covering such that

$$\cup_{z_k \in S_i(M)} Q_{r(z_k)}(z_k) \supset S_i(M).$$

Hence we need only to prove the proposition in a neighborhood $E(r_0) = \{z : dist(z, S_i(M)) < r_0\}$ with $r_0 \leq \min\{r(z_k)/100\}$. Our condition (3.12) implies that if $z \in S_i(M)$, then

$$A_1(r_1) \leq 4N\delta^{-1} J_b(r_2) \quad \text{for all} \quad r_1, r_2 < r_0.$$

We let $\kappa = (3\delta/100)^{1/b}$ and

$$K_1 = K_1(\rho, z) = \{(y,s) \in Q_\rho(z) : |\nabla u(y,s)| \leq \kappa\rho^{-2}\}$$
$$K_2 = K_2(\rho, z) = Q_\rho(z) \setminus K_1,$$

then for $\rho < r_0$, we have

$$\begin{aligned}\delta\rho^{5-2b} &\le \int_{Q_\rho(z)} |\nabla u|^b dz = \int_{K_1} \cdots dz + \int_{K_2} \cdots dz \\ &\le \delta\rho^{5-2b}/4 + \int_{K_2} \cdots dz.\end{aligned}$$

From Hölder inequality and (3.12) we have

$$J_b(\rho) \le cA_1(\rho)^{b/2} \le cN^{b/2}$$

and combining this with the previous two inequalities we see that

$$J_b(\rho) \le cN^{b/2}\delta^{-1}\rho^{2b-5} \int_{K_2} |\nabla u|^b dz.$$

Given $0 < r < r_0$ we have a covering $\{Q_{5r}(z_i)\}$ of $E(r)$ such that

$$\begin{aligned}&z_i \in H_i(M) \\ &E(r) \subset \cup_i Q_{5r}(z_i) \\ &Q_r(z_i) \cap Q_r(z_j) = \emptyset \quad \text{if} \quad i \ne j.\end{aligned}$$

We define $E_1(r) = \{z : |\nabla u(z)| > \kappa r^{-2}\} \cap E(r)$, then

$$\begin{aligned}\int_{E_r} |\nabla u|^2 dz &\le \sum_i \int_{Q_{5r}(z_i)} |\nabla u|^2 dz \\ &\le cN\delta^{-1}r^{2b-4} \sum_i \int_{Q_r(z_i)} |\nabla u|^b dz \\ &\le cN\delta^{-1}r^{2b-4} \sum_i \int_{K_2(r,z_i)} |\nabla u|^b dz \\ &\le cN\delta^{-1}r^{2b-4} \int_{E_1(r)} |\nabla u|^b dz.\end{aligned}$$

Let α be a positive constant specified later and denote $d_n(z) = \min\{dist(z, S_i(M)), \frac{1}{n}\}$. For a large n with $\frac{1}{n} < r_0$, we have that, after changing the order of integration,

$$
\begin{aligned}
&\int_{\frac{1}{n}}^{r_0} r^{-1-\alpha} \left(\int_{E(r)} |\nabla u|^2 dz \right) dr \qquad (2.14)\\
&\quad = \int_{E(r_0)} \frac{1}{\alpha} (d_n(z)^{-\alpha} - r_0^{-\alpha}) |\nabla u|^2 dz \\
&\quad \le c N^{(2+b)/2} \delta^{-2} \int_{\frac{1}{n}}^{r_0} r^{-1-\alpha} \left(\int_{E_1(r)} |\nabla u|^b dz \right) dr \\
&\quad \le c N^{(2+b)/2} \delta^{-2} \frac{1}{4+\alpha-2b} \int_{E(r_0)} \min\{d_n(z)^{-2}, |\nabla u|/\kappa\} |\nabla u|^b dz
\end{aligned}
$$

and clearly

$$\min\{d_n(z)^{-2}, |\nabla u|/\kappa\} |\nabla u|^b \le d_n^{-\alpha} |\nabla u|^2$$

and hence if we choose

$$\alpha = \frac{\kappa^{2-b}(4-b)\delta^2}{c N^{(2+b)/2}},$$

the righthand side of (2.14) can be absorbed into the lefthand side and

$$\int_{E(r_0)} d_n(z)^{-\alpha} |\nabla u|^2 dz \le c r_0^{-\alpha} \int_{E(r_0)} |\nabla u|^2 dz.$$

Sending n to infinity, we prove the proposition.

Setting $h(t) = t^{1-\alpha}$ in (2.2) and reminding the singularity condition in (3.12) with $b = 2$, we have

$$\Psi_{5r}(S_i(M)) \leq \frac{c}{\delta} \sum_i r^{-\alpha} \int_{Q_r(z_i)} |\nabla u|^2 dz$$

$$\leq \frac{c}{\delta} \sum_i \int_{E(r_0)} dist^{-\alpha} |\nabla u|^2 dz.$$

Letting r goes to zero, we have $\Lambda_{1-\alpha}(F_i(M) \cap S) = 0$.

Corollary 2.13. *We fix M. For $b = \frac{9}{5}$ and $\kappa = (3\delta/100)^{1/b}$, we have*

$$\Lambda_{1-\alpha}(F_i(M) \cap S) = 0, \quad \textit{for} \quad i = 1, 2, 3$$

where

$$\alpha = \frac{\kappa^{2-b}(4-b)\delta^2}{c(M^3+1)^{(2+b)/2}}.$$

We start to improve Hausdorff dimension by log factor.

We let $m : (0,1) \to R^+$ be positive monotone decreasing function such that

$$\lim_{r \to 0^+} m(r) = \infty$$

and set

$$F(m) = \{z : \limsup_{r \to 0} \frac{A_1(r)}{m(r)} \leq 1\}.$$

In fact, we need only

$$m(r) = \log(1/r).$$

Lemma 2.14. *If $z \in F(m)$, then there is c such that*

$$\limsup_{r\to 0} \frac{A_2(r)}{m^2(r)} \le c.$$

Proof. We prove by contradiction. We let $A_1(r_0) \le 2m(r_0)$. Set $g(r) = \frac{A_2(r)}{m^2(r)}$ and assume that $\lim g(r) = \infty$. Then there is a sequence r_n in (2.12) such that $2g(r_n) \ge \sup_{r\in[r_n,1]} g(r)$ and $r_n \to 0$. If $0 < r < \rho$ and $r/\rho = \lambda$ is small enough, then from the first inequality of Lemma 2.7 we deduce

$$A_3(r) \le cm(r_n)A_2^{1/2}(r_n) + \frac{1}{2}A_3(\rho).$$

Iterating this inequality we deduce that for large n and $0 < r_n \le r < r_0$

$$A_3(r) \le cm(r_n)A_2^{1/2}(r_n)$$

and from (2.12)

$$A_0(r) \le cA_3^{2/3}(2r) + cA_3(2r) \le cm(r_n)A_2^{1/2}(r_n).$$

From (2.11), we have for $r_n \le r \le \frac{\rho}{4c}$

$$A_2(r) \le cm(r_n)A_2^{1/2}(r_n) + \frac{1}{2}A_2(\rho)$$

and iterating again we have for sufficiently large n

$$A_2(r_n) \le cm(r_n)A_2^{1/2}(r_n).$$

This implies that

$$\frac{A_2(r_n)}{m^2(r_n)} \le c$$

and we conclude the proof.

Lemma 2.15. *Suppose* $9/5 \leq b < 2$*. There is* c *such that for* $z \in F(m) \cap S$

$$\liminf J_b(r)m(r)^{3-b} \geq c$$

Proof. There is r_1 such that $A_1(r) \leq cm(r)$ and $A_2(r) \leq cm^2(r)$ for $0 < r < r_1$. From the third inequality in Lemma 2.7,

$$A_3(r) \leq c(\rho/r)^2 A_2(\rho)^{(3-b)/2} J_b(\rho) + c\min\{(r/\rho)^3 A_2(\rho)^{3/2}, (r/\rho)A_3(\rho)\}$$

and for $r < r_1$ we have

$$A_3(r) \leq c(r_1/r)^2 m(r_1)^{3-b} J_b(r_1) + c(r/r_1)^3 m(r_1)^3.$$

As in the proof of local estimate of localized energy estimate

$$A_2(r) \leq c\frac{r_1}{r} A_0(r_1) + c\left(\frac{r}{r_1}\right)^2 m(r_1)^2$$

and by Theorem 2.5 and the assumption that $z \in S$

$$\epsilon \leq A_2(r)^{3/2} + A_3(r) \leq c(r_1/r)^2 m(r_1)^{3-b} J_b(r_1) + c(r/r_1)^3 m(r_1)^3.$$

After choosing r so small that $c(r/r_1)^3 m(r_1)^3 < \frac{\epsilon}{2}$, we have

$$\frac{\epsilon}{2} \leq c(r_1/r)^2 m(r_1)^{3-b} J_b(r_1).$$

and since r_1 can be arbitrarily small and ϵ is an absolute constant, we conclude that

$$\liminf J_b(r)m(r)^{3-b} \geq c.$$

This ends the proof.

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3 We let $m(t) = \log(1/t)^{\sigma}$ where σ will be specified later. We follow the proof of Proposition 3.4. Let $G_k, k = 1, 2, ...,$ the set of all $z \in F(m) \cap S$ such that for $0 < r < \frac{1}{k}$

$$c \leq m(r)^{3-b} J_b(r) \quad \text{and} \quad A_1(r) \leq 2m(r) \tag{2.15}$$

and then $F(m) \cap S = \cup G_k$. Let $\hat{d}(x,t) = \inf\{|x-y| + |t-s|^{1/2} : (y,s) \in G_k\}$ and we define $\hat{E}(r) = \{z : \hat{d}(z) < r\}$. Since Hausdorff measure is countably subadditive, we fix $r_0 = \frac{1}{k}$ and for $z \in \hat{E}(r_0)$,

$$A_1(r_1) \leq cm(r_1)m(r_2)^{3-b} J_b(r_2) \quad \text{for} \quad 0 < r_1, r_2 \leq r_0.$$

Also if c_1 is large enough and

$$K_1 = K_1(\rho, z, b) = \{z_1 \in Q_\rho(z) : |\nabla u(z_1)| \leq m(\rho)^{(b-3)/b} \frac{1}{c_1 \rho^2}\}$$

$$K_2 = Q_\rho(z) \setminus K_1,$$

then for $\rho \leq r_0$ we get

$$\begin{aligned} \rho^{5-2b} m(\rho)^{b-3} &\leq c \int_{Q_\rho} |\nabla u|^b dz \\ &= c \int_{K_1} \cdots dz + c \int_{K_2} \cdots dz \\ &\leq \frac{1}{2} \rho^{5-2b} m(\rho)^{b-3} + c \int_{K_2} \cdots dz. \end{aligned} \tag{2.16}$$

From Hölder inequality, we have

$$J_b(\rho) \leq cA_1(\rho)^{b/2} \leq cm(\rho)^{b/2}. \tag{2.17}$$

Combining (3.15) and (3.16), we obtain that

$$J_b(\rho) \le c\rho^{2b-5} m(\rho)^{3-b/2} \int_{K_2} |\nabla u|^b dz.$$

Choose the covering $\{Q_{5r}(z_i)\}$ of $\hat{E}(r)$ satisfying the conditions of Proposition 3.4 with H_k, d, E replaced by $G_k, \hat{d}, \hat{E}$. If we set

$$\hat{E}_1(r) = \{z : cm(r)^{(b-3)/b} r^{-2} < |\nabla u(z)|\} \cap \hat{E}(r)$$

then

$$\begin{aligned} \int_{\hat{E}_1(r)} |\nabla u|^2 dz &\le \sum_i \int_{Q_{5r}(z_i)} |\nabla u|^2 dz \qquad (2.18) \\ &\le cr^{2b-4} m(r)^{4-b} \sum_i \int_{Q_r(z_i)} |\nabla u|^b dz \\ &\le cr^{2b-4} m(r)^{(14-3b)/2} \sum_i \int_{K_2(r,z_i,b)} |\nabla u|^b dz \\ &\le cr^{2b-4} m(r)^{(14-3b)/2} \int_{\hat{E}_1(r)} |\nabla u|^b dz. \end{aligned}$$

As in [7] we multiply (3.17) by $\frac{1}{r}$ and integrate from n^{-1} to r_0, and if we set $\hat{d}_n = \max\{\hat{d}, n^{-1}\}$, we have

$$\begin{aligned} \int_{n^{-1}}^{r_0} \frac{1}{r} \left(\int_{\hat{E}(r)} |\nabla u|^2 dz \right) dr &= \int_{\hat{E}(r_0)} \log(r_0/\hat{d}_n) |\nabla u|^2 dz \qquad (2.19) \\ &\le c \int_{n^{-1}}^{r_0} r^{2b-5} m(r)^{(14-3b)/2} \left(\int_{\hat{E}_1(r)} |\nabla u|^b dz \right) dr \\ &\le c \int_{\hat{E}(r_0)} \min \Big\{ |\nabla u|^{2-b} m(|\nabla u|)^{(14-3b)/2+(3-b)/b}, \\ &\qquad \hat{d}_n^{2b-4} m(\hat{d}_n^{-1})^{(14-3b)/2} \Big\} |\nabla u|^b dz. \end{aligned}$$

We take σ so small that

$$\sigma\left((14-3b)/2+(3-b)/b\right)<1,$$

then

$$\min\left\{|\nabla u|^{2-b}m(|\nabla u|)^{(14-3b)/2+(3-b)/b},\hat{d}_n^{2b-4}m(\hat{d}_n^{-1})^{(14-3b)/2}\right\}|\nabla u|^b$$
$$\leq c\log\left(\frac{1}{\hat{d}_n}\right)^{\sigma(14-3b)/2}|\nabla u|^2.$$

The righthand side of (3.18) can be absorbed by the lefthand side and we have

$$\int_{\hat{E}(r_0)}\log(1/\hat{d}_n)|\nabla u|^2dz\leq c(\sigma,b,r_0)$$

and letting $n\to\infty$ with monotone convergence theorem

$$\int_{\hat{E}(r_0)}\log(1/\hat{d})|\nabla u|^2dz\leq\infty. \tag{2.20}$$

Letting $b=2$ in (3.14), we have

$$\Psi_{5r}(G_k,tm(t))\leq c\log(1/r)\sum_i\int_{Q_r(z_i)}|\nabla u|^2dz$$
$$\leq c\int_{E(r_0)}\log(1/\hat{d})|\nabla u|^2dz$$

and letting $r\to 0$ we have $\Lambda(G_k,tm(t))=0$ for all k. Arguing like [7], we also have $\Lambda(S\setminus F(m),tm(t))=0$. This ends the proof.

Is there an isolated singular point

$$z \in F(M)?$$

§3. Boundary regularity for the suitable solutions

§3..1 Introduction

In this section, we prove boundary partial regularity of Navier-Stokes equations for the suitable weak solution. After Scheffer [36] considered the boundary partial regularity of the suitable weak solution, Seregin [37] established a criterion of ϵ regularity similar to [3] on the boundary.

All the previous results rely on the boundary localized energy inequality and higher integrability of pressure like $L^{3/2}$ of the suitable weak solutions. From the definition of the suitable weak solution, the pressure satisfies Poisson equation and is represented by Newtonian potential with density made of velocity. Therefore, we need complicated estimates of pressure in various forms. However, we obtain boundary localized estimates of velocity and pressure from the representation of solution by Green potential for the inhomogeneous Stokes equations due to Solonnikov [40].

Following [3] we introduce the suitable weak solution. Although general boundary geometric conditions are important, we consider merely an initial boundary value problem in the half space $\Omega_T = (0,T) \times R^3_+$ of Navier-Stokes equations:

$$\frac{\partial}{\partial t}u - \Delta u + \mathrm{div}(u \otimes u) + \nabla p = 0, \tag{3.1}$$

$$\mathrm{div} u = 0$$

in $(0,T) \times R^3_+$ with an initial data

$$u(x,0) = u_0(x),$$

where the velocity fields u and u_0 are three dimension solenoidal vector fields and the pressure p is a scalar field.

The terminal time T is not important in our argument. Due to viscosity, u satisfies no slip condition

$$u(x',0,t) = 0 \quad \text{for} \quad x' = (x^1, x^2) \in R^2, t > 0. \tag{3.2}$$

Denote $z = (x,t)$ and $V(\Omega_T) = L^\infty(0,T : L^2(R^3_+)) \cap L^2(0,T : H^1_0(R^3_+))$.

We say $(u,p) \in V \times L^{3/2}(\Omega_T)$ is suitable weak solution to the initial boundary value problem if for all $\phi \in C_0^\infty$

$$\int u\cdot\phi_t dz + \int \nabla u : \nabla\phi dz + \int u\otimes u : \nabla\phi dz - \int p\mathrm{div}\phi dz = 0 \tag{3.3}$$

and u is weakly divergence free for almost all time, satisfies the localized energy inequality for almost all t

$$\int |u(x,t)|^2\phi dx + 2\int_0^t \int |\nabla u|^2 \phi dx ds \tag{3.4}$$
$$\leq \int_0^t \int |u|^2(\phi_t + \Delta\phi) dx ds + \int_0^t \int (|u|^2 + 2p) u \cdot \phi dx ds$$

for all nonnegative $\phi \in C_0^\infty(R^3 \times R_+)$ and

$$\int |u(x,t) - u_0(x)|^2 dx \to 0 \quad as \quad t \to 0,$$

where the initial data u_0 is weakly divergence free in $L^2(R^3_+)$.

For a boundary point $x_0 = (x_0', 0)$ we define a half ball $B_r^+(x_0) = \{x : |x - x_0| < r, x^3 > 0\}$ and a half parabolic cylinder $Q_r^+(x_0, t_0) = B_r^+(x_0) \times (t_0 - r^2, t_0 + r_0^2)$.

The following boundary ϵ regularity theorem is due to Seregin(see Theorem 2.3 in [37]):

Theorem 3.1. *There is an absolute constant ϵ_0 such that*

$$\limsup_{r \to 0} \frac{1}{r} \int_{Q_r^+} |\nabla u|^2 dz \leq \epsilon_0$$

implies that for an r_0

$$\sup_{z \in Q_{r_0}^+} |u(z)| \leq \frac{c}{r_0}$$

for an absolute constant c.

The boundary singular set S is the set of point z in $\{x_3 = 0\} \times (0, T)$ such that u is unbounded in any neighborhood of z. Here, from the countably subadditivity of Hausdorff measure we consider the case when time is greater than a positive constant $\tau > 0$ to avoid initial boundary which is 2 dimensional set.

Theorem 3.2. *For each $\tau > 0$, define $d(z) = \inf\{|x-y| + \sqrt{|t-s|} : (y,s) \in S, s > \tau\}$ and $\tilde{d} = \min\{d, 1\}$. Then, there is c depending only on τ such that*

$$\int \log(e/\tilde{d})|\nabla u|^2 dz$$

and the boundary singular set S satisfies

$$\Lambda(S, t\log(e/t)) = 0.$$

§3..2 Hausdorff measure of boundary singular set

In this section we assume the center of cylinder z_0 is origin, and delete z_0 if obvious. We define several scale invariant quantities:

$$A_1(r) = \frac{1}{r}\int_{Q_r^+} |\nabla u|^2 dz, \quad A_2(r) = \sup_{-r^2 \le t \le r^2} \frac{1}{r}\int_{B_r^+} |u|^2 dx,$$
$$A_3(r) = \frac{1}{r^2}\int_{Q_r^+} |u|^3 dz, \quad A_4(r) = \frac{1}{r^2}\int_{Q_r^+} |p - (p)_r(s)|^{3/2} dz,$$

where $(p)_r(s) = \frac{1}{|B_r^+|}\int_{B_r^+} p(x,s)dx$.

The localized energy inequality is written by the scale invariant terms.

Lemma 3.3. *Suppose $Q_\rho^+ \in \Omega_T$, then there is an absolute constant c such that for $0 < r < \rho$*

$$A_1(r) + A_2(r) \le c\frac{\rho}{r}\left(A_3^{2/3}(\rho) + A_3(\rho) + A_3^{1/3}(\rho)A_4^{2/3}(\rho)\right). \tag{3.5}$$

For the estimate of pressure near the boundary, we follows essentially the theory for the inhomogeneous Stokes equations established by Solonnikov [40]. We consider a cutoff function ϕ in Q_1 and write the localized Navier-Stokes equations as an inhomogeneous Stokes equations

$$(u\phi)_t - \Delta(u\phi) + \nabla(p\phi) = f + g,$$

where f and g are defined as

$$\begin{aligned} f =& u\phi_t - u\Delta\phi + 2\nabla(u\nabla\phi) + p\mathrm{div}\phi + u \otimes u : \nabla\phi \\ g =& -\nabla \cdot (u \otimes u\phi) \end{aligned}$$

and if we let $\phi = 1$ in $Q_{3/4}$, $f = 0$ in $Q^+_{3/4}$.

Thus if (u_1, p_1) is a solution to the inhomogeneous Stokes equations in Q^+_1

$$(u_1)_t - \Delta u_1 + \nabla p_1 = f$$

with zero initial boundary condition, the corresponding pressure p_1 expressed by the Green potential in the half space satisfies for $r < \frac{1}{2}$

$$\int_{Q_r^+} |p_1 - (p_1)_r(t)|^{3/2} \leq cr^5 \left(\int_{Q_1^+} |u|^3 dz \right)^{1/2} \tag{3.6}$$
$$+cr^5 \int_{Q_1^+} |u|^3 dz + cr^5 \int_{Q_1^+} |p - (p)_1(t)|^{3/2} dz$$

and $(u_2, p_2) = (u - u_1, p - p_1)$ satisfies the inhomogeneous Stokes equations in Q^+_1

$$(u_2)_t - \Delta u_2 + \nabla p_2 = \nabla \cdot (u \otimes u\phi)$$

with zero initial boundary condition and $u \otimes u\phi \in L^{3/2}(Q_1^+)$. Hence, by Calderon-Zygmund type Stokes estimate of the Green potential in half space(see [40]), we also have

$$\int_{Q_1^+} |p_2 - (p_2)_1(t)|^{3/2} dz \le c \int_{Q_1^+} |u|^3 dz. \tag{3.7}$$

Therefore from (3.6) and (3.7), we have for $r < \frac{1}{2}$

$$\begin{aligned}\int_{Q_r^+} |p - (p)(t)|^{3/2} &\le cr^5 \left(\int_{Q_1^+} |u|^3 dz \right)^{1/2} \\ &+ cr^5 \int_{Q_1^+} |p - (p)_1(t)|^{3/2} dz + c \int_{Q_1^+} |u|^3 dz\end{aligned}$$

and after scaling a localized pressure estimate follows:

Lemma 3.4. *Suppose that $Q_\rho^+ \in \Omega_T$ and $0 < r < \rho$. Then there is an absolute constant c such that*

$$A_4(r) \le c \left(\frac{r}{\rho} \right)^3 A_3^{1/2}(\rho) + c \left(\frac{r}{\rho} \right)^3 A_4(\rho) + cA_3(\rho) \tag{3.8}$$

We have a proposition for the regularity criterion(see Proposition 2.4 in [37]).

Proposition 3.5. *There is an absolute constant* ϵ_0 *such that if*

$$A_3(r) + A_4(r) \le \epsilon_0,$$

then there is a constant c *such that*

$$ess \sup_{Q^+_{r/2}} |u| \le \frac{c}{r}.$$

We have an interpolation lemma for A_3 from Choe-Lewis [7]. Since the cylinder touches the boundary and $u = 0$ on the boundary, we do not need to worry of the average of u and we have even simpler inequalities. Compare with (2.6) in [7].

Lemma 3.6. *Suppose* $Q^+_\rho \subset \Omega_T$ *and* $r < \rho$ *and* $\frac{9}{5} \le b \le 2$. *we have that*

$$\begin{aligned} A_3(r) &\le c(\rho/r)^2 A_1(\rho) A_2(\rho)^{1/2} \\ A_3(r) &\le c(\rho/r)^2 A_1(\rho)^{3/4} A_2(\rho)^{3/4} \\ A_3(r) &\le c r^{-2} \rho^{2b-3} A_2(\rho)^{(3-b)/2} \int_{Q^+_\rho} |\nabla u|^b dz. \end{aligned} \tag{3.9}$$

We improve one dimensional Hausdorff measure estimate of boundary singular set by logarithmic factor. Choe-Lewis [7] proved the interior theorem and we follow a similar path to improve Hausdorff measure. We define the intermediate sets by

$$F_i(M) = \{z \in \partial\Omega_T \cap \{x_3 = 0\} : \limsup_{r\to 0} A_i(r; z) \le M\}, \quad i = 1, 2, 3.$$

To treat A_2, we need an intrinsic behavior of lim sup of functions(see (2.16) in [7]). If g is a real valued function on $(0, r_0]$ which is bounded on any closed subinterval of $(0, r_0]$ and if $\limsup_{r\to 0} g(r) = \infty$, then there is a decreasing sequence (s_k) converging to zero in $(0, r_0]$ with

$$\begin{aligned} g(s_k) &\to \infty, \quad \text{as} \quad k \to \infty \\ 2g(s_k) &\geq \sup_{s\in[s_k,r_0]} g(s). \end{aligned} \tag{3.10}$$

Theorem 1.2 implies the following theorem:

Theorem 3.7. *The Hausdorff dimension of the boundary singular set S is less than or equal to one and*

$$\Lambda_1(S) = 0.$$

We need an equivalence lemma among $F_i(M)$'s.

Lemma 3.8. *Suppose $z \in F_i(M)$ for $i = 1, 2, 3$, then there is an absolute constant c such that for all $j = 1, 2, 3$,*

$$z \in F_j(c(M^3 + 1)).$$

Proof. We start with $i = 3$. We assume $M \geq 1$ and r_0 so small that

$$A_3(r) \leq 2M$$

for $r \in (0, r_0)$. From (3.8), we have that with $0 < r < \rho < r_0$

$$A_4(r) \leq c\left(\frac{r}{\rho}\right)^3 A_3^{1/2}(\rho) + c\left(\frac{r}{\rho}\right)^3 A_4(\rho) + cA_3(\rho)$$

and we can argue like [7] to get the boundedness of A_4. Then, after choosing r small, we get

$$\limsup A_4 \leq cM.$$

If we take $\rho = 2r$, then (3.5) implies that for all r

$$A_1(r) + A_2(r) \leq c\left(A_3^{2/3}(2r) + A_3(2r) + A_3^{1/3}(2r)A_4^{2/3}(2r)\right)$$

and with the estimate of A_4

$$\limsup A_1 + \limsup A_2 \leq cM.$$

Next we prove the case for $i = 2$. From (3.9), we have for $r < r_0$

$$A_3(r) \leq cA_1(2r)^{3/4}M^{3/4}. \tag{3.11}$$

If A_4 is not bounded, then from (2.12) and (3.8) there is a sequence $s_k \to 0$ and

$$A_4(s_k) \leq c(1 + A_3(\rho))$$

and this contradicts to the unboundedness of A_4. Hence A_4 is bounded and again (3.8) implies for all $\rho < r_0$

$$\limsup A_4(r) \leq cA_3(\rho).$$

If A_3 is not bounded, we have a sequence $\{s_k\}$ satisfying (2.12) for $g = A_3$. From the localized energy inequality (3.5) in Lemma 2.1, we have

$$A_1(\frac{s_k}{2}) + A_2(\frac{s_k}{2}) \leq c\,(1 + A_3(s_k))$$

and (3.11) implies

$$A_3(s_k) \leq cA_3^{3/4}(s_k)M^{3/4}.$$

This contradicts to the unboundedness of A_3. If A_3 is bounded, we let $\{r_k\}$ be

$$\lim_k A_3(r_k) = \limsup A_3$$

and we assume $\limsup A_3 > 1$. Then, again, the localized energy inequality implies that

$$A_1(\frac{r}{2}) + A_2(\frac{r}{2}) \leq c\,(1 + A_3(r)) \leq c\,(1 + \limsup A_3)$$

for all sufficiently small r and (3.11) implies that

$$A_3(r_k) \leq cM^3$$

and considering small r in (3.5) we conclude that

$$\limsup A_1 + \limsup A_2 \leq cM^3.$$

Finally, we consider the case $i = 1$. If A_4 is not bounded, then from (2.12) and (3.8) there is a sequence $s_k \to 0$ for $g = A_4$ such that

$$A_4(s_k) \leq c(1 + A_3(\rho))$$

and this contradicts to the unboundedness of A_4. Hence A_4 is bounded and again (3.8) implies for all $\rho < r_0$

$$\limsup A_4 \le cA_3(\rho).$$

Now we prove that

$$\limsup_{r\to 0} A_2(r) < \infty.$$

We let $\{s_k\}$ be the sequence in (2.12) for $g = A_2$. By Lemma 2.4, we have that for $r < r_0$

$$A_3(r) \le cMA_2^{1/2}(2r).$$

From (3.5) and the estimate for $\limsup A_4$ by A_3, it follows that for $s_k \le r < \rho$

$$A_2(r) \le c\frac{\rho}{r}\left(1 + MA_2^{1/2}(s_k)\right)$$

and replacing $r = s_k$ and $\rho = cs_k$ and iterating we prove that

$$A_2(s_k) \le cMA_2^{1/2}(s_k)$$

and this implies that $\limsup A_2$ is bounded. Hence $\limsup A_2$ and $\limsup A_4$ are bounded and again (3.8) implies for all $\rho < r_0$

$$\limsup A_4(r) \le cA_3(\rho).$$

we have

$$\limsup A_2 \le cM^2.$$

With these estimates and Lemma 2.4, we have

$$\limsup A_3 \le cM^3.$$

This ends the proof.

For a given b satisfying $\frac{9}{5} \le b < 2$, we put a scale invariant quantity

$$J_b(r) = J(r, z, b) = r^{2b-5} \int_{Q_r^+(z)} |\nabla u|^b dz.$$

Set $S_\tau = S \cap \{z \in \partial\Omega_T : t > \tau\}$.

Lemma 3.9. *Suppose that $\tau_0 > 0$ and $z \in F_i(M) \cap S_{\tau_0}, i = 1, 2, 3$. Then there is $\delta > 0$ depending on M such that*

$$\liminf_{r \to 0} J_b(r) \ge \delta.$$

Proof. Let $N = c(M^3 + 1)$ and assume all cylinders Q_r^+ considered is contained in Ω_T. As in the proof of Lemma 2.6, we have

$$\limsup A_4 \le N.$$

We suppose that r_1 so small that $A_2(r) \le 2N$ for $r < r_1$ and Lemma 2.4 implies that for $r < \rho < r_1$

$$A_3(r) \le c(\rho/r)^2 J_b(\rho).$$

Hence if $J_b(\rho) < \delta$ for a small δ, then

$$A_3(r) \le c \left(\frac{\rho}{r}\right)^2 \delta$$

and hence for given $\epsilon_1 > 0$ there are δ and r_2 small enough so that

$$A_3(r_2) \le \epsilon_1.$$

Indeed, we choose small r_2 initially and then choose δ to make $c\left(\frac{\rho}{r_2}\right)^2 \delta$ smaller than ϵ_1. From (3.8), we have that with $0 <$

$r < r_2$

$$A_4(r) \le c\left(\frac{r}{r_2}\right)^3 \epsilon_1^{1/2} + c\left(\frac{r}{r_2}\right)^3 N + c\epsilon_1$$

and if r_3 is small enough, then

$$A_4(r_3) \le c\epsilon_1.$$

Finally, from the definition of A_3,

$$A_3(r_3) \le \left(\frac{r_1}{r_3}\right)^2 A_3(r_1) \le \left(\frac{r_1}{r_3}\right)^2 \epsilon_1$$

and we have

$$A_3(r_3) + A_4(r_3) \le c\epsilon_1 + \left(\frac{r_1}{r_3}\right)^2 \epsilon_1.$$

Therefore if ϵ_1 is so small that

$$c\epsilon_1 + \left(\frac{r_1}{r_3}\right)^2 \epsilon_1 \le \epsilon_0,$$

then z is regular point from Proposition 2.3 and we obtain contradiction.

Define the parabolic distance *dist* by

$$dist(z, A) = \inf\{|x-y| + \sqrt{|t-s|} : z = (x,t), (y,s) \in A\}$$

and denote $S_i(M) = F_i(M) \cap S_{\tau_0}$ for $\tau_0 > 0$. Since the Hausdorff measure is countably subadditive, we need only to consider the case $\tau_0 > 0$ and the appearing cylinders have radii smaller than $\sqrt{\tau_0}$. We follow the argument of [7].

Proposition 3.10. *There is $\alpha > 0$ depending only on M such that*

$$\int_{\Omega_T} dist(z, S_i(M))^{-\alpha} |\nabla u|^2 dz < \infty$$

for $i = 1, 2, 3$.

Proof. For a fixed i, we let $r(z)$ be the generic radius of z in $S_i(M)$ such that for $0 < r < r(z)$

$$J_b(r) \geq \delta/2 \quad \text{and} \quad A_1(r) \leq 2N \tag{3.12}$$

and since we can assume that $S_i(M)$ is compact, we have a finite covering such that

$$\cup_{z_k \in S_i(M)} Q_{r(z_k)}(z_k) \supset S_i(M).$$

Hence we need only to prove the proposition in a neighborhood $E(r_0) = \{z = (x', 0, t) : dist(z, S_i(M)) < r_0, t > \tau_0 > 0\}$ with $r_0 \leq \min\{r(z_k)/100\}$. We extend the boundary set $E(r_0)$ to interior by

$$D(r_0) = \{(x', x^3, t) : (x', 0, t) \in E(r_0), 0 < x^3 < r_0, \tau_0 < t < T\}$$

Our condition (3.12) implies that if $z \in S_i(M)$, then

$$A_1(r_1) \leq 4N\delta^{-1} J_b(r_2) \quad \text{for all} \quad r_1, r_2 < r_0.$$

We let $\kappa = (3\delta/100)^{1/b}$ and

$$K_1 = K_1(\rho, z) = \{(y,s) \in Q_\rho^+(z) : |\nabla u(y,s)| \le \kappa \rho^{-2}\}$$
$$K_2 = K_2(\rho, z) = Q_\rho^+(z) \setminus K_1,$$

then for $\rho < r_0$, we have

$$\delta \rho^{5-2b} \le \int_{Q_\rho^+(z)} |\nabla u|^b dz = \int_{K_1} \cdots dz + \int_{K_2} \cdots dz$$
$$\le \delta \rho^{5-2b}/4 + \int_{K_2} \cdots dz.$$

From Hölder inequality and (3.12) we have

$$J_b(\rho) \le c A_1(\rho)^{b/2} \le c N^{b/2}$$

and combining this with the previous two inequalities we see that

$$J_b(\rho) \le c N^{b/2} \delta^{-1} \rho^{2b-5} \int_{K_2} |\nabla u|^b dz.$$

Given $0 < r < r_0$ we have a covering $\{Q_{5r}(z_i)\}$ of $E(r)$ such that

$$z_i \in S_k(M), k = 1, 2, 3$$
$$E(r) \subset \cup_i Q_{5r}(z_i)$$
$$Q_r(z_i) \cap Q_r(z_j) = \emptyset \quad \text{if} \quad i \ne j.$$

We define $D_1(r) = \{z : |\nabla u(z)| > \kappa r^{-2}\} \cap D(r)$, then

$$\begin{aligned}\int_{D_r} |\nabla u|^2 dz &\le \sum_i \int_{Q^+_{5r}(z_i)} |\nabla u|^2 dz \\ &\le cN\delta^{-1} r^{2b-4} \sum_i \int_{Q^+_r(z_i)} |\nabla u|^b dz \\ &\le cN\delta^{-1} r^{2b-4} \sum_i \int_{K_2(r,z_i)} |\nabla u|^b dz \\ &\le cN\delta^{-1} r^{2b-4} \int_{D_1(r)} |\nabla u|^b dz.\end{aligned}$$

Let α be a positive constant specified later and denote $d_n(z) = \min\{dist(z, S_i(M)), \frac{1}{n}\}$. For a large n with $\frac{1}{n} < r_0$, we have that, after changing the order of integration,

$$\begin{aligned}&\int_{\frac{1}{n}}^{r_0} r^{-1-\alpha} \left(\int_{D(r)} |\nabla u|^2 dz \right) dr \qquad (3.13)\\ &= \int_{D(r_0)} \frac{1}{\alpha} (d_n(z)^{-\alpha} - r_0^{-\alpha}) |\nabla u|^2 dz \\ &\le cN^{(2+b)/2} \delta^{-2} \int_{\frac{1}{n}}^{r_0} r^{-1-\alpha} \left(\int_{D_1(r)} |\nabla u|^b dz \right) dr \\ &\le cN^{(2+b)/2} \delta^{-2} \frac{1}{4+\alpha-2b} \int_{D(r_0)} \min\{d_n(z)^{-2}, |\nabla u|/\kappa\} |\nabla u|^b dz\end{aligned}$$

and clearly

$$\min\{d_n(z)^{-2}, |\nabla u|/\kappa\} |\nabla u|^b \le d_n^{-\alpha} |\nabla u|^2$$

and hence if we choose

$$\alpha = \frac{\kappa^{2-b}(4-b)\delta^2}{cN^{(2+b)/2}},$$

the righthand side of (2.14) can be absorbed into the lefthand side and

$$\int_{D(r_0)} d_n(z)^{-\alpha}|\nabla u|^2 dz \le c r_0^{-\alpha} \int_{D(r_0)} |\nabla u|^2 dz.$$

Sending n to infinity, we prove the proposition.

Setting $h(t) = t^{1-\alpha}$ in (2.2) and reminding the singularity condition in (3.12) with $b = 2$, we have

$$\Psi_{5r}(S_i(M)) \le \frac{c}{\delta} \sum_i r^{-\alpha} \int_{Q_r^+(z_i)} |\nabla u|^2 dz$$
$$\le \frac{c}{\delta} \sum_i \int_{D(r_0)} dist^{-\alpha} |\nabla u|^2 dz.$$

Letting r goes to zero, we have $\Lambda_{1-\alpha}(S_i(M)) = 0$.

Corollary 3.11. *We fix M. For $b = \frac{9}{5}$ and $\kappa = (3\delta/100)^{1/b}$, we have for all $\tau_0 > 0$*

$$\Lambda_{1-\alpha}(F_i(M) \cap S_{\tau_0}) = 0, \quad \textit{for} \quad i = 1, 2, 3$$

where

$$\alpha = \frac{\kappa^{2-b}(4-b)\delta^2}{c(M^3+1)^{(2+b)/2}}.$$

We let $m : (0,1) \to R^+$ be positive monotone decreasing function such that

$$\lim_{r\to 0^+} m(r) = \infty$$

and set

$$F(m) = \{z : \limsup_{r\to 0} \frac{A_1(r)}{m(r)} \leq 1\}.$$

Lemma 3.12. *If $z \in F(m)$, then there is c such that*

$$\limsup_{r\to 0} \frac{A_2(r)}{m^2(r)} \leq c.$$

Proof. We prove by contradiction. We let $A_1(r_0) \leq 2m(r_0)$. Set $g(r) = \frac{A_2(r)}{m^2(r)}$ and assume that $\lim g(r) = \infty$. Then there is a sequence r_n in (2.12) such that $2g(r_n) \geq \sup_{r\in[r_n,1]} g(r)$ and $r_n \to 0$. We deduce with the first inequality in Lemma 2.4 that for large n and $0 < r_n \leq r < r_0$

$$A_3(r) \leq cm(r_n)A_2^{1/2}(r_n)$$

and from Lemma 2.2

$$A_4(r) \leq c\left(\frac{r}{\rho}\right)^3 A_3^{1/2}(\rho) + c\left(\frac{r}{\rho}\right)^3 A_4(\rho). + cA_3(\rho)$$

From our assumption $m(r_n)A_2^{1/2}(r_n)$ goes to infinity and thus for sufficiently large n

$$A_4(r) \leq cm(r_n)A_2^{1/2}(r_n).$$

From the localized energy inequality (3.5), we have for $r_n \leq r \leq \frac{\rho}{4c}$

$$A_2(r) \leq cm(r_n)A_2^{1/2}(r_n)$$

and this implies that

$$\frac{A_2(r_n)}{m^2(r_n)} \le c$$

and we conclude the proof.

Lemma 3.13. *Suppose* $9/5 \le b < 2$*. There is* c *such that for* $z \in F(m) \cap S_{\tau_0}$

$$\liminf J_b(r) m(r)^{3-b} \ge c$$

Proof. From Lemma 2.10, there is r_1 such that $A_1(r) \le cm(r)$ and $A_2(r) \le cm^2(r)$ for $0 < r < r_1$. From the third inequality in Lemma 2.4,

$$A_3(r) \le c(\rho/r)^2 A_2(\rho)^{(3-b)/2} J_b(\rho)$$

and for $r < r_1$ we have

$$A_3(r) \le c(r_1/r)^2 m(r_1)^{3-b} J_b(r_1).$$

From Lemma 2.2, for sufficiently small $r < r_1$ we have

$$A_4(r) \le A_3(r_1).$$

From Proposition 2.3 and the assumption that $z \in S$

$$\epsilon_0 \le A_3(r) + A_4(r) \le c(r_1/r)^2 m(r_1)^{3-b} J_b(r_1).$$

After choosing r so small that $c(r/r_1)^3 m(r_1)^3 < \frac{\epsilon_0}{2}$, we have

$$\frac{\epsilon_0}{2} \le c(r_1/r)^2 m(r_1)^{3-b} J_b(r_1).$$

and since r_1 can be arbitrarily small and ϵ_0 is an absolute constant, we conclude that

$$\liminf J_b(r) m(r)^{3-b} \ge c.$$

Proof of Theorem 3.3. We let $m(t) = \log(1/t)^\sigma$ where σ will be specified later. We follow a similar path to the proof of Proposition 2.8. Let $G_k, k = 1, 2, ...,$ the set of all $z \in F(m) \cap S_{\tau_0}$ such that for $0 < r < \frac{1}{k}$

$$c \leq m(r)^{3-b} J_b(r) \quad \text{and} \quad A_1(r) \leq 2m(r) \tag{3.14}$$

and then $F(m) \cap S_{\tau_0} = \cup G_k$. We need only to consider a sufficiently large k. Let $\hat{d}(x,t) = \inf\{|x-y| + |t-s|^{1/2} : (y,s) \in G_k\}$ and we define $\hat{E}(r) = \{z : \hat{d}(z) < r\}$. Since Hausdorff measure is countably subadditive, we fix $r_0 = \frac{1}{k}$ and for $z \in \hat{E}(r_0)$,

$$A_1(r_1) \leq cm(r_1)m(r_2)^{3-b} J_b(r_2) \quad \text{for} \quad 0 < r_1, r_2 \leq r_0.$$

Also if c_1 is large enough and

$$K_1 = K_1(\rho, z, b) = \{z_1 \in Q_\rho^+(z) : |\nabla u(z_1)| \leq m(\rho)^{(b-3)/b} \frac{1}{c_1 \rho^2}\}$$

$$K_2 = Q_\rho^+(z) \setminus K_1,$$

then for $\rho \leq r_0$ we get

$$\begin{aligned} \rho^{5-2b} m(\rho)^{b-3} &\leq c \int_{Q_\rho^+} |\nabla u|^b dz \\ &= c \int_{K_1} \cdots dz + c \int_{K_2} \cdots dz \\ &\leq \frac{1}{2} \rho^{5-2b} m(\rho)^{b-3} + c \int_{K_2} \cdots dz. \end{aligned} \tag{3.15}$$

From Hölder inequality, we have

$$J_b(\rho) \leq cA_1(\rho)^{b/2} \leq cm(\rho)^{b/2}. \tag{3.16}$$

Combining (3.15) and (3.16), we obtain that

$$J_b(\rho) \le c\rho^{2b-5} m(\rho)^{3-b/2} \int_{K_2} |\nabla u|^b dz.$$

Choose the covering $\{Q^+_{5r}(z_i)\}$ of $\hat{E}(r)$ satisfying the conditions of Proposition 2.8 with d, E replaced by $\hat{d}, \hat{E}$. If we set

$$\hat{E}_1(r) = \{z : cm(r)^{(b-3)/b} r^{-2} < |\nabla u(z)|\} \cap \hat{E}(r)$$

then

$$\begin{aligned}
\int_{\hat{E}_1(r)} |\nabla u|^2 dz &\le \sum_i \int_{Q^+_{5r}(z_i)} |\nabla u|^2 dz \qquad (3.17)\\
&\le cr^{2b-4} m(r)^{4-b} \sum_i \int_{Q^+_r(z_i)} |\nabla u|^b dz\\
&\le cr^{2b-4} m(r)^{(14-3b)/2} \sum_i \int_{K_2(r,z_i,b)} |\nabla u|^b dz\\
&\le cr^{2b-4} m(r)^{(14-3b)/2} \int_{\hat{E}_1(r)} |\nabla u|^b dz.
\end{aligned}$$

As in [7] we multiply (3.17) by $\frac{1}{r}$ and integrate from n^{-1} to

r_0, and if we set $\hat{d}_n = \max\{\hat{d}, n^{-1}\}$, we have

$$\int_{n^{-1}}^{r_0} \frac{1}{r}\left(\int_{\hat{E}(r)} |\nabla u|^2 dz\right) dr = \int_{\hat{E}(r_0)} \log(r_0/\hat{d}_n)|\nabla u|^2 dz \tag{3.18}$$

$$\leq c \int_{n^{-1}}^{r_0} r^{2b-5} m(r)^{(14-3b)/2} \left(\int_{\hat{E}_1(r)} |\nabla u|^b dz\right) dr$$

$$\leq c \int_{\hat{E}(r_0)} \min\{|\nabla u|^{2-b} m(|\nabla u|)^{(14-3b)/2+(3-b)/b}, \tag{3.19}$$

$$\hat{d}_n^{2b-4} m(\hat{d}_n^{-1})^{(14-3b)/2}\}|\nabla u|^b dz.$$

We take σ so small that

$$\sigma\left((14-3b)/2 + (3-b)/b\right) < 1,$$

then

$$\min\left\{|\nabla u|^{2-b} m(|\nabla u|)^{(14-3b)/2+(3-b)/b}, \hat{d}_n^{2b-4} m(\hat{d}_n^{-1})^{(14-3b)/2}\right\}|\nabla u|^b$$

$$\leq c \log\left(\frac{1}{\hat{d}_n}\right)^{\sigma(14-3b)/2} |\nabla u|^2.$$

The righthand side of (3.18) can be absorbed by the lefthand side and we have

$$\int_{\hat{E}(r_0)} \log(1/\hat{d}_n)|\nabla u|^2 dz \leq c(\sigma, b, r_0)$$

and letting $n \to \infty$ with monotone convergence theorem

$$\int_{\hat{E}(r_0)} \log(1/\hat{d})|\nabla u|^2 dz \leq \infty. \tag{3.20}$$

Reminding the definition of Ψ, we have

$$\Psi_{5r}(G_k, t\log(e/t)) \leq c\log(1/r)\sum_i \int_{Q_r^+(z_i)} |\nabla u|^2 dz$$

$$\leq c\int_{E(r_0)} \log(1/\hat{d})|\nabla u|^2 dz$$

and letting $r \to 0$ we have $\Lambda(G_k, t\log(e/t)) = 0$ for all k. Arguing like [7], we also have $\Lambda(S, t\log(e/t)) = 0$.

References

[1] R. Adams, *Sobolev Spaces,* Pure Appl. Math., Academic Press(1975).

[2] R. Brown, *The method of layer potentials for the heat equation in Lipschitz cylinders,* Amer. J. Math., Vol. 111, 339-379(1989).

[3] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math. *35*(1982), 771-831.

[4] H. J. Choe, On the regularity of parabolic equations and obstacle problems with quadratic growth nonlinearities, J. Differential Equations *102*(1993), 101-118.

[5] H. Choe and H. Kozono, *The Stokes problem for Lipschitz domains*, Indiana Univ. Math. J. **51**, no. 5, 12351260(2002).

[6] R. Coifman, A. McIntoshi and Y. Meyer, *L'Intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes,* Ann. of Math., Vol. 116, 361-387(1982).

[7] H. J. Choe and J. Lewis, On the singular set in the Navier-Stokes equations, J. Functional Analysis *175*(2000), 348-369.

[8] 4.P. CONSTANTIN, W. E, and E. S. TITI, Onsager's conjecture on the energy conservation for solutions of Euler's equation, Comm. Math. Phys. 165 (1994), 207?209.

[9] B. Dahlberg and C. Kenig, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domain,* Ann. of Math. (2), Vol.125, 437-465(1987).

[10] B. Dahlberg and C. Kenig, *L^p estimates for the three-dimensional system of elastostatics on Lipschitz domain*, Lecture Notes in Pure and Applied Math. Vol. 122, Dekker, New York 1990.

[11] B. Dahlberg, C. Kenig, J. Pipher and G. Verchota, *Area integral estimates for higher order elliptic equations and systems,* Ann. Inst. Fourier(Grenoble), Vol 47, 1425-1461(1997).

[12] B. Dahlberg, C. Kenig and G. Verchota, *Boundary value problems for the systems of elastostatics in Lipschitz domains*, Duke Math. J., Vol. 57, 795-818(1988).

[13] De Lellis, Camillo; Szekelyhidi, Laszlo, Jr. The Euler equations as a differential inclusion. Ann. of Math. (2) 170 (2009), no. 3, 1417?1436.

[14] P. Deuring and W. von Wahl, *Strong solutions of the Navier-Stokes system in Lipschitz boundary,* Math. Nachr., Vol. 171, 111-148(1995).

[15] E. Fabes, *Layer potential methods for boundary value problems on Lipschitz domain,* Lecture Notes in Math., Vol. 1344, Springer-Verlag, New York, 1987, pp. 55-80.

[16] E. Fabes, C. Kenig and G. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J., Vol. 57, 769-793(1988).

[17] E. Fabes, J. Lewis and N. Riviere, *Boundary value problems for the Navier-Stokes equations,* Amer. J. Math., Vol. 99, 626-668(1977).

[18] Fabes, Eugene; Mendez, Osvaldo; Mitrea, Marius, Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains. J. Funct. Anal. 159 (1998), no. 2, 323?368.

[19] E. Fabes and N. Riviére, *Dirichlet and Neumann problems for the heat equation in C^1-cylinders,* Proceedings Symp. Pure Math., Vol. 35 part 2, AMS(1979).

[20] E. Fabes and S. Salsa, *Estimates of caloric measure and the initial-Dirichlet problem for the heat equation in Lipschitz cylinders,* Trans. AMS, Vol. 279, No. 2, 635-650(1983)

[21] G. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol I,* Springer Tracts in Natural Philosophy, 38(1994).

[22] M. Giaquinta and G. Modica, *Nonlinear systems of the type of the stationary Navier-Stokes system*, J. reine Angew. Math., Vol. 330, 173-214(1982).

[23] Guermond, J.-L. Finite-element-based Faedo-Galerkin weak solutions to the Navier-Stokes equations in the three-dimensional torus are suitable. J. Math. Pures Appl. (9) 85 (2006), no. 3, 451?464.

[24] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr. *4*(1951), 213-231.

[25] Jerison, David; Kenig, Carlos E. The inhomogeneous Dirichlet problem in Lipschitz domains. J. Funct. Anal. 130 (1995), no. 1, 161?219.

[26] B. Jin, Caccioppoli inequality for Stokes equations, personal communications.

[27] K. Kang, *On boundary regularity of the Navier-Stokes equations*, Comm. Partial. Differential Equations, **29**, no 7-8, 955-987(2004).

[28] C. Kenig, *Elliptic boundary value problems on Lipschitz domains,* Beijing Lecture in Harmonic Analysis, Ann. of Math. Study 112, Princeton Press, 131-183(1984).

[29] J. Kinnunen and J. Lewis, Higher integrability for parabolic systems of p-Laplacian type, Duke Math. J. *102*(2000), 253-271.

[30] W. Kratz, *The maximum modulus theorem for the Stokes system in a ball'* Math. Z. **226**, no. 3, 389-403(1997).

[31] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta. Math. *63*(1934) 193-248.

[32] F. Lin, A new proof of the Caffarelli, Kohn and Nirenberg theorem, Comm. Pure Appl. Math. *51*(1998), 241-257.

[33] P. Maremonti and R. Russo, *On the maximum modulus theorem for the Stokes system*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **21**, no. 4, 629643(1994).

[34] V. Mazya and J. Rossmann, *A maximum modulus estimate for solutions of the Navier-Stokes system in domains of polyhedral type*, Math. Nachr. 282, no. 3, 459-469(2009).

[35] J. Pipher and G. Verchota, *The Dirichlet problem in L^p for biharmonic functions on Lipschitz domains*, Amer. J. Math., Vol. 114, 923-972(1992).

[36] V. Scheffer, Hausdorff measure and the Navier-Stokes equations, Comm. Math. Phys. *55*(1977), 97-112

[37] G. A. Seregin, Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary, J. Math. Fluid Mech. *4* (2002), no. 1, 1-29.

[38] Z. Shen, *Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz domains*, Amer. J. Math., Vol. 113, 293-373(1991).

[39] Z. Shen, *A note on the Dirichlet problem for the Stokes system in Lipschitz domains*, Proceedings AMS, Vol. 123, No. 3, 801-811(1995).

[40] Solonnikov, V. A, *Estimates of the solutions of the nonstationary Navier-Stokes system. (Russian) Boundary value problems of mathematical physics and related*

questions in the theory of functions, 7. Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LoMI) 38: 153-231. Translated in J. Soviet Math. 1977, 8: 47-529.

[41] Solonnikov, V. A, *On estimates of the solutions of the non-stationary Stokes problem in anisotroptc Sobolev spaces and on estimates for the resolvent of the Stokes operator*. Russian Math. Surveys 58:2 331-365(2003).

[42] E. Stein, "Singular Integrals and Differentiability Properties of Functions", Princeton Univ. Press, NJ, 1970.

[43] R. Temam, "Navier-Stokes Equations, Theory and Numerical Analysis," North-Holland, Amsterdam/New York, 1977.

[44] W. Varnhorn, *Boundary integral equations and maximum modulus estimates for the Stokes system*, Proc. Appl. Math. Mech. 7, 1042603-1042604 (2007).

[45] G. Verchota, *layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains,* J. Functional Analysis, Vol 59, 572-611(1984).