Some Progress on Fluid Dynamics in Singular Domains

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Content

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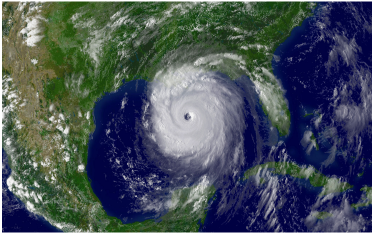
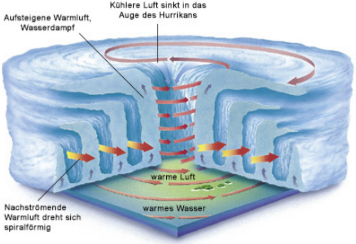
Tornado-Hurricane models

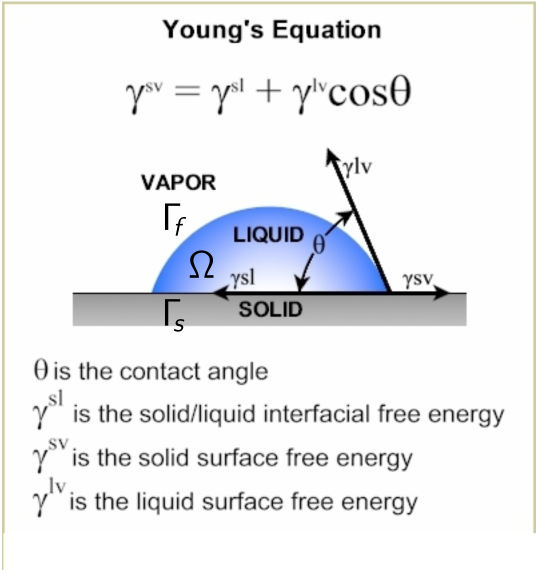




$$
\begin{cases}
\partial_t u - \operatorname{div} D(u) + (u\cdot\nabla)u + \frac{\nabla q}{\rho} + \Omega e_3 \times u - e_3 g \frac{\vartheta - \overline{\vartheta}}{\overline{\vartheta}} = 0 & \text{in } J\times\Omega,\\
\operatorname{div} \rho u = 0 & \text{in } J\times\Omega,\\
\partial_t \vartheta - \nu\Delta\vartheta + (u\cdot\nabla)\vartheta = 0 & \text{in } J\times\Omega,\\
(\alpha^v D(u)n + \beta^v u)_\tau = 0 & \text{on } J\times\partial\Omega,\\
\alpha^\vartheta \partial_n \vartheta + \beta^\vartheta \vartheta = 0 & \text{on } J\times\partial\Omega,\\
n\cdot u = 0 & \text{on } J\times\partial\Omega,\\
u|_{t=0} = u_0 & \text{in } \Omega,\\
\vartheta|_{t=0} = \vartheta_0 & \text{in } \Omega,
\end{cases}
$$

Setting: $\Omega = B(0,r)\times(0,d)$ "cylinder" and "partial slip"

Dynamic contact lines



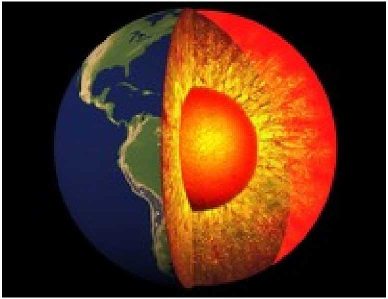
Fundamental model

$$
(CLM)\left\{\begin{array}{rcllll}
\partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla q & = & 0 & \text{in} & (0,T)\times\Omega(t) \\
\nabla\cdot v & = & 0 & \text{in} & (0,T)\times\Omega(t) \\
(T(v,q)\nu - \lambda v)_\tau & = & 0 & \text{auf} & (0,T)\times\Gamma_s \\
\nu\cdot v & = & 0 & \text{auf} & (0,T)\times\Gamma_s \\
V_\nu & = & \nu\cdot v & \text{auf} & (0,T)\times\Gamma_f(t) \\
T(v,q)\nu & = & \sigma\kappa\,\nu & \text{auf} & (0,T)\times\Gamma_f(t) \\
\angle(\Gamma_s,\Gamma_f(t)) & = & \theta & \text{in} & (0,T) \\
v\big|_{t=0} & = & v_0 & \text{in} & \Omega(t) \\
\Gamma_f\big|_{t=0} & = & \Gamma_f^0 & &
\end{array}\right.
$$

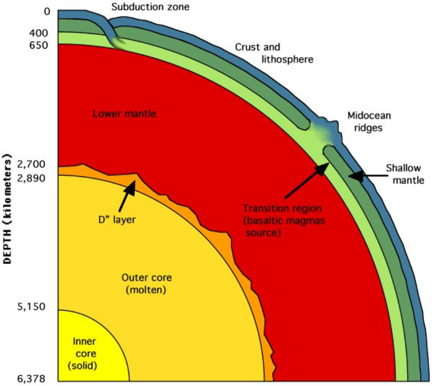
- So far analytically solved only for $\theta\in\{0,\pi/2,\pi\}$!
- Linearization leads to Stokes equations on a "wedge"
- "partial slip" on Γ_s,

  since $\int_{\Omega\cap B(x_c,\varepsilon)}|\nabla u|^2\,dx=\infty$ for Dirichlet bc. and $\theta\notin\{0,\pi\}$.
  (Huh, Scriven '71, Dussan, Davis '74, Hocking '77, Solonnikov '95)

Even rougher boundaries



core of earth



- Leads to Ekman boundary layer with "Lipschitz boundary" (G. Varet '03, J. Math. Pures Appl.)
- occasionally modeled by "partial slip" ($\rightarrow$ Farwig, Komo)

Known results (L^p-theory on singular domains)

- Elliptic and parabolic: Kondrat'ev, Grisvard, Mazya, Bogovskii, Jerison, Kenig, Fabes, Shen, Dauge, Costabel, Nicaise, Mitrea x 3, Monniaux, Gröger, Rehberg, Haller, Grubb, Schulze, Schrohe, Prüß, Simonett, ... etc.

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- Stokes:
  - small Lipschitz bound: Galdi, Simader, Sohr, Farwig, ...
  - angle $\theta = 0, \pi/2, \pi$: Friedman, Velasquez, Solonnikov, Schweizer, Bemelmans, Wilke, ...
  - weakly singular domains, reflecting bc: Köhne, Nau, S.
  - elliptic, 2D, polygon, Dirichlet: Bogovskii, Grisvard, Knüpfer, Masmoudi (Darcy), ...
  - bounded Lipschitz, perfect slip, Hodge setting: Mitrea, Monniaux, ...
  - Lipschitz, Dirichlet "Taylor conjecture": Shen

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Conclusion: There is a lack of results, in particular on

- weakly singular domains: partial slip ($\rightarrow$ tornado-hurricane equ.)
- wedge type domains: parabolic th., partial slip, maximal regularity
- Lipschitz domains: partial slip

Content

Schedule of the mini course

1. Partial slip: why and how?
2. Partial slip in the parabolic situation
   (2.1 Cylinders)
   2.2 Wedges
   2.3 (graph) Lipschitz domains
3. Partial slip in the Stokes situation
4. Related nonlinear problems

Sectoriality and H^∞-calculus

Set $\quad \Sigma_\phi := \{z \in \mathbb{C} \setminus \{0\} : \ |\arg z| < \phi\}.$

Definition

$A : D(A) \subset X \to X$, linear and closed, is sectorial ($A \in S(X)$), if

(i) $\overline{D(A)} = \overline{R(A)} = X$, $N(A) = 0$,

(ii) $\exists \psi \in (0, \pi) : \ \Sigma_\psi \subset \rho(A)$ and $\|z(z-A)^{-1}\|_{L(X)} \le C \quad (z \in \Sigma_\psi)$.

$\phi_A := \sup\{\psi : \ \text{above estimate holds}\}$ "spectral angle".

Set

$$H^\infty(\Sigma_\varphi) = \{h : \Sigma_\varphi \to \mathbb{C} : h \text{ bounded and holomorphic}\}$$

and for $\eta(z) = z/(1+z)^2$

$$H_0^\infty(\Sigma_\varphi) = \{h \in H^\infty(\Sigma_\varphi) : \ \exists C, \varepsilon > 0 \, \forall z \in \Sigma_\varphi : \ |h(z)| \le C|\eta(z)|^\varepsilon\}.$$

Sectoriality and H^∞-calculus

For $A \in S(X)$, $0 < \phi_A < \varphi$, and $h \in H_0^\infty(\Sigma_\varphi)$ set

$$h(A) := \frac{1}{2\pi i} \int_\Gamma h(\lambda)(\lambda - A)^{-1} \mathrm{d}\lambda \in L(X).$$

The map $h \mapsto h(A)$ defines algebra homomorphism.

Definition (McIntosh '86)

$A : D(A) \subset X \to X$, sectorial, admits a bounded H^∞-calculus on X ($A \in H^\infty(X)$), if

$$\|h(A)\|_{\mathscr{L}(X)} \leq C\|h\|_\infty$$

for all $h \in H_0^\infty(\Sigma_\varphi)$. Then estimate holds for all $H^\infty(\Sigma_\varphi)$.
$\phi_A^\infty := \inf\{\varphi : \text{ above estimate holds}\}$.

well-known: $H^\infty \Rightarrow$ BIP $\Rightarrow$ Max. Reg. $\Rightarrow$ analyt. HG

Many examples known: elliptic operators, Stokes operators, etc.

The Kalton-Weis theorem

Theorem (Kalton, Weis '01)

X Banach space with property (α), $A \in \mathcal{H}^\infty(X)$, $\mathcal{T} \subset \mathcal{COM}(A) \subset \mathscr{L}(X)$ $\mathcal{R}$-bounded.

$$\mathcal{RH}^\infty(\Sigma_\sigma, \mathcal{T}) := \{f \in \mathcal{H}^\infty(\Sigma_\sigma, \mathscr{L}(X));\ f(z) \in \mathcal{T}\ (z \in \Sigma_\sigma)\}.$$

Then for $\sigma > \phi_A^\infty$ we have

$$\mathcal{R}(\{f(A);\ f \in \mathcal{RH}^\infty(\Sigma_\sigma, \mathcal{T})\}) < \infty.$$

Especially:

$$\begin{aligned} f(z) &= h(z + B) \\ f(z) &= h(z \cdot B) \qquad (h \in \mathcal{H}^\infty(\Sigma_\sigma)). \end{aligned}$$

Corollary

X Banach space with property (α), $A, B \in \mathcal{H}^\infty(X)$ with $\phi_A^\infty + \phi_B^\infty < \pi$ resolvent commuting. Then

(a) $A + B \in \mathcal{RH}^\infty(X)$ *with* $\phi_{A+B}^{\mathcal{R},\infty} \le \max\{\phi_A^\infty, \phi_B^\infty\}$.

(b) *If* $0 \in \rho(A)$, *then* $AB \in \mathcal{RH}^\infty(X)$ *with* $\phi_{AB}^{\mathcal{R},\infty} \le \phi_A^\infty + \phi_B^\infty$.

The Kalton-Weis theorem (non-commuting version)

Labbas-Terreni commutator condition:

$$\left.\begin{array}{l}\text{Let } 0 \in \rho(A) \text{ and let there exist constants } c > 0,\ 0 \le \alpha < \beta < 1, \\ \psi_A > \phi_A, \psi_B > \phi_B, \psi_A + \psi_B < \pi, \\ \text{ such that for all } \lambda \in \Sigma_{\pi-\psi_A},\ \mu \in \Sigma_{\pi-\psi_B} \text{it holds that} \\ \|A(\lambda + A)^{-1}[A^{-1}, (\mu + B)^{-1}]\| \le c/(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}. \end{array}\right\} \qquad (1)$$

Here $[S, T] = ST - TS$. Then

Theorem (Prüß-Simonett '07)

Let X be a Banach space of class $\mathcal{HT}$ with property (α), let $A, B \in \mathcal{H}^\infty(X)$ and suppose that (1) holds for some angles $\psi_A > \phi_A^\infty$, $\psi_B > \phi_B^\infty$ with $\psi_A + \psi_B < \pi$. Then there exists $\delta \ge 0$ such that $A + B + \delta$ is invertible and such that $A + B + \delta \in \mathcal{RH}^\infty(X)$ with $\phi_{A+B+\delta}^\infty \le \max\{\psi_A, \psi_B\}$. In case that the resolvents commute or if c in (1) is small enough, we can take $\delta = 0$.

An operator sum result of Prüß

Theorem

Suppose the Banach space E belongs to the class $\mathcal{HT}$ and assume

1. $\omega_A + A$, $\omega_B + B \in BIP(E)$ *for some* $\omega_A, \omega_B \in \mathbb{R}$;
2. A *and* B *are resolvent commuting;*
3. $\theta_{A+\omega_A} + \theta_{B+\omega_B} < \pi$.

Then $A + B$ *with domain* $D(A+B) = D(A) \cap D(B)$ *is closed and* $\sigma(A+B) \subset \sigma(A) + \sigma(B)$. *In particular, if* $\sigma(A) \cap \sigma(-B) = \emptyset$ *then* $A + B$ *is invertible.*

Inspection of condition on first Eigenvalue of L_0

$$\lambda_1 < (2 - \frac{2}{p} - \frac{\gamma}{p})^2 \quad \Leftrightarrow$$

$$(2 - \sqrt{\lambda_1})p - 2 < \gamma < (2 + \sqrt{\lambda_1})p - 2$$

Since $\lambda_1 = \min\{1, (\frac{\pi}{\varphi_0} - 1)^2\}$, we have a closer look at the condition

$$\left(3 - \frac{\pi}{\varphi_0}\right) p - 2 < \gamma < \left(1 + \frac{\pi}{\varphi_0}\right) p - 2 \qquad (2)$$

in terms of $\gamma \in \mathbb{R}$, $p \in (1, \infty)$ and the angle $\varphi_0 \in (0, \pi)$. The following tabular displays γ-intervals for some characteristic angles φ_0.

φ_0	$\gamma \in$	$\gamma = 0:\ p \in$
$\varphi_0 \leq \frac{\pi}{2}$	$(p - 2, 3p - 2)$	$(1, 2)$
$\varphi_0 = \frac{3}{4}\pi$	$\left(\frac{5}{3}p - 2, \frac{7}{3}p - 2\right)$	$\left(1, \frac{6}{5}\right)$
$\varphi_0 = (1 - \varepsilon)\pi$	$((3 - 1/(1 - \varepsilon))p - 2, (1 + 1/(1 - \varepsilon))p - 2)$	$\left(1, \frac{2(1-\varepsilon)}{3-1}\right)$

A result of M. Mitrea '04

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a simply connected (graph) Lipschitz domain. Then there exists $p_\Omega \in [1,2)$ such that for all $p \in (1,\infty)$ with $\frac{2}{3}(1-\frac{1}{p_\Omega}) < \frac{1}{p} < \frac{1}{3}(\frac{2}{p_\Omega}+1)$, the problem

$$\begin{cases} u \in L^p(\Omega;\mathbb{R}^3), \mathrm{curl}\, u \in L^p(\Omega;\mathbb{R}^3), \mathrm{curl}\,\mathrm{curl}\, u \in L^p(\Omega;\mathbb{R}^3), \mathrm{div}\, u \in W^{1,p}(\Omega), \\ \Delta u = \eta \in L^p(\Omega;\mathbb{R}^3), \\ \nu \cdot u = 0, \\ \nu \times \mathrm{curl}\, u = g \in B^{p,p}_{-1/p}(\partial\Omega;\mathbb{R}^3), \end{cases}$$

for g such that $g = \nu \times w$ for some $w \in L^p(\Omega;\mathbb{R}^3)$ with $\mathrm{curl}\, w \in L^p(\Omega;\mathbb{R}^3)$ has a unique solution.

Since $p_\Omega \in [1,2)$, the interval $\left[\frac{3}{2}, 3\right]$ is always included. Moreover, if Ω is of class C^1, then $p_\Omega = 1$, hence $p \in (1,\infty)$ is possible.

Off-diagonal estimates

Lemma

$\exists\, C, c > 0\ \forall\, E, F \subset \Omega$ such that $\operatorname{dist}(E, F) > 0\ \forall\, \alpha \in \mathbb{R}$, $\forall\, z \in \Sigma_{\pi/2}\ \forall\, f \in L^2$ we have

$$|z|\|\chi_E u\|_2 + \sqrt{|z|}\Big(\|\chi_E \operatorname{div} u\|_2 + \|\chi_E \operatorname{curl} u\|_2\Big) \le C \exp\left(-c\operatorname{dist}(E, F)\sqrt{|z|}\right)\|\chi_F f\|_2,$$

for $u = (z + A_\alpha)^{-1}(\chi_F f)$ and

$$\|\chi_E \nabla \operatorname{div} u\|_2 + \|\chi_E \operatorname{curl}\operatorname{curl} u\|_2 \le C\left(1 + \frac{|\alpha|}{\sqrt{|z|}}\right)\exp\left(-c\operatorname{dist}(E, F)\sqrt{|z|}\right)\|\chi_F f\|_2.$$

Inequalities in Hodge spaces

Lemma (Poincaré inequality; Mitrea, Monniaux '09)

In a bounded simply connected Lipschitz domain Ω it holds

$$\|u\|_2 \leq C\left(\|\mathrm{curl}\, u\|_2 + \|\mathrm{div}\, u\|_2\right) \quad (u \in V).$$

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Lemma (Sobolev type embedding; Mitrea, Mitrea, Taylor '01)

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a $p_\Omega \in [1,2)$ such that for all $p \in (1,\infty)$ with $\frac{2}{3}(1-\frac{1}{p_\Omega}) < \frac{1}{p} < \frac{1}{3}(\frac{2}{p_\Omega}+1)$ there exists a constant $C > 0$ such that for all $u \in L^p(\Omega;\mathbb{R}^3)$ satisfying $\mathrm{div}\, u \in L^p(\Omega,\mathbb{R})$ and $\mathrm{curl}\, u \in L^p(\Omega;\mathbb{R}^3)$, the estimate

$$\|u\|_{p^*} \leq C\left(\|u\|_{L^p(\Omega)} + \|\mathrm{div}\, u\|_{L^p(\Omega)} + \|\mathrm{curl}\, u\|_{L^p(\Omega)} + \min\left\{\|\nu\cdot u\|_{L^p(\partial\Omega)}, \|\nu\times u\|_{L^p(\partial\Omega)}\right\}\right)$$

holds with $p^* = \frac{3p}{2}$ provided $\nu \cdot u \in L^p(\partial\Omega,\mathbb{R})$ or $\nu \times u \in L^p(\partial\Omega,\mathbb{R}^3)$.

Note: standard Sobolev exponent $p^* = 3p/(3-p)$!

Extrapolating sectoriality to L^p

Theorem (Monniaux, S. '13)

Let $p \in [9/7, 9/2]$. *Then* $\exists C > 0 \; \forall z \in \Sigma_{\pi-\theta} \; \forall f \in L^p_\sigma(\Omega)$*:*

$$|z| \, \|(z + A_\alpha)^{-1} f\|_p \le C \|f\|_p.$$

Furthermore, if $p \in [3/2, 3]$,

$$\sqrt{|z|} \left(\|\mathrm{div}\,(z + A_\alpha)^{-1} f\|_p + \|\mathrm{curl}\,(z + A_\alpha)^{-1} f\|_p \right) \le C \left(1 + \frac{\alpha}{\sqrt{|z|}} \right) \|f\|_p.$$

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$$\sqrt{|z|}\left(\|\mathrm{div}\,(z + A_\alpha)^{-1} f\|_p + \|\mathrm{curl}\,(z + A_\alpha)^{-1} f\|_p\right) \leq C\left(1 + \frac{\alpha}{\sqrt{|z|}}\right)\|f\|_p.$$

Corollary (18)

Let $\alpha \in \mathbb{R}$, $p \in [9/7, 9/2]$. *Then* A_α *generates an analytic* C_0*-semigroup on* $L^p(\Omega)$ *and we have* $A'_\alpha = A_{-\alpha}$, $0 \in \rho(A_\alpha)$, *and (at least) if* $p \in ((3+\varepsilon)', 3+\varepsilon)$ *that*

$$D(A_\alpha) = \left\{u \in V :\ \mathrm{curl}^2 u \in L^2,\ \mathrm{div}\, u \in W^{1,p},\ n \times \mathrm{curl}\, u = \alpha n \times u \text{ on } \partial\Omega\right\}.$$

Assumptions for fixed point argument

Set $P(u \cdot \nabla)u = \sum_{j=1}^{3} \Gamma_j G_j(u)$ with $\Gamma_j u = P\partial_j u$ and $G_j(u) = u^j u$.

(A) The estimate

$$\|e^{-t\mathcal{A}_S} u\|_p \le M \frac{\|u\|_s}{t^\sigma} \quad (u \in L^s_\sigma,\ 0 < t < \infty) \tag{3}$$

holds with $\sigma = \frac{3}{2}(\frac{1}{s} - \frac{1}{p})$, $p \ge s > 1$ and a constant M depending only on p, s.

$(N1)$ The estimate

$$\|e^{-t\mathcal{A}_S} \sum_{j=1}^{3} \Gamma_j u\|_p \le N_1 \frac{\|u\|_p}{t^{1/2}} \quad (u \in C^\infty_{c,\sigma}(G),\ 0 < t < \infty) \tag{4}$$

holds with N_1 depending only on $p \in (1, \infty)$.

$(N2)$ For the nonlinear terms $G_j(u)$ we have $G_j(0) = 0$ and the estimate

$$\|G_j(v) - G_j(w)\|_s \le N_2 \|v - w\|_p (\|v\|_p + \|w\|_p) \quad (j = 1, 2, 3) \tag{5}$$

with $1 \le s = \frac{p}{2}$ and N_2 depending only on p for $1 < p < \infty$.

Abstract result of Y. Giga '86

Theorem (Maier, S. '13)

Let $\Omega \subset \mathbb{R}^3$ be a wedge with angle $\varphi_0 \in (0, \pi)$.

(i) *(Existence and Uniqueness). Suppose $u_0 \in L^r_\sigma(\Omega)$, $r \geq 3$ and conditions (A), (N1), (N2) be satisfied. Then there is $T_0 > 0$ and a unique mild solution of the Navier-Stokes equations on $[0, T_0)$ such that*

$$u \in BC([0, T_0), L^r_\sigma) \cap L^q(0, T_0, L^p_\sigma)$$
$$t^{\frac{1}{q}} u \in BC([0, T_0), L^p_\sigma), \quad t^{\frac{1}{q}} u(t) \to 0 \quad (t \to 0)$$

with $\frac{2}{q} + \frac{3}{p} = \frac{3}{r}$, $q, p > r$. There is a positive constant ε such that if $\|u_0\|_3 < \varepsilon$ then $T_0 = \infty$.

(ii) *(Estimate for the blow-up). Let $(0, T_*)$ be the maximal interval such that u is a solution in $C((0, T_*), L^r_\sigma)$, $r > 3$. Then*

$$\|u(s)\|_r \geq \frac{c}{(T_* - s)^{(r-3)/2r}}$$

with constant $c > 0$ independent of T_* and s.

Main result on Hodge-Navier-Stokes system

Theorem (Monniaux, S. '13)

Let $p \in (3, 3+\varepsilon)$, $u_0 \in L^p_\sigma(\Omega; \mathbb{R}^3)$, $\Omega \subset \mathbb{R}^3$ bounded simply connected (graph) Lipschitz domain.

(a) *There is a $T_0 > 0$ and a unique maximal (mild) solution $u \in C([0, T_0), L^p_\sigma(\Omega; \mathbb{R}^3))$ of the Navier-Stokes equations satisfying*

$$\left(t \mapsto t^{1/2} \operatorname{curl} u(t)\right) \in C\left([0, T_0), L^p_\sigma(\Omega; \mathbb{R}^3)\right),$$

$$\|u(t) - u_0\|_{L^p(\Omega;\mathbb{R}^3)} \to 0 \quad (t \to 0).$$

(b) *There is an $\delta > 0$ such that, if $\|u_0\|_p < \delta$, then the solution u exist globally, that is, in (a) we can admit $T_0 = \infty$. Furthermore, we have the estimate*

$$\|u(t)\|_p + t^{1/2} \|\operatorname{curl} u(t)\|_p \le C e^{-\kappa t} \quad (t \ge 0),$$

hence $u \equiv 0$ is exponentially stable.

Related literature

- S. Maier, J.S., Stokes and Navier-Stokes equations with perfect slip on wedge type domains, to appear.
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- S. Monniaux, J.S., Robin-Hodge boundary conditions for the Navier-Stokes equations in 3D bounded Lipschitz domains, in preparation.
- T. Nau, J.S., H^∞-calculus for cylindrical problems, Adv. Differ. Equ., 2011.
- J. Prüß, G. Simonett, H^∞-calculus for the sum of non-commuting operators, Trans. Amer. Math. Soc., 2007.

ARIGATOU GOZAIMASS !