On the sectorial $\mathcal{R}$-boundedness of the Stokes operator for the compressible viscous fluid flow and its application

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Joint work with Prof. Yoshihiro Shibata

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Linearized problem

We consider the linearized problem in bounded domain with slip boundary condition.

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$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \gamma \operatorname{div} u = f & \text{in } \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial t} - \operatorname{Div} S(u,\rho) = g & \text{in } \Omega,\ t > 0, \\ S(u,\rho)\nu|_{\tan} = h|_{\tan} & \text{on } \Gamma,\ t > 0, \\ u \cdot \nu = 0 & \text{on } \Gamma,\ t > 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

- $u = (u_1, \ldots, u_N)$: unknown velocity field $(N \geq 2)$,
  ρ : unknown density
- $S(u,\rho) = \alpha[\nabla u + (\nabla u)^T] + [(\beta - \alpha) \operatorname{div} u - \gamma\rho]I$: stress tensor
- α, β, γ : constant. $\alpha, \gamma > 0$, $2\alpha + N\beta > 0$.
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$n|_{\tan} = n - \langle n, \nu \rangle \nu$

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2\alpha[D(u)\nu - \langle D(u)\nu, \nu\rangle\nu] = h - \langle h, \nu\rangle\nu & \text{on } \Gamma,\ t > 0, \\
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Known result and Our goal

incompressible

- Shimada (2007):
  with non-homogeneous Robin B.C.
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  $\Longrightarrow$global in time unique existence theorem for some initial data close to a constant state in bounded domain for Navier-Stokes equations.

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In order to show a global in time unique existence theorem, we prove the exponential stability of solution to (P).

Main theorem (exponential stability)

Let $1 < p, q < \infty$ and $N < r < \infty$. Assume that Ω is not rotationally symmetric. Then, there exists a constant $\gamma_0 > 0$ such that for $(\rho_0, u_0) \in E_{p,q}(\Omega)$, $(f, g) \in L_{p,\gamma_0}(\mathbb{R}_+, \dot{W}^{1,0}_q(\Omega))$ and $h \in L_{p,\gamma_0}(\mathbb{R}_+, W^1_q(\Omega)) \cap W^{1/2}_{p,\gamma_0}(\mathbb{R}_+, L_q(\Omega))$, the problem (P) admits a unique solution

$$\rho \in W^1_{p,\gamma_0}(\mathbb{R}_+, W^1_q(\Omega))$$

$$u \in W^1_{p,\gamma_0}(\mathbb{R}_+, L_q(\Omega)) \cap L_{p,\gamma_0}(\mathbb{R}_+, W^2_q(\Omega))$$

which satisfies the estimate

$$\|e^{\gamma_0 t}(\rho_t, \rho)\|_{L_p(\mathbb{R}_+, W^1_q)} + \|e^{\gamma_0 t}(u_t, u, \langle D_t\rangle^{1/2}\nabla u, \nabla^2 u)\|_{L_p(\mathbb{R}_+, L_q)}$$

$$\leq C(\|(\rho_0, u_0)\|_{E_{p,q}} + \|e^{\gamma_0 t}(f, \nabla f, g, \nabla h)\|_{L_p(\mathbb{R}_+, L_q)} + \|\langle D_t\rangle^{1/2} e^{\gamma_0 t} h\|_{L_p(\mathbb{R}_+, L_q)}).$$

$$\dot{W}^{1,0}_q(\Omega) = \{(f, g) \in W^{1,0}_q(\Omega) \mid \int_\Omega f(x)dx = 0\}, E_{p,q} = (\dot{W}^{1,0}_q(\Omega), \dot{\mathcal{D}}(\mathcal{A}))_{1-1/p,p},$$

$$L_{p,\gamma_0}(\mathbb{R}_+, X) = \{f \in L_{p,loc}(\mathbb{R}_+, X) \mid e^{\gamma_0 t} f(t) \in L_p(\mathbb{R}_+, X)\},$$

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Sketch of proof

We set $U = (\rho, u)$, $F = (f, g) \in \dot{W}_q^{1,0}(\Omega)$,

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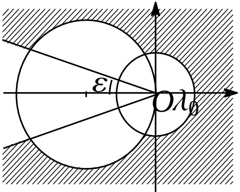
Resolvent problem:



$$\text{(RP)}\qquad \begin{cases} \lambda U - \mathcal{A}U = F & \text{in } \Omega, \\ S(u,\rho)\nu|_{\tan} = 0|_{\tan}, \quad u\cdot\nu = 0 & \text{on } \Gamma. \end{cases}$$

Theorem ($\mathcal{R}$-boundedness)

Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$. Then, there exist a $\lambda_0 \geq 1$ depending on ε, q, N and an operator $R(\lambda) \in \mathcal{L}(W_q^{1,0}(\Omega), W_q^{1,2}(\Omega))$ such that the following two assertions hold :

(i) For any $(f,g) \in W_q^{1,0}(\Omega)$ and $\lambda \in \Lambda_{\varepsilon,\lambda_0}$,
$(\rho, u) = R(\lambda)(f,g) \in W_q^{1,2}(\Omega)$ solves the problem (RP) uniquely.

(ii) There exist $\kappa > 0$ such that

$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega),\ W_q^{1,0}(\Omega))}\{\lambda R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \kappa,$$
$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega),\ L_q(\Omega)^{N^2})}(\{\lambda^{1/2}\nabla P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \kappa,$$
$$\mathcal{R}_{\mathcal{L}(W_q^{1,0}(\Omega),\ L_q(\Omega)^{N^3})}(\{\nabla^2 P_v R(\lambda) \mid \lambda \in \Lambda_{\varepsilon,\lambda_0}\}) \leq \kappa,$$

where we set $P_v R(\lambda)(f,g) = u$.

By Theorem ($\mathcal{R}$-boundedness) and homotopic argument, we have $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0\} \cup \Lambda_{\varepsilon,\lambda_0} \subset \rho(\dot{\mathcal{A}})$ and for any $(f,g) \in \dot{W}_q^{1,0}(\Omega)$ and $\lambda \in \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \geq 0\} \cup \Lambda_{\varepsilon,\lambda_0}$, $(\rho,u) = (\lambda - \dot{\mathcal{A}})^{-1}(f,g) \in \dot{\mathcal{D}}(\mathcal{A})$ satisfies the estimate:

$$(1+|\lambda|)\|(\rho,u)\|_{W_q^{1,0}(\Omega)} + (1+|\lambda|^{1/2})\|\nabla u\|_{L_q(\Omega)} + \|\nabla^2 u\|_{L_q(\Omega)} \leq C\|(f,g)\|_{W_q^{1,0}(\Omega)}.$$

Theorem (generation of analytic semigroup)

$\dot{\mathcal{A}}$ generates an analytic semigroup $\{\dot{T}(t)\}_{t\geq 0}$ on $\dot{W}_q^{1,0}(\Omega)$. Moreover, there exists a constant $\gamma_1 > 0$ and M for any $(f,g) \in W_q^{1,0}(\Omega)$, $(\rho,u) = \dot{T}(t)(f,g)$ satisfies the estimate:

$$\|\dot{T}(t)(f,g)\|_{W_q^{1,0}(\Omega)} \leq Me^{-\gamma_1 t}\|(f,g)\|_{W_q^{1,0}(\Omega)}.$$

For any $(\rho_0,u_0) \in E_{p,q}$, the problem (P1) admits a unique solution (ρ,u) satisfies the estimate:

$$\|e^{\gamma_1 t}(\rho_t,\rho)\|_{L_p(\mathbb{R}_+,W_q^1(\Omega))} + \|e^{\gamma_1 t}(u_t,u,\nabla^2 u)\|_{L_p(\mathbb{R}_+,L_q(\Omega))} \leq C\|(\rho_0,u_0)\|_{E_{p,q}}.$$

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- $||e^{\gamma_2 t}U||_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))}$

For $F \in C_0^\infty(\mathbb{R}_+, \dot{W}_q^{1,0}(\Omega))$, we set

$$U(t) = \int_0^t T(t-s)F(s)ds. \tag{1}$$

By analytic semigroup theory, U solves (P2), that is

$$U(t) - \dot{\mathcal{A}}U = F \text{ in } \Omega \times \mathbb{R}_+, \ U|_{t=0} = (0,0).$$

By (1), there exists a constant $\gamma_2 \in [0, \gamma_1/(2p))$ such that

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$$\|e^{\gamma_2 t}U\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))} \le C\|e^{\gamma_2 t}F\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))}.$$

- $\|e^{\gamma_2 t}U\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))}$

For $F \in C_0^\infty(\mathbb{R}_+, \dot{W}_q^{1,0}(\Omega))$, we set

$$U(t) = \int_0^t T(t-s)F(s)ds. \tag{1}$$

By analytic semigroup theory, U solves (P2), that is

$$U(t) - \dot{\mathcal{A}}U = F \text{ in } \Omega \times \mathbb{R}_+, \; U|_{t=0} = (0,0).$$

By (1), there exists a constant $\gamma_2 \in [0, \gamma_1/(2p))$ such that

$$\|e^{\gamma_2 t}U\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))} \leq C\|e^{\gamma_2 t}F\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))}.$$

- $\|e^{\gamma_3 t}U_t\|_{L_p(\mathbb{R}_+,W_q^{1,0}(\Omega))}$, $\|e^{\gamma_3 t}\nabla^2 u\|_{L_p(\mathbb{R}_+,L_q(\Omega))}$

$$\begin{cases} \partial_t U + 3\lambda_0 U - \dot{\mathcal{A}}U = F + 3\lambda_0 U & \text{in } \Omega,\ t>0, \\ S(u,\rho)\nu|_{\tan} = 0|_{\tan},\quad u\cdot\nu = 0 & \text{on } \Gamma,\ t>0, \\ (\rho,u)|_{t=0} = (0,0) & \text{in } \Omega, \end{cases}$$

where $U_0(t) = U(t)$ for $t \geq 0$ and $U_0(t) = (0,0)$ for $t < 0$, and $G(t) = F(t) + 3\lambda_0 U(t)$ for $t \geq 0$ and $G(t) = (0,0)$ for $t < 0$.

$$\partial_t U_0(t) = 1/(2\pi)\int_{-\infty}^{\infty} e^{\lambda t}\lambda((\lambda+3\lambda_0)I - \dot{\mathcal{A}})^{-1}\mathcal{L}[G](\lambda)d\tau$$

with $\lambda = -\gamma + i\tau$. By Theorem ($\mathcal{R}$-boundedness), there exists a constant $\gamma_3 \in [0,\gamma_4)$ such that

$$\|e^{\gamma_3 t}U_t\|_{L_p(\mathbb{R}_+,W_q^{1,0}(\Omega))} + \|e^{\gamma_3 t}\nabla^2 u\|_{L_p(\mathbb{R}_+,L_q(\Omega))} \leq C\|e^{\gamma_3 t}G\|_{L_p(\mathbb{R}_+,W_q^{1,0}(\Omega))},$$

where $\gamma_4 = \min\{\lambda_0, \gamma_1/(2p)\}$.

- $\|e^{\gamma_3 t} U_t\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))}$, $\|e^{\gamma_3 t} \nabla^2 u\|_{L_p(\mathbb{R}_+, L_q(\Omega))}$

$$\begin{cases} \partial_t U + 3\lambda_0 U - \dot{\mathcal{A}} U = F + 3\lambda_0 U & \text{in } \Omega,\ t > 0, \\ S(u,\rho)\nu|_{\tan} = 0|_{\tan},\quad u \cdot \nu = 0 & \text{on } \Gamma,\ t > 0, \\ (\rho, u)|_{t=0} = (0,0) & \text{in } \Omega, \end{cases}$$

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where $\gamma_4 = \min\{\lambda_0, \gamma_1/(2p)\}$.

- $\|e^{\gamma_3 t}U_t\|_{L_p(\mathbb{R}_+,W^{1,0}_q(\Omega))}$, $\|e^{\gamma_3 t}\nabla^2 u\|_{L_p(\mathbb{R}_+,L_q(\Omega))}$

$$\begin{cases} \partial_t U_0 + 3\lambda_0 - \dot{\mathcal{A}}U_0 = G & \text{in } \Omega,\ t \in \mathbb{R}, \\ S(u,\rho)\nu|_{\tan} = 0|_{\tan},\quad u\cdot\nu = 0 & \text{on } \Gamma,\ t\in\mathbb{R}, \\ (\rho,u)|_{t=0} = (0,0) & \text{in } \Omega. \end{cases}$$

where $U_0(t) = U(t)$ for $t \geq 0$ and $U_0(t) = (0,0)$ for $t < 0$, and $G(t) = F(t) + 3\lambda_0 U(t)$ for $t \geq 0$ and $G(t) = (0,0)$ for $t < 0$.

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$$\|e^{\gamma_3 t}U_t\|_{L_p(\mathbb{R}_+,W^{1,0}_q(\Omega))} + \|e^{\gamma_3 t}\nabla^2 u\|_{L_p(\mathbb{R}_+,L_q(\Omega))} \leq C\|e^{\gamma_3 t}G\|_{L_p(\mathbb{R}_+,W^{1,0}_q(\Omega))},$$

where $\gamma_4 = \min\{\lambda_0, \gamma_1/(2p)\}$.

- $\|e^{\gamma_3 t} U_t\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))}$, $\|e^{\gamma_3 t} \nabla^2 u\|_{L_p(\mathbb{R}_+, L_q(\Omega))}$

$$\begin{cases} \partial_t U_0 + 3\lambda_0 - \dot{\mathcal{A}} U_0 = G & \text{in } \Omega,\ t \in \mathbb{R}, \\ S(u,\rho)\nu|_{\tan} = 0|_{\tan}, \quad u \cdot \nu = 0 & \text{on } \Gamma,\ t \in \mathbb{R}, \\ (\rho, u)|_{t=0} = (0,0) & \text{in } \Omega. \end{cases}$$

where $U_0(t) = U(t)$ for $t \geq 0$ and $U_0(t) = (0,0)$ for $t < 0$, and $G(t) = F(t) + 3\lambda_0 U(t)$ for $t \geq 0$ and $G(t) = (0,0)$ for $t < 0$.

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$$\|e^{\gamma_3 t} U_t\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))} + \|e^{\gamma_3 t} \nabla^2 u\|_{L_p(\mathbb{R}_+, L_q(\Omega))} \leq C \|e^{\gamma_3 t} G\|_{L_p(\mathbb{R}_+, W_q^{1,0}(\Omega))},$$

where $\gamma_4 = \min\{\lambda_0, \gamma_1/(2p)\}$.

- $\|e^{\gamma_3 t}U_t\|_{L_p(\mathbb{R}_+,W_q^{1,0}(\Omega))}$, $\|e^{\gamma_3 t}\nabla^2 u\|_{L_p(\mathbb{R}_+,L_q(\Omega))}$

$$\begin{cases} \partial_t U_0 + 3\lambda_0 - \dot{\mathcal{A}} U_0 = G & \text{in } \Omega,\ t \in \mathbb{R}, \\ S(u,\rho)\nu|_{\tan} = 0|_{\tan}, \quad u \cdot \nu = 0 & \text{on } \Gamma,\ t \in \mathbb{R}, \\ (\rho, u)|_{t=0} = (0,0) & \text{in } \Omega. \end{cases}$$

where $U_0(t) = U(t)$ for $t \geq 0$ and $U_0(t) = (0,0)$ for $t < 0$, and $G(t) = F(t) + 3\lambda_0 U(t)$ for $t \geq 0$ and $G(t) = (0,0)$ for $t < 0$.

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with $\lambda = -\gamma + i\tau$. By Theorem ($\mathcal{R}$-boundedness), there exists a constant $\gamma_3 \in [0, \gamma_4)$ such that

$$\|e^{\gamma_3 t}U_t\|_{L_p(\mathbb{R}_+,W_q^{1,0}(\Omega))} + \|e^{\gamma_3 t}\nabla^2 u\|_{L_p(\mathbb{R}_+,L_q(\Omega))} \leq C\|e^{\gamma_3 t}G\|_{L_p(\mathbb{R}_+,W_q^{1,0}(\Omega))},$$

where $\gamma_4 = \min\{\lambda_0, \gamma_1/(2p)\}$.

Thank you for your attention.

$\mathcal{R}$-boundedness

Definition ($\mathcal{R}$-boundedness)

A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for each $m \in \mathbb{N}$, $T_j \in \mathcal{T}$, $f_j \in X$ $(j = 1,\dots,m)$ for all sequences $\{r_j(u)\}_{j=1}^m$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, there holds the inequality :

$$\int_0^1 \Big\|\sum_{j=1}^m r_j(u)T_j f_j\Big\|_Y^p du \le C \int_0^1 \Big\|\sum_{j=1}^m r_j(u) f_j\Big\|_X^p du.$$

Remark

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

For any Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the set of all bounded linear operators from X into Y. $\mathcal{L}(X) = \mathcal{L}(X,X)$.

Theorem (operator-valued Fourier multiplier theorem)

Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X,Y))$ such that

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{M(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_0 < \infty,$$
$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{\tau M'(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) = \kappa_1 < \infty.$$

If we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by the formula:

$$T_M\phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]],\ (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)).$$

Then, the operator T_M is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X),L_p(\mathbb{R},Y))} \leq C(\kappa_0 + \kappa_1)$$

for some constant $C > 0$ depending on p, X and Y.