The spin coating system in the singular limit of vanishing surface tension

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Joint work with Matthias Geissert (TU Darmstadt)

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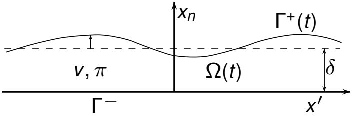
The Navier-Stokes equations with free boundary in an infinite layer

We study the dependence of solutions to the system

$$(NS_\sigma) \begin{cases} \partial_t v - \Delta v + v\nabla v + \nabla\pi = f & \text{in } \Omega(t) \\ \operatorname{div} v = 0 & \text{in } \Omega(t) \\ -S(v,\pi)\nu = \sigma\kappa\nu & \text{on } \Gamma^+(t) \\ V = v\nu & \text{on } \Gamma^+(t) \\ v = 0 & \text{on } \Gamma^- \\ v(0) = v_0 & \text{in } \Omega(0) \end{cases}$$

for $t > 0$ on the surface tension parameter $\sigma \geq 0$, in particular as $\sigma \to 0$.

- For $\sigma = 0$: Abels, Beale, Solonnikov ...
- For $\sigma > 0$: Beale, Denk et al., Shibata and Shimizu, Solonnikov ...



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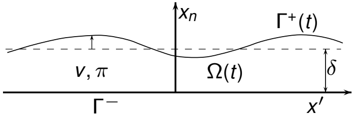

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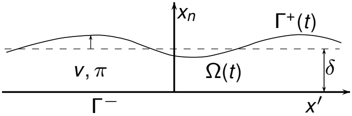

Comparison of known results

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Abels 2005 ($\sigma = 0$)

- Transformation to a fixed domain (Transformation to Lagrange coordinates)
- Analysis of the linear problem
- Fixed point argument to solve the nonlinear problem
- Obtain unique solution of the free boundary problem in L_p setting for $n < p < \infty$ on sufficiently small time intervals.

Denk, Geissert, Hieber, Saal, Sawada 2011 ($\sigma > 0$)

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Main result

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Theorem

Assume

- f, u_0 *satisfy certain compatibility and regularity conditions*
- $\Omega(0)$ *is a flat layer*
- $n \geq 2$, $n + 2 < p < \infty$
- $\sigma^* > 0$.

Then there is $T > 0$ *and* $C > 0$ *such that for* $0 \leq \sigma \leq \sigma^*$ *there is a unique solution* (u, θ) *of* (NS_σ) *with*

$$\|u\|_{H^1_p(0,T;L^p(\Omega_0))\cap L_p(0,T;H^2_p(\Omega_0))} + \|\theta\|_{L_p(0,T;\hat{H}^1_p(\Omega_0))} + \|\gamma\theta\|_{W_p^{1/2-1/2p}(0,T;L^p(\Gamma_0^+))\cap L^p(0,T;W_p^{1-1/p}(\Gamma_0^+))} \leq C\,\|(f, u_0)\|.$$

Here $T > 0$ *and* $C > 0$ *are independent of* σ.

Transformation to a fixed domain: Lagrange coordinates

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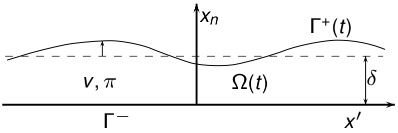
$v(t,x) = u(t,\xi)$ with

$$x = X_u(t,\xi) = \xi + \int_0^t u(\tau,\xi)\,\mathrm{d}\tau$$

and $\pi(t,x) = \theta(t,\xi)$.



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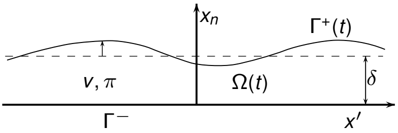

Transformation to a fixed domain: Lagrange coordinates

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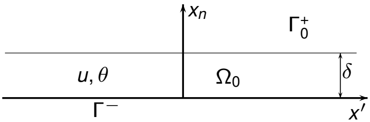
$$\begin{cases} \partial_t u - \Delta u + \nabla\theta = F_1(u,\theta) & \text{in } (0,T)\times\Omega_0 \\ \operatorname{div} u = F_d(u) & \text{in } (0,T)\times\Omega_0 \\ Eu\nu\big|_{\tan} = G^+_{\tan}(u) & \text{on } (0,T)\times\Gamma_0^+ \\ \nu\cdot S(u,\theta)\nu - \sigma(m-\Delta_{\Gamma_0^+})\eta = G^+_n(u,\theta) & \text{on } (0,T)\times\Gamma_0^+ \\ \partial_t\eta - u\cdot\nu = K^+(u) & \text{on } (0,T)\times\Gamma_0^+ \\ u = 0 & \text{on } (0,T)\times\Gamma^- \\ u(0) = u_0 & \text{in } \Omega_0 \\ \eta(0) = 0 & \text{in } \Gamma_0^+ \end{cases}$$

- $u(t,\xi) = v(t,x)$
- $\theta(t,\xi) = \pi(t,x)$
- η artificial function, $m > 0$.



Analysis of the linear problem

We look for solutions of the system

$$\begin{cases} \partial_t u - \Delta u + \nabla\theta = f_1 & \text{in } (0,T)\times\Omega_0 \\ \operatorname{div} u = f_d & \text{in } (0,T)\times\Omega_0 \\ Eu\nu\big|_{\tan} = g^+_{\tan} & \text{on } (0,T)\times\Gamma_0^+ \\ \nu\cdot S(u,\theta)\nu - \sigma(m-\Delta_{\Gamma_0^+})\eta = g_n^+ & \text{on } (0,T)\times\Gamma_0^+ \\ \partial_t\eta - u\cdot\nu = k^+ & \text{on } (0,T)\times\Gamma_0^+ \\ u = 0 & \text{on } (0,T)\times\Gamma^- \\ u(0) = u_0 & \text{in } \Omega_0 \\ \eta(0) = 0 & \text{in } \Gamma_0^+ \end{cases}$$

for data satisfying certain compatibility and regularity assumptions.

Analysis of the linear problem

Theorem

Given

- *data* f_1, f_d, g^+, k^+, u_0 *satisfying certain compatibility and regularity conditions*
- $\sigma^* > 0$, $n < p < \infty$,

for any $\sigma \in [0, \sigma^*]$ *there is a unique solution* (u, θ, η) *of the linear problem satisfying*

$$\|u\|_{H^1_p(L^p(\Omega_0)) \cap L_p(H^2_p(\Omega_0))} + \|\theta\|_{L_p(\hat{H}^1_p(\Omega_0))} + \|\gamma\theta\|_{W_p^{1/2-1/2p}(L^p(\Gamma_0^+)) \cap L^p(W_p^{1-1/p}(\Gamma_0^+))}$$
$$+ \|\eta\|_{H^1_p(J; W_p^{2-1/p}(\Gamma_0^+))} + \sigma \left\|\Delta_{\Gamma_0^+}\eta\right\|_{L^p(W_p^{1-1/p}(\Gamma_0^+))} \leq C \|(f_1, f_d, g^+, k^+, u_0)\|$$

with a constant $C > 0$ *independent of* σ.

Analysis of the linear problem

Ingredients of the proof:

- Derivation of an explicit solution formula: Fourier-Laplace transform, reduction to system of ODEs in normal direction
- Careful analysis of the representation formula and of dependence on σ: Newton polygon method, H^{∞}-calculus, Kalton-Weis theorem

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The nonlinear problem

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- Estimates for the nonlinearities F_1, F_d, G^+, K^+.
- Employ fixed point iteration to solve the nonlinear problem.

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- Generalise our results to the case where $\Omega(0)$ is sufficiently close to a layer.
- Investigate convergence of solutions (u, θ) as $\sigma \to 0$.
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