Dynamics of nematic liquid crystal systems: The quasilinear approach

Katharina Schade, TU Darmstadt

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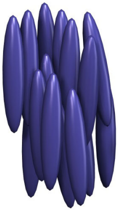
joint work with M. Hieber, M. Nesensohn and J. Prüss



8th Japanese-German International Workshop on Mathematical Fluid Dynamics

Waseda University,
June 19, 2013

1. The model

- Ω: bounded C^2 domain

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\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi = -\operatorname{div}([\nabla d]^{\mathsf T} \nabla d) & \text{in } (0,T) \times \Omega, \\
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|d| = 1 & \text{in } (0,T) \times \Omega, \\
\operatorname{div} u = 0 & \text{in } (0,T) \times \Omega, \\
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\end{cases}
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$u : (0,\infty) \times \Omega \to \mathbb{R}^n$: velocity,
$\pi : (0,\infty) \times \Omega \to \mathbb{R}$: pressure,
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- Problem: $|d| = 1$

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Theorem

Under suitable initial conditions, it holds

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$$H^1_p(J; X_0) \cap L_p(J; X_1), \quad J = [0, a],$$

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- Apply abstract result à la Clément and Li + Angenent's trick for smoothing effect

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- Uniqueness of solution of parabolic convection-reaction diffusion equation $\longrightarrow \varphi = 0$, i.e. $|d| = 1$. □

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- Temperature-dependent coefficients e.g. viscosity.

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