Navier-Stokes Flow in Infinite Cylinders with Non-Constant Cross Section

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Cylinders with Non-Constant Cross-Section

- Stokes resolvent problem

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= g && \text{in } \Omega, \qquad \text{in } L^q\text{-setting} \qquad \text{(R)} \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with $\lambda \in -\alpha + \Sigma_\theta$, $\theta \in (\pi/2, \pi)$, $\alpha \in (0, \alpha_0)$

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  - $\nabla' R$ has compact support and there is $K > 0$ s.t.

$$\|\nabla' R\|_\infty, \|\nabla'^2 R\|_\infty < K \tag{K}$$

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  - $R(\varphi, 0)$ parametrizes the boundary of a reference cross section Σ
- $\Sigma \subset \mathbb{R}^2$: bounded $C^{1,1}$-domain, star-shaped w.r.t. some ball around 0
  - $\alpha_0 > 0$: smallest eigenvalue of the Dirichlet Laplacian in Σ

Resolvent Estimate

Theorem 1

- $2 \leq q < \infty$
- smallness assumption (K)
- $(f, g) \in L^q(\Omega)^3 \times \left(W^{1,q}(\Omega) \cap \hat{W}^{-1,q}(\Omega)\right)$

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- *a priori* estimate

$$\|(\lambda + \alpha)u, \nabla^2 u, \nabla p\|_q \leq c(\|f\|_q + \|g\|_{1,q} + (|\lambda| + 1)\|g\|_{\hat{W}^{-1,q}(\Omega)})$$

- $c(q, \alpha, \theta, \Omega) > 0$ independent of λ

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Definition

- sectorial operator B on Banach space X admits bounded $H^\infty(\Sigma_\theta)$-calculus if

$$\|h(B)\|_{\mathcal{L}(X)} \leq c\|h\|_\infty$$

  for all holomorphic and bounded functions on Σ_θ
- how to define $h(B)$ $\rightsquigarrow$ Complex Analysis

$$h(B) := \frac{1}{2\pi i}\int_\Gamma h(\lambda)(\lambda - B)^{-1}\,\mathrm{d}\lambda$$

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Theorem 2

For $1 < q < \infty$ the Stokes operator A_q admits a bounded $H^\infty(\Sigma_\theta)$-calculus in $L^q_\sigma(\Omega)$ for any $\theta \in (0, \pi)$.

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Ingredients

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- Perturbation arguments:
  - Define solution operators $S(u, p) = (\lambda u - \Delta u + \nabla p, -\mathrm{div}\, u)$ on Ω and analogously on $\widetilde{\Omega}$

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- For domains with certain regularity properties such that $0 \in \rho(A_q)$, the Stokes operator admits a bounded $H^\infty(\Sigma_\theta)$-calculus by a result by Abels (2004).

Proof

Transformation back to the straight cylinder in cylindrical coordinates via

$$(\tilde{r}, \tilde{\varphi}, \tilde{z}) = \phi(r, \varphi, z) = \left(\frac{r}{P(\varphi, z)}, \varphi, z \right)$$

with $P(\varphi, z) = R(\varphi, z) / R(\varphi, 0)$.

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$$\begin{aligned} \operatorname{div} u =& \widetilde{\operatorname{div}}\, \tilde{u} \\ &+ \left(\frac{1}{\tilde{r}} \tilde{\partial}_r (\tilde{r}\, \tilde{u}_r) + \frac{1}{\tilde{r}} \tilde{\partial}_\varphi \tilde{u}_\varphi \right) \left(\frac{1}{P(\varphi, z)} - 1 \right) \\ &- \tilde{\partial}_r \tilde{u}_\varphi \left(\frac{\partial_\varphi P(\varphi, z)}{P(\varphi, z)^2} \right) - \tilde{r} \tilde{\partial}_r \tilde{u}_z \left(\frac{\partial_z P(\varphi, z)}{P(\varphi, z)} \right) \end{aligned}$$

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- $S(u, p) = (\lambda u - \Delta u + \nabla p, -\operatorname{div} u)$ is close to $\tilde{S}(\tilde{u}, \tilde{p}) = (\lambda \tilde{u} - \widetilde{\Delta} \tilde{u} + \widetilde{\nabla} \tilde{p}, -\widetilde{\operatorname{div}}\, \tilde{u})$!

Thank you very much for your attention!

Questions?