On some decay properties of solutions for the Stokes problem with surface tension

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Joint work with
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§0 Contents

§1 Introduction (Navier-Stokes equations)

(1)
$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\theta = 0, \quad \nabla\cdot u = 0 & \text{in} \quad \Omega(t), \quad t>0, \\ \partial_t h + u'\cdot\nabla' h - u_N = 0 & \text{on} \quad \Gamma(t), \quad t>0, \\ S(u,\theta)\mathbf{n} + g x_N \mathbf{n} - \sigma\kappa\mathbf{n} = 0 & \text{on} \quad \Gamma(t), \quad t>0, \\ h|_{t=0} = h_0 & \text{on} \quad \mathbf{R}^N_0 \\ u|_{t=0} = u_0 & \text{in} \quad \Omega(0). \end{cases}$$

$u = (u_1, \ldots, u_N)$: velocity, θ: pressure, h: hight function : unknown.

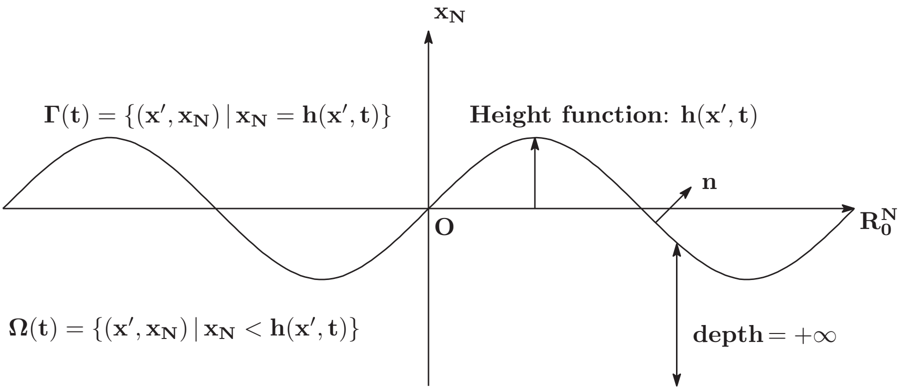
$u' = (u_1, \ldots, u_{N-1})$, $\nabla' = (\partial_1, \ldots, \partial_{N-1})$.

$S(u,\theta) = -\theta I + [\nabla u + (\nabla u)^T]$: stress tensor, I: $N\times N$ identity matrix.

$\mathbf{n}$: unit outer normal to $\Gamma(t)$, κ: mean curvature of $\Gamma(t)$.

$g > 0$: gravity constant, $\sigma > 0$: surface tension constant.

§1 Introduction (Navier-Stokes equations)



§1 Introduction (Stokes equations)

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(2) \quad \begin{cases} \partial_t u - \Delta u + \nabla\theta = F(u,\theta,h), \ \ \nabla\cdot u = F_d(u,h) & \text{in} \quad \mathbf{R}^N_+, \quad t>0, \\ \partial_t h + u_N = G(u,h) & \text{on} \quad \mathbf{R}^N_0, \quad t>0, \\ S(u,\theta)\mathbf{n}_0 + (g-\sigma\Delta')h\mathbf{n}_0 = H(u,h) & \text{on} \quad \mathbf{R}^N_0, \quad t>0, \\ \text{and Initial data } u_0, \ h_0, \end{cases}
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where $\mathbf{n}_0 = (0,\ldots,0,-1)$.

Goal

Our goal of this research is to show the global wellposedness of nonlinear problem (1) and also (2).

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Set

$$
(S(t)\,d)\,(x) = \mathcal{L}^{-1}_\lambda \left[\eta(x', x_N, \lambda)\right](t).
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Then, the solution h for (2) is given by

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h(x, t) = \int_0^t S(t-s)G(s)\,ds.
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Applying the partial Fourier transform w.r.t. x' variable to (RP) and solving ODEs w.r.t. x_N in the Fourier space, we have

$$\widehat{\eta}(\xi', 0, \lambda) = \mathcal{F}_{\xi'}[\eta(x', 0, \lambda)](\xi') = \frac{D(\xi', \lambda)}{(\sqrt{\lambda + |\xi'|^2} + |\xi'|)L(\xi', \lambda)} \widehat{d}(\xi', 0),$$

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$$D(\xi', \lambda) = B^3 + |\xi'|B^2 + 3|\xi'|^2 B - |\xi'|^3 \sim B^3,$$

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$$\lambda_\pm = \pm i g^{1/2} |\xi'|^{1/2} - 2|\xi'|^2 + O(|\xi'|^{5/2}) \quad (|\xi'| \to 0).$$

We, therefore, set

$$(S(t)\,d)(x) = (S_0(t)\,d)(x) + (S_\infty(t)\,d)(x)$$
$$:= \sum_{j=0,\infty} \mathcal{L}_\lambda^{-1} \mathcal{F}_{\xi'}^{-1} [\varphi_j(\xi') e^{-\sqrt{\lambda + |\xi'|^2} x_N} \widehat{\eta}(\xi', 0, \lambda)](x', t),$$

where $e^{-\sqrt{\lambda + |\xi'|^2} x_N}$: parabolic extension to $\mathbf{R}^N_+$, $\varphi_0(\xi')$: Low frequency cut-off function and $\varphi_\infty(\xi') = 1 - \varphi_0(\xi')$.

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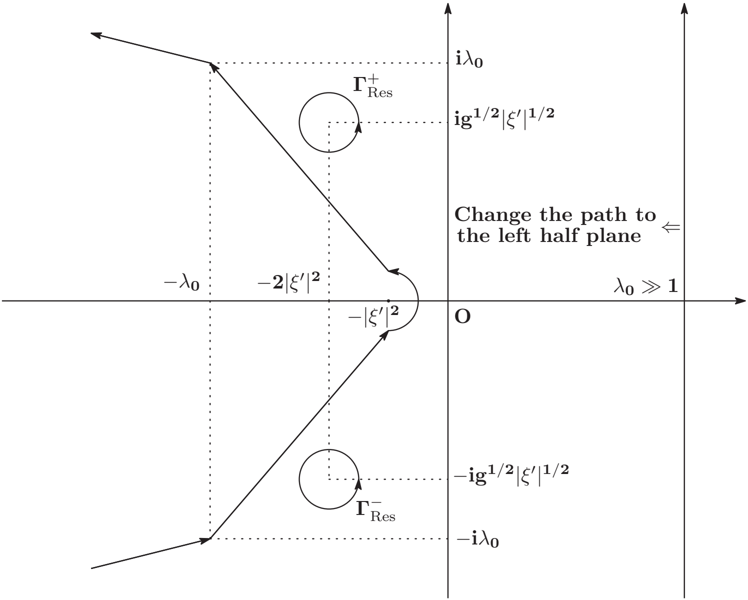

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Setting $B_\pm = \sqrt{\lambda_\pm + |\xi'|^2}$, we have $B_+ \sim |\xi'|^{1/4}$.

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$$\partial_N(S_0^+(t)\,d)(x) := \partial_N\{\text{Residue part of } (S_0(t)\,d)(x)\}$$

$$= \frac{1}{2\pi}\mathcal{F}^{-1}_{\xi'}\left[\int_{\Gamma^+_{\text{Res}}} e^{\lambda t}\frac{\varphi_0(\xi')D(\xi',\lambda)(-\sqrt{\lambda+|\xi'|^2})}{\sqrt{\lambda+|\xi'|^2}\,L(\xi',\lambda)} e^{-\sqrt{\lambda+|\xi'|^2}\,x_N}\,d\lambda\,\widehat{d}(\xi',0)\right](x')$$

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$$= \frac{1}{2\pi}\mathcal{F}_{\xi'}^{-1}\left[\int_{\Gamma^+_{\text{Res}}} e^{\lambda t}\frac{\varphi_0(\xi')D(\xi',\lambda)(-\sqrt{\lambda + |\xi'|^2})}{\sqrt{\lambda + |\xi'|^2}L(\xi',\lambda)} e^{-\sqrt{\lambda + |\xi'|^2}x_N}\, d\lambda\, \widehat{d}(\xi', 0)\right](x')$$

$$= \frac{-1}{\pi}\mathcal{F}_{\xi'}^{-1}\left[\int_{\widetilde{\Gamma}^+_{\text{Res}}} e^{(B^2 - |\xi'|^2)t}\frac{\varphi_0(\xi')D(\xi', B)Be^{-Bx_N}}{(B - B_+)(B - B_-)(B - B'_+)(B - B'_-)}\, dB\, \widehat{d}(\xi', 0)\right]$$

$$= -2i\mathcal{F}_{\xi'}^{-1}\left[e^{(B_+ - |\xi'|^2)t}\frac{\varphi_0(\xi')D(\xi', B_+)B_+e^{-B_\pm x_N}}{(B_+ - B_-)(B_+ - B'_+)(B_+ - B'_-)}\widehat{d}(\xi', 0)\right](x')$$

$$\sim \mathcal{F}_{\xi'}^{-1}\left[e^{\lambda_\pm t}\frac{\varphi_0(\xi')|\xi'|^{3/4}|\xi'|^{1/4}}{|\xi'|^{3/4}}e^{-|\xi'|^{1/4}x_N}\widehat{d}(\xi', 0)\right](x')$$

§2 Analysis of Stokes equations

Setting $B_\pm = \sqrt{\lambda_\pm + |\xi'|^2}$, we have $B_+ \sim |\xi'|^{1/4}$.
We, therefore, have by the change of variables: $B = \sqrt{\lambda + |\xi'|^2}$ and the factorization: $L(\xi', \lambda) = (B - B_+)(B - B_-)(B - B'_+)(B - B'_-)$

$$\partial_N(S_0^+(t)\,d)(x) := \partial_N\{\text{Residue part of } (S_0(t)\,d)(x)\}$$

$$= \frac{1}{2\pi}\mathcal{F}^{-1}_{\xi'}\left[\int_{\Gamma^+_{\text{Res}}} e^{\lambda t}\frac{\varphi_0(\xi')D(\xi',\lambda)(-\sqrt{\lambda+|\xi'|^2})}{\sqrt{\lambda+|\xi'|^2}L(\xi',\lambda)}e^{-\sqrt{\lambda+|\xi'|^2}x_N}\,d\lambda\,\widehat{d}(\xi',0)\right](x')$$

$$= \frac{-1}{\pi}\mathcal{F}^{-1}_{\xi'}\left[\int_{\widetilde{\Gamma}^+_{\text{Res}}} e^{(B^2-|\xi'|^2)t}\frac{\varphi_0(\xi')D(\xi',B)Be^{-Bx_N}}{(B-B_+)(B-B_-)(B-B'_+)(B-B'_-)}\,dB\,\widehat{d}(\xi',0)\right]$$

$$= -2i\mathcal{F}^{-1}_{\xi'}\left[e^{(B_+-|\xi'|^2)t}\frac{\varphi_0(\xi')D(\xi',B_+)B_+e^{-B_\pm x_N}}{(B_+-B_-)(B_+-B'_+)(B_+-B'_-)}\widehat{d}(\xi',0)\right](x')$$

$$\sim \mathcal{F}^{-1}_{\xi'}\left[e^{\lambda_\pm t}\frac{\varphi_0(\xi')|\xi'|^{3/4}|\xi'|^{1/4}}{|\xi'|^{3/4}}e^{-|\xi'|^{1/4}x_N}\widehat{d}(\xi',0)\right](x')$$

§2 Analysis of Stokes equations

Setting $B_\pm = \sqrt{\lambda_\pm + |\xi'|^2}$, we have $B_+ \sim |\xi'|^{1/4}$.
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$$\sim \mathcal{F}_{\xi'}^{-1}\left[e^{\lambda_\pm t}\frac{\varphi_0(\xi')|\xi'|^{3/4}|\xi'|^{1/4}}{|\xi'|^{3/4}}e^{-|\xi'|^{1/4}x_N}\widehat{d}(\xi',0)\right](x')$$

Setting $B_{\pm} = \sqrt{\lambda_{\pm} + |\xi'|^2}$, we have $B_+ \sim |\xi'|^{1/4}$.
We, therefore, have by the change of variables: $B = \sqrt{\lambda + |\xi'|^2}$ and the factorization: $L(\xi', \lambda) = (B - B_+)(B - B_-)(B - B'_+)(B - B'_-)$

$$\partial_N (S_0^+(t)\, d)(x) := \partial_N \{\text{Residue part of } (S_0(t)\, d)(x)\}$$

$$= \frac{1}{2\pi} \mathcal{F}_{\xi'}^{-1} \left[\int_{\Gamma^+_{\text{Res}}} e^{\lambda t} \frac{\varphi_0(\xi') D(\xi', \lambda)(-\sqrt{\lambda + |\xi'|^2})}{\sqrt{\lambda + |\xi'|^2} L(\xi', \lambda)} e^{-\sqrt{\lambda + |\xi'|^2} x_N}\, d\lambda\, \widehat{d}(\xi', 0) \right] (x')$$

$$= \frac{-1}{\pi} \mathcal{F}_{\xi'}^{-1} \left[\int_{\widetilde{\Gamma}^+_{\text{Res}}} e^{(B^2 - |\xi'|^2) t} \frac{\varphi_0(\xi') D(\xi', B) B e^{-B x_N}}{(B - B_+)(B - B_-)(B - B'_+)(B - B'_-)}\, dB\, \widehat{d}(\xi', 0) \right]$$

$$= -2i \mathcal{F}_{\xi'}^{-1} \left[e^{(B_+ - |\xi'|^2) t} \frac{\varphi_0(\xi') D(\xi', B_+) B_+ e^{-B_{\pm} x_N}}{(B_+ - B_-)(B_+ - B'_+)(B_+ - B'_-)} \widehat{d}(\xi', 0) \right] (x')$$

$$\sim \mathcal{F}_{\xi'}^{-1} \left[e^{\lambda_{\pm} t} \frac{\varphi_0(\xi') |\xi'|^{3/4} |\xi'|^{1/4}}{|\xi'|^{3/4}} e^{-|\xi'|^{1/4} x_N} \widehat{d}(\xi', 0) \right] (x')$$

§2 Analysis of Stokes equations

Setting $B_\pm = \sqrt{\lambda_\pm + |\xi'|^2}$, we have $B_+ \sim |\xi'|^{1/4}$.
We, therefore, have by the change of variables: $B = \sqrt{\lambda + |\xi'|^2}$ and the factorization: $L(\xi', \lambda) = (B - B_+)(B - B_-)(B - B'_+)(B - B'_-)$

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$$= \frac{-1}{\pi}\mathcal{F}_{\xi'}^{-1}\left[\int_{\widetilde{\Gamma}^+_{\mathrm{Res}}} e^{(B^2-|\xi'|^2)t}\frac{\varphi_0(\xi')D(\xi',B)Be^{-Bx_N}}{(B-B_+)(B-B_-)(B-B'_+)(B-B'_-)}\, dB\, \widehat{d}(\xi',0)\right]$$

$$= -2i\mathcal{F}_{\xi'}^{-1}\left[e^{(B_+-|\xi'|^2)t}\frac{\varphi_0(\xi')D(\xi',B_+)B_+e^{-B_\pm x_N}}{(B_+-B_-)(B_+-B'_+)(B_+-B'_-)}\widehat{d}(\xi',0)\right](x')$$

$$\sim \mathcal{F}_{\xi'}^{-1}\left[e^{\lambda_\pm t}\frac{\varphi_0(\xi')|\xi'|^{3/4}|\xi'|^{1/4}}{|\xi'|^{3/4}}e^{-|\xi'|^{1/4}x_N}\widehat{d}(\xi',0)\right](x')$$

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$$\sim \mathcal{F}^{-1}_{\xi'}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x')$$

$$= \mathcal{F}^{-1}_{\xi'}\left[e^{-|\xi'|^2 t}e^{\pm i g^{1/2}|\xi'|^{1/2}t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,e^{-|\xi'|^2 t}\widehat{d}(\xi',0)\right](x').$$

By L_q-L_r estimate of $N-1$ dimensions heat kernel: $\mathcal{F}^{-1}_{\xi'}[e^{-|\xi'|^2 t}](x')$,

$$\begin{aligned}
&\|\partial_N(S_0^{\pm}(t)\,d)(\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^2 t/2}e^{-|\xi'|^{1/4}x_N}e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}\\
&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.
\end{aligned}$$

And then, we have

$$\|\partial_N S_0^{\pm}(t)\,d\|_{L_q(\mathbf{R}^N_+)} \lesssim t^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(1-\frac{1}{q}\right)}\|d\|_{W^1_r(\mathbf{R}^N_+)}\ (1\le r\le 2\le q\le\infty).$$

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$$\sim \mathcal{F}^{-1}_{\xi'}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x')$$

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&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.
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$$\sim \mathcal{F}_{\xi'}^{-1}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x')$$
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$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^2 t/2}e^{-|\xi'|^{1/4}x_N}e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}$$
$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.$$

And then, we have

$$\|\partial_N S_0^{\pm}(t)\,d\|_{L_q(\mathbf{R}_+^N)} \lesssim t^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(1-\frac{1}{q}\right)}\|d\|_{W_r^1(\mathbf{R}_+^N)}\ (1\le r\le 2\le q\le\infty).$$

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$$\sim \mathcal{F}^{-1}_{\xi'}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x')$$
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&\|\partial_N(S_0^{\pm}(t)\,d)(\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}\\
&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^2 t/2}e^{-|\xi'|^{1/4}x_N}e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}\\
&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.
\end{aligned}$$

And then, we have

$$\|\partial_N S_0^{\pm}(t)\,d\|_{L_q(\mathbf{R}^N_+)} \lesssim t^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(1-\frac{1}{q}\right)}\|d\|_{W^1_r(\mathbf{R}^N_+)}\ (1\le r\le 2\le q\le\infty).$$

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$$\sim \mathcal{F}^{-1}_{\xi'}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x')$$
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By L_q-L_r estimate of $N-1$ dimensions heat kernel: $\mathcal{F}^{-1}_{\xi'}[e^{-|\xi'|^2 t}](x')$,

$$\|\partial_N(S_0^{\pm}(t)\,d)(\cdot, x_N)\|_{L_q(\mathbf{R}^{N-1})}$$
$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^2 t/2}e^{-|\xi'|^{1/4}x_N}e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}$$
$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.$$

And then, we have

$$\|\partial_N S_0^{\pm}(t)\,d\|_{L_q(\mathbf{R}^N_+)} \lesssim t^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(1-\frac{1}{q}\right)}\|d\|_{W^1_r(\mathbf{R}^N_+)}\ (1\le r\le 2\le q\le\infty).$$

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$$
\begin{aligned}
&\sim \mathcal{F}_{\xi'}^{-1}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x') \\
&= \mathcal{F}_{\xi'}^{-1}\left[e^{-|\xi'|^2 t}e^{\pm i g^{1/2}|\xi'|^{1/2}t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,e^{-|\xi'|^2 t}\widehat{d}(\xi',0)\right](x').
\end{aligned}
$$

By L_q-L_r estimate of $N-1$ dimensions heat kernel: $\mathcal{F}_{\xi'}^{-1}[e^{-|\xi'|^2 t}](x')$,

$$
\begin{aligned}
&\|\partial_N(S_0^{\pm}(t)\,d)(\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})} \\
&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^2 t/2}e^{-|\xi'|^{1/4}x_N}e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})} \\
&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.
\end{aligned}
$$

And then, we have

$$
\|\partial_N S_0^{\pm}(t)\,d\|_{L_q(\mathbf{R}_+^N)} \lesssim t^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(1-\frac{1}{q}\right)}\|d\|_{W_r^1(\mathbf{R}_+^N)}\ (1 \le r \le 2 \le q \le \infty).
$$

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By L_q-L_r estimate of $N-1$ dimensions heat kernel: $\mathcal{F}_{\xi'}^{-1}[e^{-|\xi'|^2 t}](x')$,

$$\|\partial_N(S_0^{\pm}(t)\,d)(\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}$$
$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^2 t/2}e^{-|\xi'|^{1/4}x_N}e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}$$
$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.$$

And then, we have

$$\|\partial_N S_0^{\pm}(t)\,d\|_{L_q(\mathbf{R}^N_+)} \lesssim t^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(1-\frac{1}{q}\right)}\|d\|_{W_r^1(\mathbf{R}^N_+)}\ (1\le r\le 2\le q\le\infty).$$

§2 Analysis of Stokes equations

$$\sim \mathcal{F}^{-1}_{\xi'}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x')$$
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$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.$$

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$$\sim \mathcal{F}^{-1}_{\xi'}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x')$$
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&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^2 t/2}e^{-|\xi'|^{1/4}x_N}e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}\\
&\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.
\end{aligned}$$

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$$\sim \mathcal{F}_{\xi'}^{-1}\left[e^{\left(\pm i g^{1/2}|\xi'|^{1/2}-2|\xi'|^2\right)t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,\widehat{d}(\xi',0)\right](x')$$
$$= \mathcal{F}_{\xi'}^{-1}\left[e^{-|\xi'|^2 t}e^{\pm i g^{1/2}|\xi'|^{1/2}t}\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^{1/4}x_N}\,e^{-|\xi'|^2 t}\widehat{d}(\xi',0)\right](x').$$

By L_q-L_r estimate of $N-1$ dimensions heat kernel: $\mathcal{F}_{\xi'}^{-1}[e^{-|\xi'|^2 t}](x')$,

$$\|\partial_N(S_0^{\pm}(t)\,d)(\cdot,x_N)\|_{L_q(\mathbf{R}^{N-1})}$$
$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|\varphi_0(\xi')|\xi'|^{1/4}e^{-|\xi'|^2 t/2}e^{-|\xi'|^{1/4}x_N}e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}$$
$$\lesssim t^{-\frac{N-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\frac{\|\varphi_0(\xi')e^{-|\xi'|^2 t/2}\widehat{d}(\xi',0)\|_{L_2(\mathbf{R}^{N-1})}}{t^{1/8}+x_N}.$$

And then, we have

$$\|\partial_N S_0^{\pm}(t)\,d\|_{L_q(\mathbf{R}_+^N)} \lesssim t^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(1-\frac{1}{q}\right)}\|d\|_{W_r^1(\mathbf{R}_+^N)}\ (1\le r\le 2\le q\le\infty).$$

§2 Analysis of Stokes equations

Theorem

Let $1 \leq r \leq 2 < q \leq \infty$. Then, we have for any $t > 0$

$$
\begin{aligned}
&\|(\nabla')^{k+1} S_0(t)\, d\|_{L_q(\mathbf{R}^N_+)} \\
&\lesssim (t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(4-\frac{1}{q}\right)-\frac{k}{2}}(1+|\log t|e^{-\lambda_0 t})\|d\|_{W^1_r(\mathbf{R}^N_+)}, \\
&\|(\nabla')^{k}\, \partial_N^{\ell} S_0(t)\, d\|_{L_q(\mathbf{R}^N_+)} \\
&\lesssim (t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(\ell-\frac{1}{q}\right)-\frac{k}{2}}(1+t^{-\frac{\ell}{2}+\frac{1}{2q}}e^{-\lambda_0 t})\|d\|_{W^1_r(\mathbf{R}^N_+)}, \\
&\|\partial_t S_0(t)\, d\|_{L_q(\mathbf{R}^N_+)} \\
&\lesssim (t+1)^{-\frac{N-1}{2}\left(\frac{1}{r}-\frac{1}{q}\right)-\frac{1}{8}\left(2-\frac{1}{q}\right)}(1+t^{-1+\frac{1}{2q}}e^{-\lambda_0 t})\|d\|_{W^1_r(\mathbf{R}^N_+)},
\end{aligned}
$$

where $k \in \mathbf{N} \cup \{0\}$ and $\ell = 1, 2, 3$.