JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Error estimates for the Peterlin viscoelastic model

Hana Mizerová, Mária Lukáčová

JSPS - DFG IRTG 1529 "Mathematical Fluid Dynamics"

June 18, 2013

Viscoelastic model

Equations describing the motion of an incompressible viscoelastic fluid:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \eta \Delta \mathbf{v} + \text{div } \mathbf{T} - \nabla p \qquad (1)$$

$$\text{div } \mathbf{v} = 0$$

$$\mathbf{T} = \text{tr } \mathbf{C}.\mathbf{C}$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C} - (\nabla \mathbf{v})\mathbf{C} - \mathbf{C}(\nabla \mathbf{v})^T = \text{tr } \mathbf{C}.\mathbf{I} - (\text{tr } \mathbf{C})^2 \mathbf{C} + \epsilon \Delta \mathbf{C} \qquad (2)$$

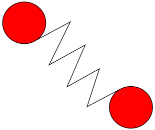
Boundary conditions:

$$\mathbf{v} = \mathbf{0} \qquad \frac{\partial \mathbf{C}}{\partial \mathbf{n}} = \mathbf{0}$$

Initial conditions:

$$\mathbf{v}(0) = \mathbf{v}_0 \qquad \mathbf{C}(0) = \mathbf{C}_0$$

$\mathbf{v}$ = velocity
p = pressure
$\mathbf{T}$ = elastic stress tensor
$\mathbf{C}$ = conformation tensor; positive semidefinite



Michael Renardy: Mathematical Analysis of Viscoelastic Flows

Energy estimates

- multiply (1) by $\mathbf{v}$ and integrate

$$\frac{\rho}{2}\int_\Omega |\mathbf{v}|^2\, d\mathbf{x} \;-\frac{\rho}{2}\int_\Omega |\mathbf{v}_0|^2\, d\mathbf{x} \;+\eta\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\, d\mathbf{x}\, dt \;= \\ = -\int_0^t\int_\Omega \nabla\mathbf{v} : \mathrm{tr}\,\mathbf{C}.\mathbf{C}\, d\mathbf{x}\, dt \qquad (3)$$

- multiply (2) by $\frac{1}{2}\mathrm{tr}\,\mathbf{C}.\mathbf{I}$ and integrate

$$\frac{1}{4}\int_\Omega |\mathrm{tr}\,\mathbf{C}|^2\, d\mathbf{x} \;-\frac{1}{4}\int_\Omega |\mathrm{tr}\,\mathbf{C}_0\,|^2\, d\mathbf{x} + \\ +\frac{\epsilon}{2}\int_0^t\int_\Omega |\nabla\mathrm{tr}\,\mathbf{C}|^2\, d\mathbf{x}\, dt \;+\frac{1}{2}\int_0^t\int_\Omega |\mathrm{tr}\,\mathbf{C}|^4\, d\mathbf{x}\, dt \;= \\ =\frac{1}{2}\int_0^t\int_\Omega \mathrm{tr}\,\mathbf{C}.\mathrm{tr}\left[(\nabla\mathbf{v})\mathbf{C}+\mathbf{C}(\nabla\mathbf{v})^T\right]\, d\mathbf{x}\, dt + \\ +\frac{1}{2}\int_0^t\int_\Omega 3|\mathrm{tr}\,\mathbf{C}|^2\, d\mathbf{x}\, dt \qquad (4)$$

- multiply (1) by $\mathbf{v}$ and integrate

$$\frac{\rho}{2}\int_\Omega |\mathbf{v}|^2\, d\mathbf{x} \; - \frac{\rho}{2}\int_\Omega |\mathbf{v}_0|^2\, d\mathbf{x} \; + \eta\int_0^t\int_\Omega |\nabla \mathbf{v}|^2\, d\mathbf{x}\, dt \; = $$
$$= -\int_0^t\int_\Omega \nabla\mathbf{v} : \mathrm{tr}\,\mathbf{C}.\mathbf{C}\; d\mathbf{x}\; dt \qquad (3)$$

- multiply (2) by $\frac{1}{2}\mathrm{tr}\,\mathbf{C}.\mathbf{I}$ and integrate

$$\frac{1}{4}\int_\Omega |\mathrm{tr}\,\mathbf{C}|^2\, d\mathbf{x} \; - \frac{1}{4}\int_\Omega |\mathrm{tr}\,\mathbf{C}_0\,|^2\, d\mathbf{x} \, +$$
$$+ \frac{\epsilon}{2}\int_0^t\int_\Omega |\nabla \mathrm{tr}\,\mathbf{C}|^2\, d\mathbf{x}\; dt \; + \frac{1}{2}\int_0^t\int_\Omega |\mathrm{tr}\,\mathbf{C}|^4\, d\mathbf{x}\; dt \; =$$
$$= +\frac{1}{2}\int_0^t\int_\Omega \mathrm{tr}\,\mathbf{C}.\mathrm{tr}\left[(\nabla\mathbf{v})\mathbf{C} + \mathbf{C}(\nabla\mathbf{v})^T\right]\; d\mathbf{x}\; dt \, +$$
$$+ \frac{1}{2}\int_0^t\int_\Omega 3|\mathrm{tr}\,\mathbf{C}|^2\; d\mathbf{x}\; dt \qquad (4)$$

$$= 0$$

Thus, adding (3) and (4) yields

$$\frac{\rho}{2}\int_\Omega |\mathbf{v}|^2\, d\mathbf{x} + \frac{1}{4}\int_\Omega |\mathrm{tr}\ \mathbf{C}|^2\, d\mathbf{x} + \eta\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\, d\mathbf{x}\, dt +$$
$$+\frac{\epsilon}{2}\int_0^t\int_\Omega |\nabla \mathrm{tr}\ \mathbf{C}|^2\, d\mathbf{x}\, dt + \frac{1}{2}\int_0^t\int_\Omega |\mathrm{tr}\ \mathbf{C}|^4\, d\mathbf{x}\, dt =$$
$$= \frac{\rho}{2}\int_\Omega |\mathbf{v}_0|^2\, d\mathbf{x} + \frac{1}{4}\int_\Omega |\mathrm{tr}\ \mathbf{C}_0|^2\, d\mathbf{x} + \frac{1}{2}\int_0^t\int_\Omega 3|\mathrm{tr}\ \mathbf{C}|^2\, d\mathbf{x}\, dt,$$

which gives us the a priori bound

$$\mathrm{tr}\ \mathbf{C} \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;L^4(\Omega))$$
$$\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)).$$

Energy estimates

Thus, adding (3) and (4) yields

$$\frac{\rho}{2}\int_\Omega |\mathbf{v}|^2 \, d\mathbf{x} \; + \frac{1}{4}\int_\Omega |\mathrm{tr}\; \mathbf{C}|^2 \, d\mathbf{x} \; + \eta \int_0^t \int_\Omega |\nabla \mathbf{v}|^2 \, d\mathbf{x}\; dt \; +$$
$$+ \frac{\epsilon}{2}\int_0^t \int_\Omega |\nabla \mathrm{tr}\; \mathbf{C}|^2 \, d\mathbf{x}\; dt \; + \frac{1}{2}\int_0^t \int_\Omega |\mathrm{tr}\; \mathbf{C}|^4 \, d\mathbf{x}\; dt \; =$$
$$= \frac{\rho}{2}\int_\Omega |\mathbf{v}_0|^2 \, d\mathbf{x} \; + \frac{1}{4}\int_\Omega |\mathrm{tr}\; \mathbf{C}_0 \;|^2 \, d\mathbf{x} \; + \frac{1}{2}\int_0^t \int_\Omega 3|\mathrm{tr}\; \mathbf{C}|^2 \, d\mathbf{x}\; dt \;,$$

which gives us the a priori bound

$$\mathrm{tr}\; \mathbf{C} \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;L^4(\Omega))$$
$$\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)).$$

Using these bounds and the positive semidefinitness of **C** we also get

$$\mathbf{C} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)) \cap \mathbf{L}^4(0,T;\mathbf{L}^4(\Omega)).$$

Existence of weak solution

Theorem

There exists a weak solution

$$\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,div}(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2_{div}(\Omega))$$
$$\mathbf{C} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)) \cap \mathbf{L}^4(0,T;\mathbf{L}^4(\Omega))$$

satisfying $\mathbf{v}(0) = \mathbf{v}_0,\ \mathbf{C}(0) = \mathbf{C}_0$ *and*

$$\rho \int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{w}\ d\mathbf{x} + \rho \int_\Omega (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}\ d\mathbf{x} + \eta \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w}\ d\mathbf{x} =$$
$$= - \int_\Omega \nabla \mathbf{w} : \mathrm{tr}\ \mathbf{C}.\mathbf{C}\ d\mathbf{x}$$
$$\forall\ \mathbf{w} \in \mathbf{H}^1_{0,div}(\Omega),\ \text{a.e. } t \in (0,T)$$
$$\int_\Omega \frac{\partial \mathbf{C}}{\partial t} : \mathbf{D}\ d\mathbf{x} + \int_\Omega (\mathbf{v} \cdot \nabla)\mathbf{C} : \mathbf{D}\ d\mathbf{x} + \epsilon \int_\Omega \nabla \mathbf{C} : \nabla \mathbf{D}\ d\mathbf{x} =$$
$$= \int_\Omega [(\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T] : \mathbf{D}\ d\mathbf{x} + \int_\Omega [\mathrm{tr}\ \mathbf{C}.\mathbf{I} - (\mathrm{tr}\ \mathbf{C})^2\, \mathbf{C}] : \mathbf{D}\ d\mathbf{x}$$
$$\forall\ \mathbf{D} \in \mathbf{H}^1(\Omega),\ \text{a.e. } t \in (0,T).$$

Proof: the Galerkin approximation and the classical energy estimates

Let us assume

- dimension d=2
- $\mathbf{W}_h$, L_h, $\mathbf{X}_h$ are finite-dimensional subspaces of $\mathbf{W}_0^{1,2}(\Omega)$, $L_0^2(\Omega)$, $\mathbf{W}^{1,2}(\Omega)$ respectively
- $\mathbf{W}_h$ and L_h satisfy the inf-sup condition:

$$\forall \eta_h \in L_h : \quad \sup_{0 \neq \mathbf{v}_h \in \mathbf{W}_h} \frac{(\eta_h, \operatorname{div} \mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,2}} \geq c\|\eta_h\|_2$$

- there exists an interpolation operator

  $\Pi_h^v : \mathbf{W}_0^{1,2}(\Omega) \to W_h$ such that

$$\forall\ \mathbf{w} \in \mathbf{W}_0^{1,2}(\Omega) \cap \mathbf{W}^{l+1,2}(\Omega), 1 \leq l \leq k, r \in \{0,1\} :$$
$$\|\Pi_h^v \mathbf{w} - \mathbf{w}\|_{r,2} \leq Ch^{l+1-r}\|\mathbf{w}\|_{l+1,2}$$

$$\forall\ \mathbf{w} \in \mathbf{W}_0^{1,2}(\Omega)\ \forall\ q_h \in L_h : (q_h, \operatorname{div}\ \Pi_h^v \mathbf{w}) = (q_h, \operatorname{div}\ \mathbf{w})$$

Let us assume

- there exists an interpolation operator

  $\Pi_h^p : L_0^2(\Omega) \to L_h$ such that

  $$\forall\ q \in L_0^2(\Omega) \cap W^{l+1,2}(\Omega), 1 \le l \le k, r \in \{0,1\} :$$
  $$\|\Pi_h^p q - q\|_{r,2} \le C h^{l+1-r} \|q\|_{l+1,2},$$

- there exists an interpolation operator preserving the positive semi-definitness

  $\Pi_h^C : \mathbf{W}^{1,2}(\Omega) \to X_h$ such that

  $$\forall\ \mathbf{D} \in \mathbf{W}^{1,2}(\Omega) \cap \mathbf{W}^{l+1,2}(\Omega),\ 1 \le l \le k, r \in \{0,1\} :$$
  $$\|\Pi_h^v \mathbf{D} - \mathbf{D}\|_{r,2} \le C h^{l+1-r} \|\mathbf{D}\|_{l+1,2},$$

- $h \in (0,1)$ is the mesh size

Let $(\mathbf{v}, p, \mathbf{C})$ be such that

$$\mathbf{v} \in \mathbf{L}^2(0, T; \mathbf{W}_0^{1,2}(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)),$$
$$p \in L^2(0, T; L_0^2(\Omega))$$
$$\mathbf{C} \in \mathbf{L}^2(0, T; \mathbf{W}^{1,2}(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^4(0, T; \mathbf{L}^4(\Omega))$$

satisfying $\mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{C}(0) = \mathbf{C}_0$ and

$$\int_0^T \int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \varphi \; d\mathbf{x} \; dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \varphi \; d\mathbf{x} \; dt + \int_0^T \int_\Omega \nabla \mathbf{v} : \nabla \varphi \; d\mathbf{x} \; dt =$$
$$= - \int_0^T \int_\Omega \nabla \varphi : \mathrm{tr}\, \mathbf{C}.\mathbf{C} \; d\mathbf{x} \; dt + \int_0^T \int_\Omega p \, \mathrm{div}\, \varphi \; d\mathbf{x} \; dt$$
$$\int_\Omega \mathrm{div}\, \mathbf{v}.\psi \; d\mathbf{x} = 0$$
$$\int_0^T \int_\Omega \frac{\partial \mathbf{C}}{\partial t} : \xi \; d\mathbf{x} \; dt + \int_0^T \int_\Omega (\mathbf{v} \cdot \nabla)\mathbf{C} : \xi \; d\mathbf{x} \; dt + \epsilon \int_0^T \int_\Omega \nabla \mathbf{C} : \nabla \xi \; d\mathbf{x} \; dt =$$
$$= \int_0^T \int_\Omega \left((\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T \right) : \xi \; d\mathbf{x} \; dt + \int_0^T \int_\Omega \left(\mathrm{tr}\, \mathbf{C}.\mathbf{I} - (\mathrm{tr}\, \mathbf{C})^2 \, \mathbf{C} \right) : \xi \; d\mathbf{x} \; dt$$

$$\forall \; \varphi \in \mathbf{L}^2(0, T; \mathbf{W}_0^{1,2}(\Omega)), \; \forall \; \psi \in L^2(0, T; L_0^2(\Omega)), \; \forall \; \xi \in \mathbf{L}^2(0, T; \mathbf{W}^{1,2}(\Omega)).$$

Find $(\mathbf{v}_h, p_h, \mathbf{C}_h) \in \mathbf{L}^2(0, T; \mathbf{W}_h) \times L^2(0, T; L_h) \times \mathbf{L}^2(0, T; \mathbf{X}_h)$ satisfying

$\mathbf{v}_h(0) = \Pi_h^v \mathbf{v}_0, \quad \mathbf{C}_h(0) = \Pi_h^C \mathbf{C}_0$ and

$$\int_0^T \int_\Omega \frac{\partial \mathbf{v}_h}{\partial t} \cdot \varphi_h \, d\mathbf{x}\, dt + \int_0^T \int_\Omega (\mathbf{v}_h \cdot \nabla)\mathbf{v}_h \cdot \varphi_h \, d\mathbf{x}\, dt + \int_0^T \int_\Omega \nabla \mathbf{v}_h : \nabla \varphi_h \, d\mathbf{x}\, dt =$$

$$= - \int_0^T \int_\Omega \nabla \varphi_h : \mathrm{tr}\, \mathbf{C}_h . \mathbf{C}_h \, d\mathbf{x}\, dt + \int_0^T \int_\Omega p \,\mathrm{div}\, \varphi_h \, d\mathbf{x}\, dt$$

$$\int_\Omega \mathrm{div}\, \mathbf{v}_h . \psi_h \, d\mathbf{x} = 0$$

$$\int_0^T \int_\Omega \frac{\partial \mathbf{C}_h}{\partial t} : \xi_h \, d\mathbf{x}\, dt + \int_0^T \int_\Omega (\mathbf{v}_h \cdot \nabla)\mathbf{C}_h : \xi_h \, d\mathbf{x}\, dt + \epsilon \int_0^T \int_\Omega \nabla \mathbf{C}_h : \nabla \xi_h \, d\mathbf{x}\, dt =$$

$$= \int_0^T \int_\Omega \left((\nabla \mathbf{v}_h)\mathbf{C}_h + \mathbf{C}_h (\nabla \mathbf{v}_h)^T \right) : \xi_h \, d\mathbf{x}\, dt + \int_0^T \int_\Omega \left(\mathrm{tr}\, \mathbf{C}_h . \mathbf{I} - (\mathrm{tr}\, \mathbf{C}_h)^2 \, \mathbf{C}_h \right) : \xi_h \, d\mathbf{x}\, dt$$

$\forall\ \varphi_h \in \mathbf{L}^2(0, T; \mathbf{W}_h)$, $\forall\ \psi_h \in L^2(0, T; L_h)$ and $\forall\ \xi_h \in \mathbf{L}^2(0, T; \mathbf{X}_h)$.

Theorem

Let $(\mathbf{v}, p, \mathbf{C})$ *be the weak solution (WS) satisfying additionally the following assumptions*

$$\begin{aligned}
&\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{W}^{2,2}(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{W}^{1,2}(\Omega))\\
&\mathbf{v}_t \in \mathbf{L}^2(0,T;\mathbf{W}^{1,2}(\Omega))\\
&\mathbf{C} \in \mathbf{L}^2(0,T;\mathbf{W}^{2,2}(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{W}^{1,2}(\Omega))\\
&\mathbf{C}_t \in \mathbf{L}^2(0,T;\mathbf{W}^{1,2}(\Omega))\\
&p \in L^2(0,T;W^{1,2}(\Omega)).
\end{aligned}$$

Let the set of finite-dimensional spaces $\{(\mathbf{W}_h, L_h, \mathbf{X}_h)\}_{h>0}$ *satisfy the above hypothesis. Then there is a constant* $C > 0$ *independent of* h *such that*

$$\sup_{\tau\in(0,T)} \left(\|\mathbf{v}(\tau,\cdot) - \mathbf{v}_h(\tau,\cdot)\|_2^2 + \|\mathbf{C}(\tau,\cdot) - \mathbf{C}_h(\tau,\cdot)\|_2^2 \right) + \\
+ \int_0^T \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_2^2 + \|\nabla(\mathbf{C} - \mathbf{C}_h)\|_2^2 \, dt \; + \int_0^T \|p - p_h\|_2^2 \, dt \; \leq Ch^2,$$

where $(\mathbf{v}_h, p_h, \mathbf{C}_h)$ *is the approximate solution (AS).*

Proof: We start by writing

$$\frac{1}{2}\left(\|\delta_v(T)\|_2^2 - \|\delta_v(0)\|_2^2\right) + \frac{1}{4}\left(\|\mathrm{tr}\ \delta_C(T)\|_2^2 - \|\mathrm{tr}\ \delta_C(0)\|_2^2\right) +$$
$$+ \int_0^T \|\nabla\delta_v\|_2^2\ dt\ + \frac{\epsilon}{2}\int_0^T \|\nabla \mathrm{tr}\ \delta_C\|_2^2\ dt\ =$$
$$= \int_0^T (\partial_t\delta_v, \delta_v) + (\nabla\delta_v, \nabla\delta_v) + \left(\partial_t\delta_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla\delta_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ =$$
$$= \int_0^T (\partial_t e_v, \delta_v) + (\nabla e_v, \nabla\delta_v) + \left(\partial_t e_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla e_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ -$$
$$- \int_0^T (\partial_t\eta_v, \delta_v) + (\nabla\eta_v, \nabla\delta_v) + \left(\partial_t\eta_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla\eta_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ ,$$

where

$$e_v = \mathbf{v} - \mathbf{v}_h = (\mathbf{v} - \Pi_h^v\mathbf{v}) + (\Pi_h^v\mathbf{v} - \mathbf{v}_h) = \eta_v + \delta_v$$

and similarly for e_p, e_C.

Proof: We start by writing

$$\frac{1}{2}\left(\|\delta_v(T)\|_2^2 - \overset{=0}{\|\delta_v(0)\|_2^2}\right) + \frac{1}{4}\left(\|\mathrm{tr}\ \delta_C(T)\|_2^2 - \|\mathrm{tr}\ \overset{=0}{\delta_C(0)\|_2^2}\right) +$$
$$+ \int_0^T \|\nabla \delta_v\|_2^2\ dt\ + \frac{\epsilon}{2}\int_0^T \|\nabla \mathrm{tr}\ \delta_C\|_2^2\ dt\ =$$
$$= \int_0^T (\partial_t \delta_v, \delta_v) + (\nabla \delta_v, \nabla \delta_v) + \left(\partial_t \delta_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla \delta_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ =$$
$$= \int_0^T (\partial_t e_v, \delta_v) + (\nabla e_v, \nabla \delta_v) + \left(\partial_t e_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla e_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ -$$
$$- \int_0^T (\partial_t \eta_v, \delta_v) + (\nabla \eta_v, \nabla \delta_v) + \left(\partial_t \eta_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla \eta_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ ,$$

where

$$e_v = \mathbf{v} - \mathbf{v}_h = (\mathbf{v} - \Pi_h^v \mathbf{v}) + (\Pi_h^v \mathbf{v} - \mathbf{v}_h) = \eta_v + \delta_v$$

and similarly for e_p, e_C.

Proof: We start by writing

$$\frac{1}{2}\|\delta_v(T)\|_2^2 + \frac{1}{4}\|\text{tr }\delta_C(T)\|_2^2 +$$
$$+\int_0^T \|\nabla\delta_v\|_2^2\, dt + \epsilon\int_0^T \|\nabla \text{tr }\delta_C\|_2^2\, dt =$$
$$= \int_0^T (\partial_t\delta_v,\delta_v) + (\nabla\delta_v,\nabla\delta_v) + \left(\partial_t\delta_C, \frac{1}{2}\text{tr }\delta_C.\mathbf{I}\right) + \epsilon\left(\nabla\delta_C, \nabla\left(\frac{1}{2}\text{tr }\delta_C.\mathbf{I}\right)\right) dt =$$
$$= \int_0^T (\partial_t e_v,\delta_v) + (\nabla e_v,\nabla\delta_v) + \left(\partial_t e_C, \frac{1}{2}\text{tr }\delta_C.\mathbf{I}\right) + \epsilon\left(\nabla e_C, \nabla\left(\frac{1}{2}\text{tr }\delta_C.\mathbf{I}\right)\right) dt -$$
$$-\int_0^T (\partial_t\eta_v,\delta_v) + (\nabla\eta_v,\nabla\delta_v) + \left(\partial_t\eta_C, \frac{1}{2}\text{tr }\delta_C.\mathbf{I}\right) + \epsilon\left(\nabla\eta_C, \nabla\left(\frac{1}{2}\text{tr }\delta_C.\mathbf{I}\right)\right) dt \quad ,$$

where
$$e_v = \mathbf{v} - \mathbf{v}_h = (\mathbf{v} - \Pi_h^v\mathbf{v}) + (\Pi_h^v\mathbf{v} - \mathbf{v}_h) = \eta_v + \delta_v$$

and similarly for e_p, e_C.

Proof: We start by writing

$$\frac{1}{2}\|\delta_v(T)\|_2^2 + \frac{1}{4}\|\mathrm{tr}\ \delta_C(T)\|_2^2 +$$
$$+ \int_0^T \|\nabla\delta_v\|_2^2\ dt\ + \epsilon\int_0^T \|\nabla \mathrm{tr}\ \delta_C\|_2^2\ dt\ =$$
$$= \int_0^T (\partial_t\delta_v, \delta_v) + (\nabla\delta_v, \nabla\delta_v) + \left(\partial_t\delta_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla\delta_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ =$$
$$= \int_0^T (\partial_t e_v, \delta_v) + (\nabla e_v, \nabla\delta_v) + \left(\partial_t e_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla e_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ -$$
$$- \int_0^T (\partial_t\eta_v, \delta_v) + (\nabla\eta_v, \nabla\delta_v) + \left(\partial_t\eta_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon\left(\nabla\eta_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ ,$$

where

$$e_v = \mathbf{v} - \mathbf{v}_h = (\mathbf{v} - \Pi_h^v\mathbf{v}) + (\Pi_h^v\mathbf{v} - \mathbf{v}_h) = \eta_v + \delta_v$$

and similarly for e_p, e_C.

Taking the difference of (WS) and (AS) allows us to write:

$$
\begin{aligned}
&\int_0^T (\partial_t e_v, \delta_v) + (\nabla e_v, \nabla \delta_v)\ dt\ + \\
&+ \int_0^T \left(\partial_t \mathrm{tr}\ e_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right) + \epsilon \left(\nabla \mathrm{tr}\ e_C, \nabla\left(\frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\right)\ dt\ = \\
&= -\int_0^T ((\mathbf{v}\cdot\nabla)e_v + (e_v\cdot\nabla)\mathbf{v}_h, \delta_v)\ dt\ - \\
&- \int_0^T \left((\mathbf{v}\cdot\nabla)e_C + (e_v\cdot\nabla)\mathbf{C}_h, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\ dt\ - \\
&- \int_0^T (\mathrm{tr}\ e_C.\mathbf{C} + \mathrm{tr}\ \mathbf{C}_h.e_C, \nabla\delta_v)\ dt\ + \int_0^T \left(\mathrm{tr}\ e_C.\mathbf{I}, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\ dt\ - \\
&- \int_0^T \left(\mathrm{tr}\ e_C.\mathrm{tr}\ \mathbf{C}.\mathbf{C} + \mathrm{tr}\ \mathbf{C}_h.\mathrm{tr}\ e_C.\mathbf{C} + (\mathrm{tr}\ \mathbf{C}_h)^2\, e_C, \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\ dt\ + \\
&+ \int_0^T \left((\nabla e_v)\mathbf{C} + \mathbf{C}(\nabla e_v)^T + (\nabla \mathbf{v}_h)e_C + e_C(\nabla\mathbf{v}_h), \frac{1}{2}\mathrm{tr}\ \delta_C.\mathbf{I}\right)\ dt\ + \\
&+ \int_0^T (e_p, \mathrm{div}\ \delta_v)\ dt
\end{aligned}
$$

By estimating the integrals we get:

$$\frac{1}{2}\|\delta_v(\tau)\|_2^2 + \frac{1}{4}\|\mathrm{tr}\ \delta_C(\tau)\|_2^2 + (1 - C_0\alpha)\int_0^\tau \|\nabla\delta_v\|_2^2 + \|\nabla\mathrm{tr}\ \delta_C\|_2^2 \leq$$

$$\leq C_1 h^2 + C_2 \int_0^\tau \left(\|\delta_v\|_2^2 + \|\mathrm{tr}\ \delta_C\|_2^2\right) b(s),$$

where $C_i > 0$ are constants and $b(s)$ is integrable.
We choose $\alpha > 0$ to be small enough.

By estimating the integrals we get:

$$\frac{1}{2}\|\delta_v(\tau)\|_2^2 + \frac{1}{4}\|\mathrm{tr}\ \delta_C(\tau)\|_2^2 + (1 - C_0\alpha)\int_0^\tau \|\nabla\delta_v\|_2^2 + \|\nabla \mathrm{tr}\ \delta_C\|_2^2 \leq$$
$$\leq C_1 h^2 + C_2 \int_0^\tau \left(\|\delta_v\|_2^2 + \|\mathrm{tr}\ \delta_C\|_2^2\right) b(s),$$

where $C_i > 0$ are constants and $b(s)$ is integrable.
We choose $\alpha > 0$ to be small enough.

The Gronwall inequality thus yields:

$$\sup_{\tau\in(0,T)} \left(\|\delta_v(\tau)\|_2^2 + \|\mathrm{tr}\ \delta_C(\tau)\|_2^2\right) + \int_0^T \|\nabla\delta_v\|_2^2 + \|\nabla \mathrm{tr}\ \delta_C\|_2^2\, dt \ \leq Ch^2.$$

By estimating the integrals we get:

$$\frac{1}{2}\|\delta_v(\tau)\|_2^2 + \frac{1}{4}\|\mathrm{tr}\ \delta_C(\tau)\|_2^2 + (1 - C_0\alpha)\int_0^\tau \|\nabla\delta_v\|_2^2 + \|\nabla \mathrm{tr}\ \delta_C\|_2^2 \leq$$

$$\leq C_1 h^2 + C_2 \int_0^\tau \left(\|\delta_v\|_2^2 + \|\mathrm{tr}\ \delta_C\|_2^2\right) b(s),$$

where $C_i > 0$ are constants and $b(s)$ is integrable.
We choose $\alpha > 0$ to be small enough.

The Gronwall inequality thus yields:

$$\sup_{\tau\in(0,T)} \left(\|\delta_v(\tau)\|_2^2 + \|\mathrm{tr}\ \delta_C(\tau)\|_2^2\right) + \int_0^T \|\nabla\delta_v\|_2^2 + \|\nabla \mathrm{tr}\ \delta_C\|_2^2\ dt \ \leq Ch^2.$$

The estimate

$$\sup_{\tau\in(0,T)} \|\delta_C(\tau)\|_2^2 + \int_0^T \|\nabla\delta_C\|_2^2\ dt \ \leq Ch^2$$

is found similarly using the above result.

Since

$$(\delta_p, \text{div } \delta_v) = -(\eta_p, \text{div } \delta_v) + (\partial_t e_v, \delta_v) + (\nabla e_v, \nabla \delta_v) + \\ + ((\mathbf{v} \cdot \nabla) e_v, \delta_v) + ((e_v \cdot \nabla)\mathbf{v}_h, \delta_v) + \\ + (\text{tr } e_C . \mathbf{C}, \nabla \delta_v) + (\text{tr } \mathbf{C}_h . e_C, \nabla \delta_v),$$

the error estimate for the pressure is obtained using the inf-sup condition, i.e.

$$c\|\delta_p\|_2 \leq \sup_{0 \neq \delta_v \in \mathbf{W}_h} \frac{(\delta_p, \text{div } \delta_v)}{\|\delta_v\|_{1,2}}$$

and the already known error estimates for **v** and **C**.

Since

$$(\delta_p, \text{div } \delta_v) = -(\eta_p, \text{div } \delta_v) + (\partial_t e_v, \delta_v) + (\nabla e_v, \nabla \delta_v) + \\ + ((\mathbf{v} \cdot \nabla) e_v, \delta_v) + ((e_v \cdot \nabla) \mathbf{v}_h, \delta_v) + \\ + (\text{tr } e_C . \mathbf{C}, \nabla \delta_v) + (\text{tr } \mathbf{C}_h . e_C, \nabla \delta_v),$$

the error estimate for the pressure is obtained using the inf-sup condition, i.e.

$$c \|\delta_p\|_2 \leq \sup_{0 \neq \delta_v \in \mathbf{W}_h} \frac{(\delta_p, \text{div } \delta_v)}{\|\delta_v\|_{1,2}}$$

and the already known error estimates for **v** and **C**.

Thus, we get

$$\int_0^T \|\delta_p\|_2^2 \, dt \ \leq Ch^2.$$

Let us recall

$$e_v = \mathbf{v} - \mathbf{v}_h = (\mathbf{v} - \Pi_h^v \mathbf{v}) + (\Pi_h^v \mathbf{v} - \mathbf{v}_h) = \eta_v + \delta_v$$

and similarly for e_p, e_C.

Then,

$$\begin{aligned}
&\sup_{\tau\in(0,T)} \left(\|\mathbf{v}(\tau,\cdot) - \mathbf{v}_h(\tau,\cdot)\|_2^2 + \|\mathbf{C}(\tau,\cdot) - \mathbf{C}_h(\tau,\cdot)\|_2^2\right) + \\
&+ \int_0^T \|\nabla(\mathbf{v} - \mathbf{v}_h)\|_2^2 + \|\nabla(\mathbf{C} - \mathbf{C}_h)\|_2^2 \, dt \; + \int_0^T \|p - p_h\|_2^2 \, dt \; \leq \\
&\leq \sup_{\tau\in(0,T)} \left(\|\eta_v\|_2^2 + \|\delta_v\|_2^2 + \|\eta_C\|_2^2 + \|\delta_C\|_2^2\right) \\
&+ \int_0^T \|\nabla\eta_v\|_2^2 + \|\nabla\delta_v\|_2^2 + \|\nabla\eta_C\|_2^2 + \|\nabla\delta_C\|_2^2 \, dt \; + \\
&+ \int_0^T \|\eta_p\|_2^2 + \|\delta_p\|_2^2 \, dt \; \leq \\
&\leq Ch^2.
\end{aligned}$$

□

Thank you for your attention!

Funded by

Deutsche Forschungsgemeinschaft DFG

JG|U