# On a Conjectured Pointwise Bound for Solutions of the Stokes Equations in Nonsmooth Domains

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- **In brief:** We wish to prove, for **arbitrary open sets** $\Omega \subset R^3$, that solutions of the steady Stokes problem

$$-\triangle \mathbf{u} = -\nabla p + \mathbf{f} \ , \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{\partial\Omega} = 0,$$

*along with zero flux conditions when needed,*

satisfy (on setting $\widetilde{\triangle}\mathbf{u} \equiv -P\mathbf{f}$ )

$$\sup_{\Omega} |\mathbf{u}|^2 \leq \frac{1}{3\pi} \|\nabla \mathbf{u}\| \left\| \widetilde{\triangle}\mathbf{u} \right\| .$$

- In trying to prove this, solutions of the Poisson problem

$$-\triangle u = f \ , \quad u|_{\partial\Omega} = 0,$$

were proven (in the 1991 thesis of Wenzheng Xie) to satisfy

$$\sup_{\Omega} |u|^2 \leq \frac{1}{2\pi} \|\nabla u\| \, \|\triangle u\| .$$

- To be precise: Let $\Omega$ be an arbitrary open subset of $R^3$. Let $\mathbf{D}(\Omega) \equiv \{\boldsymbol{\varphi} \in \mathbf{C}_0^\infty(\Omega) : \nabla \cdot \boldsymbol{\varphi} = 0\}$. Let $\mathbf{J}(\Omega)$ and $\mathbf{J}_0(\Omega)$ be the completions of $\mathbf{D}(\Omega)$ in the $\mathbf{L}^2$-norm $\|\cdot\|$ and the Dirichlet-norm $\|\nabla \cdot\|$, respectively. Then, given $\mathbf{u} \in \mathbf{J}_0(\Omega)$, there is at most one $\mathbf{f} \in \mathbf{J}(\Omega)$ such that $(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) = -(\mathbf{f}, \boldsymbol{\varphi})$ for all $\boldsymbol{\varphi} \in \mathbf{D}(\Omega)$. If such a function $\mathbf{f}$ exists, it is denoted by $\widetilde{\triangle}\mathbf{u}$, and we wish to prove that

$$\sup_\Omega |\mathbf{u}|^2 \leq \frac{1}{3\pi} \|\nabla \mathbf{u}\| \left\| \widetilde{\triangle}\mathbf{u} \right\|. \tag{1}$$

- Important applications of (1), beginning with

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|^2 + v\left\|\widetilde{\triangle}\mathbf{u}\right\|^2 &= \left(\mathbf{u}\cdot\nabla\mathbf{u}, \widetilde{\triangle}\mathbf{u}\right) \leq \sup_\Omega |\mathbf{u}| \, \|\nabla\mathbf{u}\| \left\|\widetilde{\triangle}\mathbf{u}\right\| \\ &\leq \frac{1}{\sqrt{3\pi}} \|\nabla\mathbf{u}\|^{3/2} \left\|\widetilde{\triangle}\mathbf{u}\right\|^{3/2} \leq \cdots \end{aligned} \tag{2}$$

  would settle many problems for general domains by circumventing the use of domain dependent inequalities, like the famous Solonnikov inequality $\left\|D^2\mathbf{u}\right\| \leq c_\Omega \left(\left\|\widetilde{\triangle}\mathbf{u}\right\| + \|\nabla\mathbf{u}\|\right)$, in some key arguements.

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \widetilde{\Delta}\mathbf{u}, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}\big|_{\partial\Omega} = 0, \quad \mathbf{u}\big|_{t=0} = u_0$$

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|^2 = 0$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 + \left\|\widetilde{\Delta}\mathbf{u}\right\|^2 = \left(\mathbf{u}\cdot\nabla\mathbf{u}, \widetilde{\Delta}\mathbf{u}\right) \le \tfrac{1}{2}\left\|\widetilde{\Delta}\mathbf{u}\right\|^2 + c\,\|\nabla\mathbf{u}\|^6$$

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \|\nabla\mathbf{u}_t\|^2 \le c\,\|\mathbf{u}\|^2\,\|\mathbf{u}_t\|^4$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_t\|^2 + \left\|\widetilde{\Delta}\mathbf{u}_t\right\|^2 \le \cdots$$

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_{tt}\|^2 + \|\nabla\mathbf{u}_{tt}\|^2 \le \cdots$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}_{tt}\|^2 + \left\|\widetilde{\Delta}\mathbf{u}_{tt}\right\|^2 \le \cdots \qquad \text{e.t.c., e.t.c.}$$

Assuming regularity of $\partial\Omega$, and $u_0 \in \mathbf{J}_0(\Omega)$ we can integrate all of these, starting with the second, on the same interval common $(0, T)$ and get (for bounded $\Omega$) $u \in C^\infty\left((0, T); W_2^2(\Omega)\right)$. Then more spatial regularity.

Among the results that would be freed of assumptions about the boundary:

- The existence and uniqueness of a nonstationary solution for any $\mathbf{u}_0 \in \mathbf{J}_0(\Omega)$ : "Theorem" For any open $\Omega \subset R^3$ and $\mathbf{u}_0 \in \mathbf{J}_0(\Omega)$ there exists a solution $\mathbf{u}, p \in \mathbf{C}^\infty(\Omega \times (0, T))$ of
  $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \triangle \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}(0) = \mathbf{u}_0.$
  Further, $T = \frac{256\pi^2\nu^3}{27\|\nabla \mathbf{u}_0\|^4}$ is independent of $\Omega$, $\|\nabla \mathbf{u}(t)\|^2 \leq \frac{\|\nabla \mathbf{u}_0\|^2}{\sqrt{1-T/t}}$ and $\sup_\Omega |\mathbf{u}(t)| \leq t^{-1/2} b(t)$, where $b(t)$ is continuous on $[0, T)$.
- For any open $\Omega \subset R^3$, and any $\mathbf{u}_0 \in \mathbf{J}_0(\Omega) \cap \mathbf{W}_2^2(\Omega)$, Ladyzhenskaya proved the existence and uniqueness of a ‘generalized solution’. But, it’s full natural regularity has depended on the regularity of the entire boundary $\partial\Omega$. Given (1), her solution is identical to that discussed for $\mathbf{u}_0 \in \mathbf{J}_0(\Omega)$, and no less regular.
- The existence of steady solutions satisfying $\sup\limits_\Omega |\mathbf{u}| < \infty$.
- Etc., etc.

- Considering the Poisson problem for the Laplacian as a model problem, Wenzheng Xie (in his 1991 thesis) proved

$$\sup_{\Omega} |u|^2 \leq \frac{1}{2\pi} \|\nabla u\| \, \|\triangle u\| \,, \tag{3}$$

for any open $\Omega \subset R^3$, and any $u \in \widehat{H}_0^1(\Omega)$ with $\triangle u \in L^2(\Omega)$, where $\widehat{H}_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the Dirichlet norm $\|\nabla u\|$.

- The only point in Xie's proof of (3) that doesn't carry over to a proof of (1) is his use of the maximum principle to show that

$$\int_{\Omega} G_\mu^2(x,y)\, dx \leq \int_{\Omega} g_\mu^2(x,y)\, dx \,, \tag{4}$$

for all $y \in \Omega$, where $G_\mu$ is the Green's function for $-\triangle + \mu$, and $g_\mu = \dfrac{e^{-\sqrt{\mu}|x-y|}}{4\pi |x-y|}$ is the corresponding fundamental singularity.

We begin Xie's proof of (3) under three simplifying assumptions, considering first

- Smoothly bounded domains $\Omega$.
- Functions $u_m(x) = \sum_{n=1}^{m} c_n \varphi_n(x)$, where $\{\varphi_n\}$ are the $L^2$-orthonormal eigenfunctions, $-\triangle \varphi_n = \lambda_n \varphi_n,\ \varphi_n|_{\partial\Omega} = 0$.
- An arbitrary fixed choice of $y \in \Omega$, henceforth considered fixed.

For this fixed $y$, and any fixed $m$, the ratio $R_m(y)$ of the two sides of (3)

$$R_m(y) \equiv \frac{u_m^2(y)}{\|\nabla u_m\| \, \|\triangle u_m\|} = \frac{\left(\sum_{n=1}^{m} c_n \varphi_n(y)\right)^2}{\left(\sum_{n=1}^{m} \lambda_n c_n^2\right)^{1/2} \left(\sum_{n=1}^{m} \lambda_n^2 c_n^2\right)^{1/2}} \tag{5}$$

is a homogeneous function of $(c_1, c_2, \cdots, c_m) \in R^m$, constant on lines through the origin and smooth except at the origin. If $(\overline{c}_1, \overline{c}_2, \cdots, \overline{c}_m)$ is a point at which it attains its maximum $\overline{R}_m(y)$, then, at that point, $\partial R_m(y) / \partial c_n = 0$, or equivalently $\partial (\log R_m(y)) / \partial c_n = 0$, for $n = 1, \cdots, m$.

Setting to zero the derivative with respect to $c_n$ of

$$\log R_m(y) = 2\log\left(\sum_{n=1}^{m} c_n\varphi_n(y)\right) - \frac{1}{2}\log\left(\sum_{n=1}^{m}\lambda_n c_n^2\right) - \frac{1}{2}\log\left(\sum_{n=1}^{m}\lambda_n^2 c_n^2\right)$$

we obtain

$$\frac{2\varphi_n(y)}{\sum_{n=1}^{m}\overline{c}_n\varphi_n(y)} = \frac{\lambda_n\overline{c}_n}{\sum_{n=1}^{m}\lambda_n\overline{c}_n^2} + \frac{\lambda_n^2\overline{c}_n}{\sum_{n=1}^{m}\lambda_n^2\overline{c}_n^2}$$

or

$$\frac{2\varphi_n(y)}{\overline{u}_m(y)} = \frac{\lambda_n\overline{c}_n}{\|\nabla\overline{u}_m\|^2} + \frac{\lambda_n^2\overline{c}_n}{\|\triangle\overline{u}_m\|^2}\,. \tag{6}$$

Introducing

$$\overline{\mu}_m \equiv \frac{\|\triangle\overline{u}_m\|^2}{\|\nabla\overline{u}_m\|^2}$$

one can rewrite (6) as

$$\frac{2\varphi_n(y)}{\overline{u}_m(y)} = \left(\overline{\mu}_m + \lambda_n\right)\frac{\lambda_n\overline{c}_n}{\|\triangle\overline{u}_m\|^2}$$

which, on multiplying by $\overline{u}_m(y) / (\overline{\mu}_m + \lambda_n)$ becomes

becomes

$$\frac{2\varphi_n(y)}{\overline{\mu}_m+\lambda_n}=\frac{\overline{u}_m(y)}{\|\triangle\overline{u}_m\|^2}\lambda_n\overline{c}_n\ .$$

Squaring and summing gives

$$4\sum_{n=1}^{m}\left(\frac{\varphi_n(y)}{\overline{\mu}_m+\lambda_n}\right)^2=\left(\frac{\overline{u}_m(y)}{\|\triangle\overline{u}_m\|^2}\right)^2\sum_{n=1}^{m}\lambda_n^2\overline{c}_n^2=\frac{\overline{u}_m^2(y)}{\|\triangle\overline{u}_m\|^2}\,.$$

Multiplying by $\sqrt{\overline{\mu}_m}\equiv\|\triangle\overline{u}_m\|\,/\,\|\nabla\overline{u}_m\|$ and changing sides, one obtains

$$\overline{R}_m(y)\equiv\sqrt{\overline{\mu}_m}\frac{\overline{u}_m^2(y)}{\|\triangle\overline{u}_m\|^2}=4\sqrt{\overline{\mu}_m}\sum_{n=1}^{m}\left(\frac{\varphi_n(y)}{\overline{\mu}_m+\lambda_n}\right)^2\,. \tag{7}$$

Miracle of miracles, we recognize this! Since the $\{\varphi_n\}$ satisfy

$$(-\triangle+\mu)\,\varphi_n=(\mu+\lambda_n)\,\varphi_n\,,\qquad \varphi_n|_{\partial\Omega}=0\,,$$

they can be represented in terms of the Green's function $G_\mu(x,y)$ for the Helmholtz operator $-\triangle+\mu$. That is,

$$\varphi_n(y)=\int_\Omega G_\mu(x,y)\,(\mu+\lambda_n)\,\varphi_n(x)\;dx\,,$$

or dividing by $(\mu+\lambda_n)$,

$$\frac{\varphi_n(y)}{\mu+\lambda_n}=\int_\Omega G_\mu(x,y)\,\varphi_n(x)\,dx,$$

which is the $n^{th}$ Fourier coefficient of $G_\mu(\cdot,y)$. Thus (7) implies

$$\overline{R}_m(y)=4\sqrt{\overline{\mu}_m}\sum_{n=1}^{m}\left(\frac{\varphi_n(y)}{\overline{\mu}_m+\lambda_n}\right)^2\le 4\sqrt{\overline{\mu}_m}\int_\Omega G^2_{\overline{\mu}_m}\,dx.$$

The fundamental singularity for the Helmholtz operator $-\triangle+\mu$ is

$$g_\mu(x,y)\equiv\frac{e^{-\sqrt{\mu}|x-y|}}{4\pi|x-y|},$$

and $G_\mu(x,y)=g_\mu(x,y)-h_\mu(x,y)$, where $h_\mu$ is the unique solution of $\triangle h_\mu=\mu h_\mu$ in $\Omega$ satisfying $h_\mu(x,y)\big|_{\partial\Omega}=g_\mu(x,y)\big|_{\partial\Omega}$. Clearly, $h_\mu(x,\cdot)$ is positive on $\partial\Omega$ and cannot have a negative minmum. Therefore, $h_\mu(x,\cdot)$ is positive throughout $\overline{\Omega}$, and $G_\mu(x,y)<g_\mu(x,y)$ throughout $\overline{\Omega}$. Hence

$$\int_\Omega G^2_\mu\,dx\le\int_\Omega g^2_\mu\,dx=\int_0^\infty\left(\frac{e^{-\sqrt{\mu}r}}{4\pi r}\right)4\pi r^2dr=\frac{1}{8\pi\sqrt{\mu}}.$$

In summary, setting

$$\overline{u}_m = \sum_{n=1}^{m} \overline{c}_n \varphi_n \qquad \text{and} \qquad \overline{\mu}_m \equiv \frac{\left\| \triangle \overline{u}_m \right\|^2}{\left\| \nabla \overline{u}_m \right\|^2} \tag{8}$$

Xie found that

$$\begin{aligned} \overline{R}_m(y) &= \cdots = 4\sqrt{\overline{\mu}_m} \sum_{n=1}^{m} \left( \frac{\varphi_n(y)}{\overline{\mu}_m + \lambda_n} \right)^2 \\ &= 4\sqrt{\overline{\mu}_m} \sum_{n=1}^{m} \left( \int_\Omega G_{\overline{\mu}_m} \varphi_n \, dx \right)^2 \\ &\le 4\sqrt{\overline{\mu}_m} \int_\Omega G_{\overline{\mu}_m}^2 \, dx \le 4\sqrt{\overline{\mu}_m} \int_{R^3} g_{\overline{\mu}_m}^2 \, dx = \frac{1}{2\pi}, \end{aligned} \tag{9}$$

End of proof. The only point which isn't known to carry over to Stokes is the needed analogue of $\int_\Omega G_\mu^2 \, dx \le \int_\Omega g_\mu^2 \, dx$, which is **'Xie's conjecture'.** It is noteworthy that we have gained no information about $\overline{\mu}_m$.

Xie proved the constant $\frac{1}{2\pi}$ is optimal, first for $\Omega = R^3$, by observing that

$$u(x) \equiv 4\pi \left( g_0(|x|) - g_1(|x|) \right) = \begin{cases} 1 \ , & x = 0 \\ \frac{1-e^{-|x|}}{|x|} \ , & x \neq 0 \end{cases}$$

is locally $H^2$ and satisfies

$$\|\triangle u\|^2 = 2\pi, \quad \|\nabla u\|^2 = 2\pi.$$

Further, this constant is attained for the whole family of rescaled functions

$$v_\alpha(x) \equiv u(\alpha x)\ , \text{ for } \alpha > 0,$$

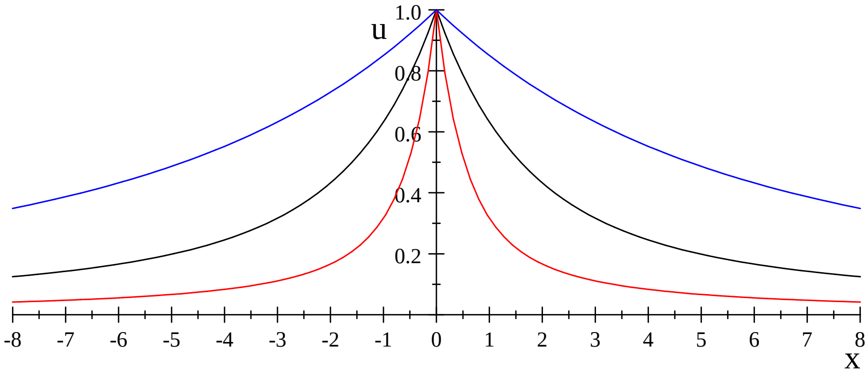
since

$$\|\triangle v_\alpha\|^2 = 2\pi\alpha^4\alpha^{-3} = 2\pi\alpha, \quad \|\nabla v_\alpha\|^2 = 2\pi\alpha^2\alpha^{-3} = 2\pi\alpha^{-1}.$$

Increasing $\alpha$ rescales $v_\alpha$ into increasingly 'pointy' functions, for which

$$\mu_\alpha \equiv \frac{\|\triangle v_\alpha\|^2}{\|\nabla v_\alpha\|^2} = \alpha^2 \to \infty\ , \text{ as } \alpha \to \infty.$$

Graphs of $u(x)$, $u(3x)$ and $u(x/3)$, in black, red and blue.




Consider now an arbitrary open set $\Omega \subset R^3$, and point $y \in \Omega$. The functions $u(\alpha(x-y))$ all equal 1 at $x = y$. As $\alpha \to \infty$, they become increasingly singular at $y$, and smaller and flater away from $y$, near $\partial\Omega$, permiting truncation with minimal changes to the ratio $\frac{u^2(y)}{\|\nabla u\| \|\triangle u\|} = \frac{1}{2\pi}$. The optimality of the constant $\frac{1}{3\pi}$ in (1) is shown similarly. I expect the maximum ratio can only be approached by functions close to these.

## Proof of (3) for any $u \in \widehat{H}_0^1(\Omega)$ with $\triangle u \in L^2(\Omega)$.

At this point we are still assuming $\Omega$ is bounded and that $\partial\Omega$ is smooth. So $\widehat{H}_0^1(\Omega) = H_0^1(\Omega)$. By elliptic regularity and a Sobolev embedding $u \in H_0^2(\Omega) \subset C(\Omega)$. In terms of its eigenfunction expansion $u = \sum_{n=1}^{\infty} c_n \varphi_n$, with $c_n = \int_\Omega u \varphi_n \, dx$, the norms of $u$ satisfy

$$\|u\|^2 = \sum_{n=1}^{\infty} c_n^2, \quad \|\nabla u\|^2 = \sum_{n=1}^{\infty} \lambda_n c_n^2, \quad \|\triangle u\|^2 = \sum_{n=1}^{\infty} \lambda_n^2 c_n^2.$$

So each $u_m = \sum_{n=1}^{m} c_n \varphi_n$ satisfies $\|\nabla u_m\| \leq \|\nabla u\|$ and $\|\triangle u_m\| \leq \|\triangle u\|$. Thus if (3) were not true, there would be some $x_0 \in \Omega$ such that

$$\sup_\Omega |u_m|^2 \leq \frac{1}{2\pi} \|\nabla u_m\| \|\triangle u_m\| \leq \frac{1}{2\pi} \|\nabla u\| \|\triangle u\| < |u(x_0)|^2. \quad (10)$$

Therefore $|u(x_0)|^2$ would exceed $\sup_\Omega |u_m|^2$, for all $m$, by at least the difference between $\frac{1}{2\pi} \|\nabla u\| \|\triangle u\|$ and $|u(x_0)|^2$. Since $u$ is continuous, that would make impossible the $L^2$-convergence $\lim_{m\to\infty} \|u_m - u\| = 0$ of the eigenfunction expansion. This completes the proof of (3) for bounded smooth domains.

**Lemma.** For any open $\Omega \subset R^n$, there exist smoothly bounded open sets $\Omega_1 \subset \Omega_2 \subset \cdots$ such that $\cup_{i=1}^{\infty} \Omega_i = \Omega$.

**Proof:** Let $f \in C_0^\infty(\Omega)$ be real valued. According to the Morse-Sard theorem, the set of values taken at critical points has measure zero. Therefore, almost every level surface $f(x) = const.$ is everywhere smooth. Now, for every $\varepsilon > 0$, let

$$\widehat{f}_\varepsilon(x) = \begin{cases} 1 & \text{if } dist(x, \partial\Omega) > 2\varepsilon \text{ and } |x| < 1/\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

and let $f_\varepsilon$ be the mollification of $\widehat{f}_\varepsilon$ with smoothing radius $\varepsilon$. Then $f_\varepsilon(x) = 0$ if $dist(x, \partial\Omega) < \varepsilon$ or $|x| > 1/\varepsilon + \varepsilon$. And $f_\varepsilon(x) = 1$ if $dist(x, \partial\Omega) > 3\varepsilon$ and $|x| < 1/\varepsilon - \varepsilon$. So $f_\varepsilon$ is constant except in its region of variation

$$\{x : \varepsilon < dist(x, \partial\Omega) < 3\varepsilon \text{ or } 1/\varepsilon + \varepsilon < |x| < 1/\varepsilon + \varepsilon\}$$

Let $\varepsilon_1 = 1$, $\varepsilon_2 = 1/3$, $\varepsilon_3 = 1/9$, $\varepsilon_4 = 1/27$, e.t.c. Then the $\{f_{\varepsilon_n}\}$ have disjoint regions of variation And by Sard's theorem almost every level curve of $f_{\varepsilon_n}$ is smooth. For each $n$, choose one and let $\Omega_n$ be the enclosed set of points.

**Finally, let $\Omega \subset R^3$ be an arbitrary open set,** and suppose $u \in \widehat{H}_0^1(\Omega)$ has $\triangle u \in L^2(\Omega)$. Choose smoothly bounded $\Omega_1 \subset \Omega_2 \subset \cdots$ such that $\cup_{n=1}^{\infty} \Omega_n = \Omega$.

**Lemma.** *There exists $\{u_n\}$ such that $u_n \in \widehat{H}_0^1(\Omega_n)$, $\triangle u_n = \triangle u$ in $\Omega_n$, $\|\nabla u_n\|_{L^2(\Omega_n)} \leq \|\nabla u\|$, and $\|u_n - u\|_{L^6(\Omega)} \to 0$ as $n \to \infty$.*

**Proof:** For each $n$ there exists a unique $u_n \in \widehat{H}_0^1(\Omega_n)$ such that

$$\int_{\Omega_n} \nabla u_n \cdot \nabla \varphi \, dx = \int_{\Omega_n} \nabla u \cdot \nabla \varphi \, dx \,, \quad \text{for every } \varphi \in \widehat{H}_0^1(\Omega_n) \,, \qquad (11)$$

by the Riesz representation theorem. Integrating the right side of (11) by parts, we obtain

$$\int_{\Omega_n} \nabla u_n \cdot \nabla \varphi \, dx = -\int_{\Omega_n} (\triangle u) \varphi \, dx \,, \quad \text{for every } \varphi \in \widehat{H}_0^1(\Omega_n) \,,$$

which implies $\triangle u_n = \triangle u$ in $\Omega_n$, and therefore $\|\triangle u_n\|_{L^2(\Omega_n)} \leq \|\triangle u\|$.
We get $\|\nabla u_n\|_{L^2(\Omega_n)} \leq \|\nabla u\|$ by letting $\varphi = u_n$ in (11) and using the Schwarz inequality. Setting $u_n$ equal to zero in $\Omega \setminus \Omega_n$ we get $u_n \in \widehat{H}_0^1(\Omega)$. Therefore (11) implies

implies

$$\lim_{n\to\infty}\int_\Omega \nabla u_n\cdot\nabla\varphi\,dx=\int_\Omega \nabla u\cdot\nabla\varphi\,dx\,,\quad \text{for every } \varphi\in C_0^\infty(\Omega)\ ,$$

which is the weak convergence $u_n \to u$ in $\widehat{H}_0^1(\Omega)$. That along with the bound $\|\nabla u_n\| \le \|\nabla u\|$ implies the strong convergence $u_n \to u$ in $\widehat{H}_0^1(\Omega)$. Therefore, $\|u_n - u\|_{L^6(\Omega)} \to 0$, in view of the Sobolev inequality $\|\varphi\|_{L^6(\Omega)} \le c\,\|\nabla\varphi\|$. So the lemma is proven.

Now, if (3) were not true for the domain $\Omega$, there would be some $x_0 \in \Omega$ such that

$$\sup_{\Omega_n} |u_n|^2 \le \frac{1}{2\pi}\|\nabla u_n\|_{L^2(\Omega_n)}\|\triangle u_n\|_{L^2(\Omega_n)} \le \frac{1}{2\pi}\|\nabla u\|\,\|\triangle u\| < |u(x_0)|^2\ .$$

Since $u \in H^2_{loc}(\Omega) \subset C(\Omega)$ by the elliptic regularity theorem, this would make impossible the convergence $\|u_n - u\|_{L^6(\Omega)} \to 0$ that we just proved. This completes the proof of the inequality (3) for arbitrary open sets $\Omega \subset R^3$ and functions $u \in \widehat{H}_0^1(\Omega)$ with $\triangle u \in L^2(\Omega)$.

References (for first lecture):

W. Xie, A sharp pointwise bound for functions with $L^2$ Laplacians and zero boundary values on arbitrary three-dimensional domains, Indiana Univ. Math. J., 40 (1991), 1185-1192.

J. G. Heywood, The Navier-Stokes equations: on the existence, regularity and decay of solutions, Indiana Univ. Math. J., 29 (1980), 639-681.

J.G. Heywood and W. Xie, Smooth solutions of the vector Burgers equation in nosmooth domains, Diff. andInt. Equ. 10 (1997), 961-974

**Second Lecture: Attempt to Circumvent Xie's Conjecture**

$$\int_\Omega G_\mu^2\, dx \leq \int_\Omega g_\mu^2\, dx$$

by proving

$$\frac{\displaystyle\int_\Omega G_\mu^2\, dx}{\displaystyle\int_{R^3} g_\mu^2\, dx} \longrightarrow 1\ , \quad \text{as} \quad \mu \to \infty$$

and

$$\overline{\mu}_m \equiv \frac{\left\|\triangle \overline{u}_m\right\|^2}{\left\|\nabla \overline{u}_m\right\|^2} \longrightarrow \infty, \quad \text{as} \quad m \to \infty.$$

Let's first review Xie's argument in the context of the Stokes problem. As for the Laplacian, the Stokes eigenvalue problem

$$-\triangle \boldsymbol{\varphi}_n = -\nabla p + \lambda_n \boldsymbol{\varphi}_n \, , \quad \nabla \cdot \boldsymbol{\varphi}_n = 0 \, , \quad \boldsymbol{\varphi}_n |_{\partial\Omega} = 0$$

better written as

$$-\widetilde{\triangle} \boldsymbol{\varphi}_n = \lambda_n \boldsymbol{\varphi}_n \, , \quad \nabla \cdot \boldsymbol{\varphi}_n = 0 \, , \quad \boldsymbol{\varphi}_n |_{\partial\Omega} = 0$$

has a system of eigenfunctions $\{\boldsymbol{\varphi}_n\}$ which are complete in both $\mathbf{J}(\Omega)$ and $\mathbf{J}_0(\Omega)$, and orthonormal in $\mathbf{J}(\Omega)$. The eigenvalues $\{\lambda_n\}$ are positive. Corresponding to the Helmholtz operator $-\triangle + \mu$ and its fundamental singularity $g_\mu$ and Green's function $G_\mu$ we have the "spectral Stokes operator" $-\widetilde{\triangle} + \mu$ and its fundamental singularity $\mathbf{g}_{\mu,\mathbf{e}}$ and Green's function $\mathbf{G}_{\mu,\mathbf{e}}$. Both $\mathbf{g}_{\mu,\mathbf{e}}$ and $\mathbf{G}_{\mu,\mathbf{e}}$ depend not only on the point of singularity $y$, but also on a directional unit vector $\mathbf{e}$. The singularity is

$$\mathbf{g}_{\mu,\mathbf{e}}(x,y) = \frac{e^{-\sqrt{\mu}|x-y|}}{4\pi |x-y|}\mathbf{e} - (\mathbf{e}\cdot\nabla)\nabla \frac{e^{-\sqrt{\mu}|x-y|} - 1}{4\pi\mu |x-y|}$$

which is the solution of

the problem (in which $y$ is the point of singularity, considered fixed)

$$(-\triangle + \mu)\, \mathbf{g}_{\mu,\mathbf{e}}(x,y) + \nabla p = \delta(x-y)\,\mathbf{e} \quad \text{in } \Omega,$$
$$\nabla \cdot \mathbf{g}_{\mu,\mathbf{e}}(x,y) = 0\ , \qquad \lim_{|x|\to\infty} \mathbf{g}_{\mu,\mathbf{e}}(x,y)\big|_{\partial\Omega} = 0\ .$$

The Green's function is $\mathbf{G}_{\mu,\mathbf{e}}(x,y) = \mathbf{g}_{\mu,\mathbf{e}}(x,y) - \mathbf{h}_{\mu,\mathbf{e}}(x,y)$ where $\mathbf{h}_{\mu,\mathbf{e}}(x,y)$ is the solution of

$$(-\triangle + \mu)\, \mathbf{h}_{\mu,\mathbf{e}}(x,y) + \nabla p = 0 \quad \text{in } \Omega,$$
$$\nabla \cdot \mathbf{h}_{\mu,\mathbf{e}}(x,y) = 0\ , \qquad \mathbf{h}_{\mu,\mathbf{e}}(x,y)\big|_{\partial\Omega} = \mathbf{g}_{\mu,\mathbf{e}}(x,y)\big|_{\partial\Omega}\ .$$

There are probably domains which contain points $x$ and $y$ for which $\left|\mathbf{G}_{\mu,\mathbf{e}}(x,y)\right| > \left|\mathbf{g}_{\mu,\mathbf{e}}(x,y)\right|$. Certainly, I can find two domains, one within the other, containing specific points $x$ and $y$ for which $\left|\mathbf{G}_{\mu,\mathbf{e}}(x,y)\right|$ is larger for the smaller domain. Because of such difficulties, the crucial inequality

$$\int_\Omega \mathbf{G}^2_{\mu,\mathbf{e}}(x,y)\ dx \le \int_{R^3} \mathbf{g}^2_{\mu,\mathbf{e}}(x,y)\ dx\ .$$

remains "Xie's conjecture". But I would bet anything it's true!

To review in the Stokes context, note that $\sup_{\Omega} |\mathbf{u}| = \sup_{y\in\Omega} \sup_{|\mathbf{e}|=1} \mathbf{e}\cdot\mathbf{u}(y)$ .

Fixing $y$ and $\mathbf{e}$, and temporarily $m$, consider functions $\mathbf{u}_m = \sum_{n=1}^{m} c_n \boldsymbol{\varphi}_n$ depending on $(c_1, c_2, \cdots, c_m)$ . Let $(\overline{c}_1, \overline{c}_2, \cdots, \overline{c}_m)$ maximize

$$R_m(y,\mathbf{e}) \equiv \frac{(\mathbf{e}\cdot\mathbf{u}_m(y))^2}{\|\nabla\mathbf{u}_m\| \left\|\widetilde{\triangle}\mathbf{u}_m\right\|} = \frac{\left(\sum_{n=1}^{m} c_n \mathbf{e}\cdot\boldsymbol{\varphi}_n(y)\right)^2}{\left(\sum_{n=1}^{m} \lambda_n c_n^2\right)^{1/2}\left(\sum_{n=1}^{m} \lambda_n^2 c_n^2\right)^{1/2}}. \quad (12)$$

Setting $\overline{\mathbf{u}}_m = \sum_{n=1}^{m} \overline{c}_n \boldsymbol{\varphi}_n$ and $\overline{\mu}_m \equiv \frac{\left\|\widetilde{\triangle}\overline{\mathbf{u}}_m\right\|^2}{\|\nabla\overline{\mathbf{u}}_m\|^2}$ **and using Xie's conjecture**

$$\begin{aligned}\overline{R}_m(y,\mathbf{e}) &\equiv \frac{(\mathbf{e}\cdot\overline{\mathbf{u}}_m(y))^2}{\|\nabla\overline{\mathbf{u}}_m\| \left\|\widetilde{\triangle}\overline{\mathbf{u}}_m\right\|} = \cdots = 4\sqrt{\overline{\mu}_m}\sum_{n=1}^{m}\left(\frac{\mathbf{e}\cdot\boldsymbol{\varphi}_n(y)}{\overline{\mu}_m+\lambda_n}\right)^2 \\ &= 4\sqrt{\overline{\mu}_m}\sum_{n=1}^{m}\left(\int_{\Omega}\mathbf{G}_{\overline{\mu}_m,\mathbf{e}}\cdot\boldsymbol{\varphi}_n\,dx\right)^2 \\ &\leq 4\sqrt{\overline{\mu}_m}\int_{\Omega}\mathbf{G}^2_{\overline{\mu}_m,\mathbf{e}}\,dx \leq 4\sqrt{\overline{\mu}_m}\int_{R^3}\mathbf{g}^2_{\overline{\mu}_m,\mathbf{e}}\,dx = \frac{1}{3\pi}.\end{aligned} \quad (13)$$

The optimality of the constant $\frac{1}{3\pi}$ in the conjectured inequality

$$\sup_{\Omega} |\mathbf{u}|^2 \leq \frac{1}{3\pi} \|\nabla \mathbf{u}\| \left\| \widetilde{\triangle} \mathbf{u} \right\|$$

is proven analgously to that of the constant in the inequality

$$\sup_{\Omega} |u|^2 \leq \frac{1}{2\pi} \|\nabla u\| \|\triangle u\| .$$

The role in the latter played by the difference of singularities

$$u(x) \equiv 4\pi \left(g_0(x,0) - g_1(x,0)\right) = \frac{1-e^{|x|}}{|x|}$$

is replaced by

$$\begin{aligned} \mathbf{u}(x) &\equiv 4\pi \left(\mathbf{g}_{0,\mathbf{e}}(x,0) - \mathbf{g}_{1,\mathbf{e}}(x,0)\right) \\ &= \frac{1-e^{-|x|}}{|x|}\mathbf{e} - (\mathbf{e}\cdot\nabla)\nabla\left(\frac{|x|}{2} + \frac{1-e^{-|x|}}{|x|}\right), \end{aligned}$$

where $\mathbf{g}_{\mu,\mathbf{e}}(x,y)$ is the singular solution introduced earlier, satisfying

$$(-\triangle + \mu)\,\mathbf{g} + \nabla P = \delta(x-y)\,\mathbf{e}, \;\; \nabla\cdot\mathbf{g} = 0 .$$

Perhaps, if the maximum principle had not been available in proving (3), the last line of Xie's proof of (3), namely

$$\leq 4\sqrt{\overline{\mu}_m}\int_\Omega G^2_{\overline{\mu}_m}\,dx \leq 4\sqrt{\overline{\mu}_m}\int_{R^3} g^2_{\overline{\mu}_m}\,dx = \frac{1}{2\pi},$$

could have been replaced by arguing that

$$4\sqrt{\overline{\mu}_m}\int_\Omega G^2_{\overline{\mu}_m}\,dx = \frac{4\sqrt{\overline{\mu}_m}\int_\Omega G^2_{\overline{\mu}_m}\,dx}{8\pi\sqrt{\overline{\mu}_m}\int_{R^3} g^2_{\overline{\mu}_m}\,dx} \to \frac{1}{2\pi}\,, \quad \text{as} \quad \overline{\mu}_m \to \infty,$$

and that

$$\overline{\mu}_m \equiv \frac{\left\|\triangle \overline{u}_m\right\|^2}{\left\|\nabla \overline{u}_m\right\|^2} \to \infty, \quad \text{as} \quad m \to \infty.$$

If this reasoning were to prove successful in dealing with the Poisson problem, it might very well carry over to Stokes problem. I will now describe some of my efforts in this direction.

For $-\triangle+\mu$, the exponential decay of the fundamental singularity $g_\mu(r)=\frac{e^{-\sqrt{\mu}r}}{4\pi r}$ makes easy the proof that

$$\frac{\int_\Omega G_\mu^2\,dx}{\int_{R^3} g_\mu^2\,dx}\to 1\ ,\ \text{ as }\ \mu\to\infty.$$

For the spectral Stokes operator $-\widetilde{\triangle}+\mu$, the fundamental singularity

$$\mathbf{g}_\mu(r)=\frac{e^{-\sqrt{\mu}r}}{4\pi r}\mathbf{e}-(\mathbf{e}\cdot\nabla)\nabla\frac{e^{-\sqrt{\mu}r}-1}{4\pi\mu r}$$

presents more of a challenge, but the desired result

$$\frac{\int_\Omega \mathbf{G}_\mu^2\,dx}{\int_{R^3} \mathbf{g}_\mu^2\,dx}\to 1\ ,\ \text{ as }\ \mu\to\infty.$$

was proven in my 2000 "Ferrara" paper. Thus to prove (1), it only remains to show $\overline{\mu}_m\to\infty$ as $m\to\infty$. I am persuaded of this by considering Xie's proofs of the optimality of the constants in the inequalities (3) and (1).

To determine $\overline{\mu}_m$ for the Poisson problem, reconsider the maximization of

$$R_m \equiv \frac{u_m^2(y)}{\|\nabla u_m\| \, \|\triangle u_m\|} = \frac{\left(\sum_{n=1}^{m} c_n \varphi_n(y)\right)^2}{\left(\sum_{n=1}^{m} \lambda_n c_n^2\right)^{1/2} \left(\sum_{n=1}^{m} \lambda_n^2 c_n^2\right)^{1/2}}$$

by the method of Lagrange multipliers. Since $R_m$ is homogeneous, the maximizing point $(\overline{c}_1, \overline{c}_2, \cdots, \overline{c}_m)$ can be chosen to satisfy the constraint

$$g(c_1, \cdots, c_m) \equiv \sum_{n=1}^{m} c_n \varphi_n(y) = 1,$$

while minimizing the denominator of $R_m$, or its square

$$f(c_1, \cdots, c_m) \equiv \left(\sum_{n=1}^{m} \lambda_n c_n^2\right) \left(\sum_{n=1}^{m} \lambda_n^2 c_n^2\right).$$

Setting $\nabla f = \lambda \nabla g$ and eliminating the Lagrange multiplier $\lambda$, it is shown in my 2001 "Advances" paper that $\overline{\mu}_m$ is a root of the equation

$$f_m(\mu) \equiv \sum_{n=1}^{m} \frac{\varphi_n^2(y)}{(\mu + \lambda_n)^2} \left[\frac{\mu}{\lambda_n} - 1\right] = 0.$$

Obviously,

$$f_m(\mu) \equiv \sum_{n=1}^{m} \frac{\varphi_n^2(y)}{(\mu+\lambda_n)^2}\left[\frac{\mu}{\lambda_n}-1\right]$$

is negative for $\mu \in [0, \lambda_1)$ and positive for $\mu \in (\lambda_m, \infty)$. Hence $\lambda_1 \leq \overline{\mu}_m \leq \lambda_m$. Furthermore, the infinite series

$$f_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{\varphi_n^2(y)}{(\mu+\lambda_n)^2}\left[\frac{\mu}{\lambda_n}-1\right] \tag{14}$$

is absolutely and uniformly covergent for $\mu$ in any bounded subinterval of $[0, \infty)$.

**Conjecture:** For every $y \in \Omega$, we suspect that $f_\infty(\mu) < 0$, for all $\mu \geq 0$.

This conjecture implies $\overline{\mu}_m \to \infty$ as $m \to \infty$, which implies (3) without using the maximum principle. All of this reasoning carries over to a similar conjecture that implies (1) for the Stokes problem.

One indication that $f_\infty(\mu) < 0$, for all $\mu \geq 0$, follows from the comparison

$$f_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{\varphi_n^2(y)}{(\mu+\lambda_n)^2}\left[\frac{\mu}{\lambda_n} - 1\right]$$

$$\widehat{f}_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{V^{-1}}{(\mu+n^{2/3})^2}\left[\frac{\mu}{n^{2/3}} - 1\right]$$

$$f_{int}(\mu) \equiv \int_0^\infty \frac{V^{-1}}{(\mu+x^{2/3})^2}\left[\frac{\mu}{x^{2/3}} - 1\right] dx = 0,$$

suggested by the asymptotics (after normalizing the volume of $\Omega$ to be $V = 6\pi^2$)

$$\lambda_n \sim n^{2/3}$$

'Large averages' of $\left\{\varphi_n^2(y)\right\} \to V^{-1}$.

It can be shown that $\widehat{f}_\infty(\mu) < f_{int}(\mu) = 0$, suggesting that $f_\infty(\mu) < 0$.

But seeking a proof I've pushed further with another approach.

The function $f_\infty$ can be expressed in terms of the Green's functions

$$G_0(x,y) = \frac{1}{4\pi r} + w_0(x,y) \ , \qquad G_\mu(x,y) = \frac{e^{-\sqrt{\mu}r}}{4\pi r} + w_\mu(x,y)$$

for $-\triangle$ and $-\triangle + \mu$ by rearranging it as follows:

$$\begin{aligned} f_\infty(\mu, y) &\equiv \sum_{n=1}^{\infty} \frac{\varphi_n^2(y)}{(\mu+\lambda_n)^2} \left[\frac{\mu}{\lambda_n} - 1\right] \\ &= \sum_{n=1}^{\infty} \frac{\varphi_n^2(y)}{\lambda_n(\mu+\lambda_n)} - 2\sum_{n=1}^{\infty} \frac{\varphi_n^2(y)}{(\mu+\lambda_n)^2} \\ &= \int_\Omega G_\mu(x,y)\, G_0(x,y)\, dx - 2\left\| G_\mu(\cdot, y)\right\|^2 . \end{aligned}$$

In attempting to evaluate the first term, note that

$$-\triangle\left(G_0(x,y) - G_\mu(x,y)\right) = \mu G_\mu(x,y)\,,$$

suggesting the representation of $G_0(x,y) - G_\mu(x,y)$ in terms of $G_0$ ,

$$\mu \int_\Omega G_\mu(z,y)\, G_0(z,x)\, dz = G_0(x,y) - G_\mu(x,y) \ , \quad \text{for all } x, y \in \Omega.$$

In fact, the representation

$$\mu \int_\Omega G_\mu(z,y)\, G_0(z,x)\ dz = G_0(x,y) - G_\mu(x,y)\ , \qquad \text{for all } x, y \in \Omega$$

is justified, since, for fixed $y$ and $x \neq y$,

$$-\triangle \left( G_0(x,y) - G_\mu(x,y) \right) = \mu G_\mu(x,y)$$

and the singularity is not very bad. Indeed,

$$\lim_{r \to 0} \left( \frac{1}{4\pi r} - \frac{e^{-\sqrt{\mu} r}}{4\pi r} \right) = \frac{\sqrt{\mu}}{4\pi}$$

is finite and $G_0(x,y) - G_\mu(x,y) \in H^2_{loc}\left(R^3\right)$. Thus, on taking the limit as $x \longrightarrow y$, one obtains

$$\int_\Omega G_\mu(z,y)\, G_0(z,y)\ dz = \frac{1}{4\pi\sqrt{\mu}} + \frac{w_0(y,y) - w_\mu(y,y)}{\mu}$$

and therefore

$$f_\infty(\mu, y) = \frac{1}{4\pi\sqrt{\mu}} + \frac{w_0(y,y) - w_\mu(y,y)}{\mu} - 2\left\| G_\mu(\cdot, y) \right\|^2 .$$

I haven't a general proof that $f_\infty(\mu, y) < 0$ for all $\mu \geq 0$, for all $y \in \Omega$.

But suppose $\Omega$ is the sphere $|x| < R$ and $y = 0$. Then we know

$$w_0(x,0) = -\frac{1}{4\pi R} \quad \text{and} \quad w_\mu(x,0) = -\left(\frac{e^{-\sqrt{\mu}R}}{e^{\sqrt{\mu}R} - e^{-\sqrt{\mu}R}}\right)\frac{e^{\sqrt{\mu}r} - e^{-\sqrt{\mu}r}}{4\pi r}.$$

Setting $a = \frac{e^{-\sqrt{\mu}R}}{e^{\sqrt{\mu}R} - e^{-\sqrt{\mu}R}}$ , we find after a rather lengthy calculation that

$$\int_\Omega G_\mu(x,0)\, G_0(x,0)\; dx = \frac{1}{4\pi\sqrt{\mu}} - \frac{1}{4\pi\mu R} + a\frac{1}{2\pi\sqrt{\mu}}\;,$$

$$2\left\| G_\mu(\cdot,0)\right\|^2 = 2\int_0^R \left(\frac{e^{-\sqrt{\mu}r}}{4\pi r} - a\frac{e^{\sqrt{\mu}r} - e^{-\sqrt{\mu}r}}{4\pi r}\right)^2 4\pi r^2 dr\;,$$

and finally, combining these, the following identity. For all $\mu \geq 0$,

$$f_\infty(\mu,0) = \sum_{n=1}^{\infty} \frac{\varphi_n^2(0)}{(\mu+\lambda_n)^2}\left[\frac{\mu}{\lambda_n} - 1\right] = -\frac{1}{4\pi\mu R} + \frac{R}{4\pi\sinh^2\sqrt{\mu}R} < 0.$$

This proves, for at least the center of a sphere, that $f_\infty(\mu,0) < 0$ for all $\mu \geq 0$, and hence that $\overline{\mu}_m \equiv \left\|\triangle \overline{u}_m\right\|^2 / \left\|\nabla \overline{u}_m\right\|^2 \to \infty$, as $m \to \infty$, in Xie's proof of the inequality (3) for the Poisson problem. Using the maximum principle I've gone a bit further.

**Proposition.** Let $y \in \Omega \subset R^3$ and suppose that for some $R > 0$ the balls of radii $R$ and $R/2$ centered at $y$ satisfy $B_{R/2} \subset \Omega \subset B_R$. Then $f_\infty(\mu, y) < 0$ for all $\mu \geq 0$, and hence $\overline{\mu}_m \to \infty$ in Xie's first proof of (3).

**Corollary.** If $\Omega$ is a ball of radius $R$, then, for all points within a distance $R/3$ of its center, there holds $f_\infty(\mu, y) < 0$ for all $\mu \geq 0$, and hence $\overline{\mu}_m \to \infty$ in Xie's first proof of (3).

**Homework.** For my next lecture, I would be greatful for help proving that, for every fixed nonzero point $y \in (0, \pi)$, the early terms in the sequence $\{\sin^2 ny\}$ do not dominate the later ones, in the sense that

$$f_\infty(\mu, y) \equiv \sum_{n=1}^{\infty} \frac{\sin^2 ny}{\left(\mu + n^{2/3}\right)^2} \left[\frac{\mu}{n^{2/3}} - 1\right] < 0, \quad \text{for all } \mu \in [0, \infty) . \quad (15)$$

This, of course, is an analogue of the conjecture

$$f_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{\varphi_n^2(y)}{(\mu + \lambda_n)^2} \left[\frac{\mu}{\lambda_n} - 1\right] < 0, \quad \text{for all } \mu \in [0, \infty)$$

that I failed to completely prove above. It may be insightful and easier to first consider the corresponding integral, and to try to show that

$$\int_0^\infty \frac{\sin^2 xy}{\left(\mu + x^{2/3}\right)^2} \left[\frac{\mu}{x^{2/3}} - 1\right] dx < 0, \quad \text{for all } \mu \in [0, \infty) . \quad (16)$$

Remember that if $\sin^2 xy$ is replaced by 1, this integral is identically zero for all $\mu$.

References (for second lecture):

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J. G. Heywood, An alternative to Xie's conjecture concerning the Stokes problem in nonsmooth domains, Annali dell' Universita' di Ferrara set VII, n. 46, 2000, 267-284.

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J. G. Heywood, Seeking a proof of Xie's inequality: on the conjecture that $\mu_m \to \infty$, preprint, submitted for the volume in honour of Y. Shibata (2013), 16 pages.

## Third Lecture: Other Applications of the Preceding Arguments

For example:

For a Fourier cosine series $f(x) = \sum_{n=1}^{\infty} c_n \cos nx$ , we prove

$$\sup_{(-\pi,\pi)} |f(x)|^2 \leq 3 \left\| f^{(1/3)} \right\|_{L^2(-\pi,\pi)} \left\| f^{(2/3)} \right\|_{L^2(-\pi,\pi)}$$

and for a Fourier sine series $f(x) = \sum_{n=1}^{\infty} c_n \sin nx$ , we conjecture

$$\sup_{(-\pi,\pi)} |f(x)|^2 \leq \frac{3}{2} \left\| f^{(1/3)} \right\|_{L^2(-\pi,\pi)} \left\| f^{(2/3)} \right\|_{L^2(-\pi,\pi)} .$$

The proofs and difficulties are analogous to those we encountered for

$$\sup_{\Omega} |u|^2 \leq \tfrac{1}{2\pi} \|\nabla u\| \, \|\triangle u\| \qquad \text{Poisson problem}$$

$$\sup_{\Omega} |\mathbf{u}|^2 \leq \tfrac{1}{3\pi} \|\nabla \mathbf{u}\| \left\| \widetilde{\triangle} \mathbf{u} \right\| \qquad \text{Stokes problem.}$$

I've found Xie's first argument can be used to prove other inequalities. One is a short, simple proof of Hölder's inequality for series. Another is:

**Theorem 1.** For any sequence $c_1, c_2, \cdots$ of real numbers,

$$\left(\sum_{n=1}^{\infty} c_n\right)^2 \leq 3\pi \left(\sum_{n=1}^{\infty} n^{2/3} c_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^{4/3} c_n^2\right)^{1/2} . \tag{17}$$

**Corollary.** If $f(x) = \sum_{n=1}^{\infty} c_n \cos nx$ , then

$$\sup_{(-\pi,\pi)} |f(x)|^2 \leq 3 \left\| f^{(1/3)} \right\|_{L^2(-\pi,\pi)} \left\| f^{(2/3)} \right\|_{L^2(-\pi,\pi)} .$$

Clarifying Definitions: For any $\alpha \geq 0$, the $L^2$-norm of the fractional derivative of $f$ of order $\alpha$ is $\left\| f^{(\alpha)} \right\|_{L^2(-\pi,\pi)} = \left(\pi \sum_{n=1}^{\infty} c_n^2 n^{2\alpha}\right)^{1/2}$, in agreement with the standard definitions when $\alpha$ is a non-negative integer. Clearly, $\sup_{(-\pi,\pi)} |f| = f(0) = \sum_{n=1}^{\infty} c_n$ , if $c_n \geq 0$ for all $n$. If some of the $c_n$ are negative, the supremum of $f$ will be only smaller.

Notice that the direct application of Hölder's inequality (first for $c_n \geq 0$)

$$\sum_{n=1}^{m} c_n = \sum_{n=1}^{m} \frac{1}{\sqrt{n}} \left(n^{1/6}\sqrt{c_n}\right)\left(n^{2/6}\sqrt{c_n}\right)$$
$$\leq \left(\sum_{n=1}^{m} \frac{1}{n}\right)^{1/2} \left(\sum_{n=1}^{m} n^{2/3} c_n^2\right)^{1/4} \left(\sum_{n=1}^{m} n^{4/3} c_n^2\right)^{1/4}$$

implies an inequality similar to (17) for finite sums,

$$\left(\sum_{n=1}^{m} c_n\right)^2 \leq \left(\sum_{n=1}^{m} \frac{1}{n}\right) \left(\sum_{n=1}^{m} n^{2/3} c_n^2\right)^{1/2} \left(\sum_{n=1}^{m} n^{4/3} c_n^2\right)^{1/2},$$

but with the $m$-dependent constant $\sum_{n=1}^{m} \frac{1}{n}$ instead of $3\pi$.

**Proof of Theorem:** We follow Xie's argument line by line. For fixed $m$, the ratio of the left to right sides of (17), namely

$$R_m \equiv \frac{\left(\sum_{n=1}^{m} c_n\right)^2}{\left(\sum_{n=1}^{m} n^{2/3} c_n^2\right)^{1/2} \left(\sum_{n=1}^{m} n^{4/3} c_n^2\right)^{1/2}}, \tag{18}$$

is a smooth function of $(c_1, c_2, \cdots, c_m) \in R^m \setminus \{0\}$. Since it is constant on lines through the origin, it has a maximum value which is attained on every sphere about the origin. Let $(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m) \neq 0$ be a point where the maximum is attained. Differentiating

$$\log R_m = 2 \log \left(\sum_{n=1}^{m} c_n\right) - \frac{1}{2} \log \left(\sum_{n=1}^{m} n^{2/3} c_n^2\right) - \frac{1}{2} \log \left(\sum_{n=1}^{m} n^{4/3} c_n^2\right)$$

with respect to $c_n$ at the critical point $(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m)$ gives

$$\frac{2}{\left(\sum_{n=1}^{m} \bar{c}_n\right)} = \frac{n^{2/3} \bar{c}_n}{\left(\sum_{n=1}^{m} n^{2/3} \bar{c}_n^2\right)} + \frac{n^{4/3} \bar{c}_n}{\left(\sum_{n=1}^{m} n^{4/3} \bar{c}_n^2\right)}. \tag{19}$$

Setting

$$\bar{\mu}_m = \left( \sum_{n=1}^{m} n^{4/3} \bar{c}_n^2 \right) / \left( \sum_{n=1}^{m} n^{2/3} \bar{c}_n^2 \right) , \tag{20}$$

and proceeding exactly as before, etc., etc., one obtains

$$\begin{aligned} \bar{R}_m &= \frac{\left( \sum_{n=1}^{m} n^{4/3} \bar{c}_n^2 \right)^{1/2}}{\left( \sum_{n=1}^{m} n^{2/3} \bar{c}_n^2 \right)^{1/2}} \frac{\left( \sum_{n=1}^{m} \bar{c}_n \right)^2}{\left( \sum_{n=1}^{m} n^{4/3} \bar{c}_n^2 \right)} \\ &= \sqrt{\bar{\mu}_m} \sum_{n=1}^{m} \frac{4}{\left(\bar{\mu}_m + n^{2/3}\right)^2} < \sqrt{\bar{\mu}_m} \sum_{n=1}^{\infty} \frac{4}{\left(\bar{\mu}_m + n^{2/3}\right)^2} \\ &< \sqrt{\bar{\mu}_m} \int_0^{\infty} \frac{4\,dx}{\left(\bar{\mu}_m + x^{2/3}\right)^2} = \sqrt{\bar{\mu}_m} \frac{3\pi}{\sqrt{\bar{\mu}_m}} = 3\pi. \end{aligned} \tag{21}$$

The comparison of the series with an integral takes the place of the inequality $\int_\Omega G^2 dx \leq \int_{R^3} g^2 dx$ or of Xie's conjecture. Nothing is learned about the $\bar{c}_n$ or $\bar{\mu}_m$, and the optmality of the constant $3\pi$ is not proven.

**Theorem 2.** *In the preceeding proof,* $\bar{\mu}_m \to \infty$ *as* $m \to \infty$.

**Proof:** Since $R_m(c_1, \cdots, c_m)$ is constant on lines through the origin, its maxima $(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m)$ can be chosen to satisfy the constraint $\sum_{n=1}^{m} \bar{c}_n = 1$. Assuming that is done, the problem of maximizing $R_m$ subject to this constraint (that its numerator should equal one) is equivalent to that of minimizing its denominator (or the square of its denominator) subject to this constraint. Thus the maximizing critical points in the 'normalized' proof are points where

$$f(c_1, \cdots, c_m) \equiv \left( \sum_{n=1}^{m} n^{2/3} c_n^2 \right) \left( \sum_{n=1}^{m} n^{4/3} c_n^2 \right) \qquad (22)$$

is minimized subject to the constraint

$$g(c_1, \cdots, c_m) \equiv \sum_{n=1}^{m} c_n = 1. \qquad (23)$$

The maximizing critical points can be found by the method of Lagrange multipliers. The proof is line for line like that we developed trying to prove $\bar{\mu}_m \to \infty$ *as* $m \to \infty$ in Xie's arguments for the Laplacian and Stokes operator. Since we didn't give the details in that context we will now.

**Lemma 1.** *In the 'normalized' proof of Theorem 1, the maximizing coefficients $\bar{c}_n$ satisfy the equations (25) below.*

**Proof:** If $(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m)$ minimizes $\left(\sum_{n=1}^{m} n^{2/3} c_n^2\right)\left(\sum_{n=1}^{m} n^{4/3} c_n^2\right)$ subject to the constraint $\sum_{n=1}^{m} c_n = 1$, there exists a Lagrange multiplier $\lambda$ such that

$$2\bar{c}_n n^{2/3} B_m + 2\bar{c}_n n^{4/3} A_m = \lambda\,, \quad \text{for } n = 1, \cdots, m, \tag{24}$$

where

$$A_m = \sum_{n=1}^{m} n^{2/3} \bar{c}_n^2\,, \quad B_m = \sum_{n=1}^{m} n^{4/3} \bar{c}_n^2\,.$$

Multiplying (24) by $\bar{c}_n$, summing, and using the constraint gives

$$4 A_m B_m = \lambda.$$

Substituting this value for $\lambda$ into (24) gives

$$\bar{c}_n = \frac{2 A_m B_m}{n^{2/3} B_m + n^{4/3} A_m}\,, \quad \text{for } n = 1, \cdots, m. \tag{25}$$

**Lemma 2.** *$\bar{\mu}_m \equiv B_m / A_m$ is a root of the equation*

$$f_m(\mu) \equiv \sum_{n=1}^{m} \frac{1}{(\mu + n^{2/3})^2} \left[ \frac{\mu}{n^{2/3}} - 1 \right] = 0 \,. \tag{26}$$

Proof. Square (25), multiply by $n^{2/3}$, and sum to obtain

$$\begin{aligned} A_m &= 4A_m^2 B_m^2 \sum_{n=1}^{m} \frac{n^{2/3}}{(n^{2/3} B_m + n^{4/3} A_m)^2} \\ &= 4A_m B_m \sum_{n=1}^{m} \frac{1}{(\bar{\mu}_m + n^{2/3})^2} \left( \frac{\bar{\mu}_m}{n^{2/3}} \right) . \end{aligned} \tag{27}$$

Similarly, square (25), multiply by $n^{4/3}$, and sum to obtain

$$\begin{aligned} B_m &= 4A_m^2 B_m^2 \sum_{n=1}^{m} \frac{n^{4/3}}{(n^{2/3} B_m + n^{4/3} A_m)^2} \\ &= 4B_m^2 \sum_{n=1}^{m} \frac{1}{(\bar{\mu}_m + n^{2/3})^2} \,. \end{aligned} \tag{28}$$

Finally, multiply (28) by $A_m / B_m$ and subtract from (27) to obtain (26) with $\bar{\mu}_m$ in place of $\mu$.

So $\mu = \bar{\mu}_m$ is a root of

$$f_m(\mu) \equiv \sum_{n=1}^{m} \frac{1}{(\mu + n^{2/3})^2} \left[\frac{\mu}{n^{2/3}} - 1\right] = 0 \,.$$

**Lemma 3.** *Since $f_m(\mu) < 0$ for $\mu < 1$, and $f_m(\mu) > 0$ for $\mu > m^{2/3}$, all roots of $f_m(\mu) = 0$ lie in the interval $\left[1, m^{2/3}\right]$. Furthermore, the convergence*

$$f_m(\mu) \to f_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{1}{(\mu + n^{2/3})^2} \left[\frac{\mu}{n^{2/3}} - 1\right] \tag{29}$$

*is uniform for $\mu$ in any bounded subinterval of $[0, \infty)$.*

**Proof:** These claims are obvious.

**Lemma 4.** *$f_\infty(\mu) < 0$ for all $\mu \in [0, \infty)$.*

Proof: Obviously, $f_\infty(\mu) < 0$ for $\mu \in [0, 1]$. For every $\mu > 0$, we can compare $f_\infty(\mu)$ with the integral

$$f_{int}(\mu) \equiv \int_0^\infty \frac{1}{(\mu + x^{2/3})^2} \left[\frac{\mu}{x^{2/3}} - 1\right] dx \,. \tag{30}$$

Remarkably, for all $\mu > 0$, the integral in $x$,

$$f_{int}(\mu) \equiv \int_0^\infty \frac{1}{(\mu + x^{2/3})^2} \left[ \frac{\mu}{x^{2/3}} - 1 \right] dx$$
$$= \left[ 3\frac{x^{5/3}}{\mu^3 + x^2} - 3\mu \frac{x}{\mu^3 + x^2} + 3\mu^2 \frac{x^{1/3}}{\mu^3 + x^2} \right]_0^\infty = 0\,.$$

The series $f_\infty(\mu) < f_{int}(\mu)$ for all $\mu > 1$, but the proof is a bit tedious because the integrand is not monotonically decreasing in $x$. It decreases monotonically from $+\infty$ to a negative value of $\left(17\sqrt{17} - 71\right)/16\mu^2$ at $x = \frac{1}{8}\mu^{3/2}\left(\sqrt{17} + 3\right)^{3/2}$, and then increases monotonically toward 0 as $x \to \infty$. Fortunately, the integral over $(0, 1]$ exceeds the first term of the series by an amount that dominates the comparison.

We can now complete the proof of Theorem 2. Since the sequence of functions $f_m(\mu)$ is uniformly convergent to the negative function $f_\infty(\mu)$ on every bounded subinterval of $[0, \infty)$, the roots of $f_m(\mu) = 0$ are pushed out further and further to the right as $m$ increases. Thus $\bar{\mu}_m \to \infty$, as $m \to \infty$.

**Next** we seek conditions on a given sequence $\{a_n\}$ that will ensure

$$\left(\sum_{n=1}^{\infty} a_n c_n\right)^2 \leq 3\pi \left(\sum_{n=1}^{\infty} n^{2/3} c_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^{4/3} c_n^2\right)^{1/2}$$

for every other sequence $\{c_n\}$. Simply following the proof of Theorem 1, in which every $a_n = 1$, we find at the end that it suffices to asume

$$\limsup_{\mu\to\infty} \sum_{n=1}^{\infty} \frac{4a_n^2}{\left(\mu^{3/4} + \mu^{-1/4} n^{2/3}\right)^2} \leq 3\pi \;, \tag{31}$$

along with an assumption replacing Lemma 4, to ensure that $\bar{\mu}_m \to \infty$ as $m \to \infty$. We interpret the sum in (31) as a weighted average, noting that

$$\int_0^{\infty} \frac{4\,dx}{\left(\mu^{3/4} + \mu^{-1/4} x^{2/3}\right)^2} = 3\pi \;, \quad \text{for all } \mu > 0, \tag{32}$$

and that the integrand is a decreasing function of $x$, which becomes smaller and more nearly constant over ever longer intervals of $x$, as $\mu \to \infty$. The assumption replacing Lemma 4 is that

$$f_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{a_n^2}{\left(\mu + n^{2/3}\right)^2} \left[\frac{\mu}{n^{2/3}} - 1\right] < 0, \quad \text{for all } \mu \in [0, \infty)\,.$$

**Theorem 3.** *Let $a_1, a_2, \cdots$ be a sequence of real numbers satisfying (34) and (35) below. Then, for any other sequence $c_1, c_2, \cdots$, one has*

$$\left(\sum_{n=1}^{\infty} a_n c_n\right)^2 \leq 3\pi \left(\sum_{n=1}^{\infty} n^{2/3} c_n^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^{4/3} c_n^2\right)^{1/2} . \qquad (33)$$

*The first assumption about $\{a_n\}$ is that large averages of the numbers $\{a_n^2\}$ tend to values less than or equal to 1, in the sense that*

$$\limsup_{\mu\to\infty} \sum_{n=1}^{\infty} \frac{4a_n^2}{\left(\mu^{3/4} + \mu^{-1/4} n^{2/3}\right)^2} \leq 3\pi . \qquad (34)$$

*The second hypothesis is that the 'first terms' of the sequence $\{a_n\}$, which are weighted positively, i.e., those for $n < \mu^{3/2}$, should not be too large relative to the 'later terms', which are weighted negatively, in the sense that*

$$f_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{a_n^2}{\left(\mu + n^{2/3}\right)^2} \left[\frac{\mu}{n^{2/3}} - 1\right] < 0, \quad \text{for all } \mu \in [0, \infty) . \qquad (35)$$

Note that (35) is a test to be satisfied for every 'dividing point' $\mu^{3/2} \geq 0$.

**Proof:** We begin the proof as we did Theorem 1, following Xie's argument for the Laplacian line by line. For fixed $m$, let $(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m)$ maximize

$$R_m \equiv \frac{\left(\sum_{n=1}^{m} a_n c_n\right)^2}{\left(\sum_{n=1}^{m} n^{2/3} c_n^2\right)^{1/2} \left(\sum_{n=1}^{m} n^{4/3} c_n^2\right)^{1/2}} ,$$

etc., etc., etc.. In the end, without yet using our hypotheses, we find that

$$\begin{aligned} \bar{R}_m &= \sqrt{\bar{\mu}_m} \sum_{n=1}^{m} \frac{4a_n^2}{\left(\bar{\mu}_m + n^{2/3}\right)^2} \\ &\leq \sum_{n=1}^{\infty} \frac{4a_n^2}{\left(\bar{\mu}_m^{3/4} + \bar{\mu}_m^{-1/4} n^{2/3}\right)^2} \equiv B_m . \end{aligned} \tag{36}$$

Thus, $\{\bar{R}_m\}$ is an increasing sequence of numbers bounded by another sequence $\{B_m\}$. Our hypothesis (34) is that $\limsup_{m\to\infty} B_m \leq 3\pi$ provided $\bar{\mu}_m \to \infty$ as $m \to \infty$. It therefore follows that $\bar{R}_m \leq 3\pi$ for all $m$, if $\bar{\mu}_m \to \infty$ as $m \to \infty$. We will show this now using our hypothesis (35).

**Lemma 5.** *$\bar{\mu}_m \to \infty$ as $m \to \infty$ in the proof of Theorem 3.*

**Proof:** (similarly to Theorem 2 and its lemmas 1,2,3,4). Since $R_m(c_1, \cdots, c_m)$ is constant on lines through the origin, its maxima $(\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m)$ can be 'normalized' to satisfy the constraint $\sum_{n=1}^{m} a_n c_n = 1$, that its numerator should equal 1, while the square of its denominator $\left(\sum_{n=1}^{m} n^{2/3} c_n^2\right)\left(\sum_{n=1}^{m} n^{4/3} c_n^2\right)$ is minimized. The normalized critical points are shown by the method of Lagrange multipliers to satisfy

$$\bar{c}_n = \frac{2a_n A_m B_m}{n^{2/3} B_m + n^{4/3} A_m}, \quad \text{for } n = 1, \cdots, m, \tag{37}$$

and then, continuing as before, $\bar{\mu}_m$ is shown to be a root of the equation

$$f_m(\mu) \equiv \sum_{n=1}^{m} \frac{a_n^2}{\left(\mu + n^{2/3}\right)^2} \left[\frac{\mu}{n^{2/3}} - 1\right] = 0\,.$$

Such roots are 'pushed' to $\infty$ using our hypothesis (35) that

$$f_\infty(\mu) \equiv \sum_{n=1}^{\infty} \frac{a_n^2}{\left(\mu + n^{2/3}\right)^2} \left[\frac{\mu}{n^{2/3}} - 1\right] < 0, \quad \text{for all } \mu \in [0, \infty)\,.$$

**Let us try to prove, as a corollary of Theorem 3,** that a Fourier sine series $f(y) = \sum_{n=1}^{\infty} c_n \sin ny$ satisfies the inequality

$$\sup_{(-\pi,\pi)} |f|^2 \leq \frac{3}{2} \left\| f^{(1/3)} \right\|_{L^2(-\pi,\pi)} \left\| f^{(2/3)} \right\|_{L^2(-\pi,\pi)} . \tag{38}$$

To that end, consider a fixed value of $y \in (-\pi, \pi)$, and let $a_n = \sin ny$. We need to verify two hypotheses. Note that the proposed constant is half of what it was for a cosine series. Consequently the limit in the first hypothesis, (34), must be reduced by half. For the proof of the following lemma I am greatful to my colleague, Professor John Fournier.

**Lemma 6.** *For every nonzero* $y \in (-\pi, \pi)$,

$$\sum_{n=1}^{\infty} \frac{4 \sin^2 ny}{\left(\mu^{3/4} + \mu^{-1/4} n^{2/3}\right)^2} \to \frac{3\pi}{2}, \quad \text{as } \mu \to \infty. \tag{39}$$

**Proof:** Using the identity $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$, one has

$$\sqrt{\mu} \sum_{n=1}^{\infty} \frac{\sin^2 ny}{\left(\mu + n^{2/3}\right)^2} = \frac{\sqrt{\mu}}{2} \sum_{n=1}^{\infty} \frac{1}{\left(\mu + n^{2/3}\right)^2} - \frac{\sqrt{\mu}}{2} \sum_{n=1}^{\infty} \frac{\cos 2ny}{\left(\mu + n^{2/3}\right)^2} . \tag{40}$$

The first sum on the right tends to $3\pi/8$ as $\mu \to \infty$, since

$$\frac{\sqrt{\mu}}{2}\int_0^\infty \frac{dx}{(\mu + x^{2/3})^2} = \frac{3\pi}{8}\,, \quad \text{for all } \mu > 0,$$

and

$$0 < \int_0^\infty \frac{dx}{(\mu + x^{2/3})^2} - \sum_{n=1}^\infty \frac{1}{(\mu + n^{2/3})^2} < \frac{1}{2\mu^2}\,.$$

The second term on the right of (40) tends to zero as $\mu \to \infty$, for every nonzero $y \in (-\pi, \pi)$. To show this, let $z = e^{i2y}$, so that $z^n = e^{i2ny}$ and $\mathrm{Re}\, z^n = \cos 2ny$. Then

$$\sqrt{\mu}\sum_{n=1}^\infty \frac{\cos 2ny}{(\mu + n^{2/3})^2} = \mathrm{Re}\,\sqrt{\mu}\sum_{n=1}^\infty \frac{z^n}{(\mu + n^{2/3})^2}\,,$$

and it will suffice to show that

$$\sqrt{\mu}\,(1 - z)\sum_{n=1}^\infty \frac{z^n}{(\mu + n^{2/3})^2} \to 0\,, \quad \text{as } \mu \to \infty, \tag{41}$$

for all $z \in C$ satisfying $|z| = 1$ and $z \neq 1$. To this end, let $c_n(\mu) \equiv \dfrac{\sqrt{\mu}}{\left(\mu + n^{2/3}\right)^2}$ . Then

$$\begin{aligned}
\sqrt{\mu}\,(1 - z) \sum_{n=1}^{\infty} \frac{z^n}{\left(\mu + n^{2/3}\right)^2} = \\
&= (1 - z)\left(c_1 z^1 + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots\right) \\
&= c_1 z^1 + \left[(c_2 - c_1)\, z^2 + (c_3 - c_2)\, z^3 + (c_4 - c_3)\, z^4 + \cdots\right] .
\end{aligned}$$

Since $c_{n+1}(\mu) - c_n(\mu) < 0$ and $|z| = 1$,

$$\begin{aligned}
\left| \sqrt{\mu}\,(1 - z) \sum_{n=1}^{\infty} \frac{z^n}{\left(\mu + n^{2/3}\right)^2} \right| \le \\
&\le |c_1(\mu)|\left|z^1\right| + |c_2(\mu) - c_1(\mu)|\left|z^2\right| + |c_3(\mu) - c_2(\mu)|\left|z^3\right| + \cdots \\
&= c_1(\mu) + (c_1(\mu) - c_2(\mu)) + (c_2(\mu) - c_3(\mu)) + \cdots \\
&= 2c_1(\mu) \to 0\,, \quad \text{as } \mu \to \infty.
\end{aligned}$$

Thus the second term on the right of (40) tends to zero as $\mu \to \infty$, completing the proof of Lemma 6.

**The second hypothesis** to be checked is that, for every fixed nonzero point $y \in (-\pi, \pi)$, the early terms in the sequence $\{\sin^2 ny\}$ do not dominate later ones, in the sense that

$$f_\infty(\mu, y) \equiv \sum_{n=1}^{\infty} \frac{\sin^2 ny}{\left(\mu + n^{2/3}\right)^2} \left[\frac{\mu}{n^{2/3}} - 1\right] < 0, \quad \text{for all } \mu \in [0, \infty). \quad (42)$$

It is the analogue of this hypothesis that remains to be verified in the pde context, and which is the principal motivation for this study.

It may be insightful and easier to first consider the corresponding integral, and to try to show that

$$\int_0^\infty \frac{\sin^2 xy}{\left(\mu + x^{2/3}\right)^2} \left[\frac{\mu}{x^{2/3}} - 1\right] dx < 0, \quad \text{for all } \mu \in [0, \infty). \quad (43)$$

**At this time we haven't a proof of either** (43) or (42), but there is strong numerical evidence and good intuitive reasoning to support them. Of course, a proof of (42) will complete the proof of (38) for a sine series.

**How widely can Xie's method be applied?** What happens if one varies the powers of $n$, say to $n^{1/2}$ and $n^{3/2}$, and tries to prove

$$\left(\sum_{n=1}^{\infty} c_n\right)^2 \leq c\left(\sum_{n=1}^{\infty} n^{1/2} c_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} n^{3/2} c_n^2\right)^{1/2} ??? \qquad (44)$$

Can Xie's argument be used to prove Hölder's inequality for two sequences $\{a_n\}$ and $\{b_n\}$, and $p$ and $q$ satisfying $\frac{1}{p}+\frac{1}{q}=1$,

$$\sum_{n=1}^{\infty} a_n b_n \leq \left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q} \quad ??? \qquad (45)$$

**In fact, one gets a simple proof of** (45): Regard the numbers $a_1, a_2, \cdots$ as given and fixed, while the numbers $b_1, b_2, \cdots$ are varied. Let

$$R_m\left(b_1, b_2, \cdots, b_m\right) \equiv \frac{\sum_{n=1}^{m} a_n b_n}{\left(\sum_{n=1}^{m} b_n^q\right)^{1/q}}. \qquad (46)$$

Since $R$ is constant along rays emanating from the origin, it attains its maximum at points where it is smooth. At such a point $\left(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_m\right)$, and for $k = 1, \cdots, m$ one has

$$0 = \frac{\partial}{\partial b_k} \log R_m = \frac{\partial}{\partial b_k} \log \sum_{n=1}^{m} a_n b_n - \frac{\partial}{\partial b_k} \frac{1}{q} \log \sum_{n=1}^{m} b_n^q$$
$$= \frac{a_k}{\sum_{n=1}^{\infty} a_n b_n} - \frac{b_k^{q-1}}{\sum_{n=1}^{\infty} b_n^q} \quad \text{for } k = 1, 2, \cdots, \tag{47}$$

and hence

$$a_k \sum_{n=1}^{m} \tilde{b}_n^q = \tilde{b}_k^{q-1} \sum_{n=1}^{m} a_n \tilde{b}_n \,. \tag{48}$$

Taking the $p^{th}$ power of this, noting that $(q-1)\,p = q$, and summing over $k$ gives

$$\sum_{k=1}^{m} a_k^p \left( \sum_{n=1}^{m} \tilde{b}_n^q \right)^p = \sum_{k=1}^{\infty} \tilde{b}_k^q \left( \sum_{n=1}^{m} a_n \tilde{b}_n \right)^p \,. \tag{49}$$

Finally, taking the $p^{th}$ root of this, one finds that (45) holds with equality for any choice of $(b_1, b_2, \cdots, b_m)$ that maximizes $R_m$. This completes the proof.

Reference (for the third lecture)

J. G. Heywood, Seeking a proof of Xie's inequality: analogues for series and Fourier series, preprint which will be submitted for publication in the Ferrara Univ. Math. journal volume in honour of M. Padula (2013), 20 pages.

## Fourth Lecture: Xie's Potential Theoretic Arguments

Using potential theoretic methods, Xie reproved

$$\sup_{\Omega} |u|^2 \leq \tfrac{1}{2\pi} \|\nabla u\| \, \|\triangle u\| \qquad \text{Poisson problem}$$

for any open $\Omega \subset R^3$, and finally, at least for $\Omega = R^3$, proved

$$\sup_{R^3} |\mathbf{u}|^2 \leq \tfrac{1}{3\pi} \|\nabla \mathbf{u}\| \left\| \widetilde{\triangle} \mathbf{u} \right\| \qquad \text{Stokes problem.}$$

He also obtained two-dimensional results, proving

$$\sup_{\Omega} |u|^2 \leq \tfrac{1}{2\pi} \left( \|u\| \, \|\triangle u\| + \|\nabla u\|^2 \right) \qquad \text{Poisson problem}$$

for any open $\Omega \subset R^2$, and for the special domain $\Omega = R^2$

$$\sup_{R^3} |\mathbf{u}|^2 \leq \tfrac{1}{4\pi} \left( \|\mathbf{u}\| \left\| \widetilde{\triangle} \mathbf{u} \right\| + \|\nabla \mathbf{u}\|^2 \right) \qquad \text{Stokes problem.}$$

**Theorem** (Xie 1995) *For any open* $\Omega \subset R^3$ *and every* $u \in \widehat{H}_0^1(\Omega)$ *with* $\triangle u \in L^2(\Omega)$,

$$\sup_{\Omega} |u|^2 \leq \frac{1}{2\pi} \|\nabla u\| \|\triangle u\| . \quad (50)$$

*The constant is optimal for any domain and equality holds if and only if*

$$\Omega = R^3 \text{ and } u(x) = c \frac{1 - e^{-\sqrt{\mu}|x-y|}}{|x-y|}, \quad (51)$$

*for some* $y \in R^3$, $\mu > 0$ *and* $c \in R$.

**Theorem** (Xie 1996) *For* $\Omega = R^3$, *and* $\mathbf{u} \in \mathbf{J}_0(\Omega)$ *with* $\widetilde{\triangle}\mathbf{u} \in J(\Omega)$,

$$\sup_{R^3} |\mathbf{u}|^2 \leq \frac{1}{3\pi} \|\nabla \mathbf{u}\| \left\|\widetilde{\triangle}\mathbf{u}\right\| . \quad (52)$$

*Equality holds if and only if*

$$\mathbf{u}(x) = (\triangle - \nabla\nabla\cdot) \left( \frac{e^{-\sqrt{\mu}|x-y|} - 1}{\mu |x-y|} - \frac{|x-y|}{2} \right) \mathbf{c}, \quad (53)$$

*for some* $y \in R^3$, $\mu > 0$ *and vector* $\mathbf{c} \in \mathbf{R}^3$.

## Proof for the Laplacian in a smoothly bounded domain $\Omega \subset R^3$.

As before, $y$ will be an arbitrary fixed point of $\Omega$. And, of course, the fundamental singularity for the Helmholtz operator $-\triangle + \mu$ is

$$g_\mu(x;y) = \frac{e^{-\sqrt{\mu}|x-y|}}{4\pi|x-y|}, \tag{54}$$

and satisfies

$$\int_{R^3} g_\mu^2(x;y)\, dx = \int_0^\infty \left(\frac{e^{-\sqrt{\mu}r}}{4\pi r}\right)^2 4\pi r^2\, dr = \frac{1}{8\pi\sqrt{\mu}}.$$

The corresponding Green's function $G_\mu(x,y)$ for $\Omega$ was seen to satisfy

$$0 < G_\mu(x;y) < g_\mu(x;y) \tag{55}$$

by the maximum principle, and therefore

$$\left\|G_\mu(\cdot;y)\right\|^2 < \frac{1}{8\pi\sqrt{\mu}}, \tag{56}$$

with $\left\|G_\mu(\cdot;y)\right\|^2 \to \frac{1}{8\pi\sqrt{\mu}}$, as $dist(y,\partial\Omega) \to \infty$.

**Lemma 1.** *One has*

$$\left\|\nabla\left(G_0 - G_\mu\right)\right\|^2 < \int_\Omega \mu g_\mu \left(g_0 - g_\mu\right)\, dx = \frac{\sqrt{\mu}}{8\pi}\,, \qquad (57)$$

with the inequality approaching equality as $dist\left(y, \partial\Omega\right) \to \infty$.

**Proof:** We have reasoned that $G_0\left(\cdot; y\right) - G_\mu\left(\cdot; y\right) \in H^2_{loc}\left(\Omega\right)$ and thus

$$-\triangle\left(G_0 - G_\mu\right) = \mu G_\mu\,, \quad \text{and} \quad G_0 - G_\mu\big|_{\partial\Omega} = 0\,, \qquad (58)$$

justifying the representation

$$G_0\left(x; y\right) - G_\mu\left(x; y\right) = \mu \int_\Omega G_0\left(z; x\right) G_\mu\left(z; y\right)\, dz\,. \qquad (59)$$

Similarly, if $\Omega = R^3$,

$$g_0\left(x; y\right) - g_\mu\left(x; y\right) = \mu \int_{R^3} g_0\left(z; x\right) g_\mu\left(z; y\right)\, dz\,. \qquad (60)$$

By the maximum principle, the right side of (60) exceeds that of (59), and therefore the left sides are in the same relation,

$$0 \leq G_0\left(x; y\right) - G_\mu\left(x; y\right) < g_0\left(x; y\right) - g_\mu\left(x; y\right)\,. \qquad (61)$$

Lemma 1 is completed by integrating by parts and using (58) and (61):

$$\begin{aligned}\left\|\nabla\left(G_0-G_\mu\right)\right\|^2 &= -\int_\Omega \triangle\left(G_0-G_\mu\right)\left(G_0-G_\mu\right)\,dx \\ &= \int_\Omega \mu G_\mu\left(G_0-G_\mu\right)\,dx \qquad \text{using (58)} \\ &< \int_{R^3} \mu g_\mu\left(g_0-g_\mu\right)\,dx \qquad \text{using (61)} \\ &= \int_0^\infty \mu\frac{e^{-\sqrt{\mu}r}}{4\pi r}\frac{1-e^{-\sqrt{\mu}r}}{4\pi r}4\pi r^2 dr = \frac{\sqrt{\mu}}{8\pi}\,.\end{aligned}$$

Proceeding now, we are assuming $\Omega$ is bounded and $\partial\Omega$ is smooth. So by elliptic regularity our given function $u \in \widehat{H}_0^1(\Omega)$ with $\triangle u \in L^2(\Omega)$ belongs to $H_0^1(\Omega) \cap H^2(\Omega)$ and by the Poisson formula we have

$$\begin{aligned}u(y) &= -\int_\Omega G_0 \triangle u\,dx \\ &= -\int_\Omega G_\mu \triangle u\,dx - \int_\Omega \left(G_0-G_\mu\right)\triangle u\,dx \\ &= -\int_\Omega G_\mu \triangle u\,dx - \int_\Omega \nabla\left(G_0-G_\mu\right)\cdot\nabla u\,dx\,.\end{aligned}$$

Writing this last line again, we have

$$u(y) = -\int_\Omega G_\mu \triangle u \, dx - \int_\Omega \nabla \left( G_0 - G_\mu \right) \cdot \nabla u \, dx \,, \tag{62}$$

and can apply the Schwarz inequality to obtain

$$|u(y)| \le \left\| G_\mu \right\| \|\triangle u\| + \left\| \nabla \left( G_0 - G_\mu \right) \right\| \|\nabla u\| \,. \tag{63}$$

Then, remembering our estimates (56) and (57) for $\left\| G_\mu \right\|$ and $\left\| \nabla \left( G_0 - G_\mu \right) \right\|$, we have

$$|u(y)| \le \sqrt{\frac{1}{8\pi\sqrt{\mu}}} \|\triangle u\| + \sqrt{\frac{\sqrt{\mu}}{8\pi}} \|\nabla u\| \,. \tag{64}$$

The minimum with respect to $\mu$ occurs when $\mu = \|\triangle u\|^2 / \|\nabla u\|^2$. Substituting that value of $\mu$ into (64) gives

$$|u(y)| \le \frac{1}{\sqrt{2\pi}} \|\nabla u\|^{1/2} \|\triangle u\|^{1/2} \,, \tag{65}$$

proving, for smoothly bounded domains, the inequality (50) claimed in the theorem. We pass from such domains to arbitrary open sets as before.

It remains to prove the uniqueness of the maximizing functions claimed in line (51) of the theorem. Equality is approached in (65) only by approaching equality in (64). That requires, first of all, that the Green's functions $G_0$ and $G_\mu$ must approach $g_0$ and $g_\mu$, in order that $\left\| G_\mu \right\|$ and $\left\| \nabla \left( G_0 - G_\mu \right) \right\|$ will approach their limits $\sqrt{\frac{1}{8\pi\sqrt{\mu}}}$ and $\sqrt{\frac{\sqrt{\mu}}{8\pi}}$. But then, further, in order for the application of the Schwarz inequality to be sharp, in its application above to

$$u(y) = -\int_\Omega G_\mu \triangle u \, dx - \int_\Omega \nabla \left( G_0 - G_\mu \right) \cdot \nabla u \, dx \,,$$

it is necessary that

$$-\triangle u = c_1 G_\mu = c_1 g_\mu \quad \text{and} \quad \nabla u = c_2 \nabla \left( G_0 - G_\mu \right) = c_2 \nabla \left( g_0 - g_\mu \right) \,.$$

One need only consider the second of these to conclude that $u = c \left( g_0 - g_\mu \right)$, since both $u$ and $g_0 - g_\mu$ belong to $\widehat{H}_0^1 (\Omega)$. That completes the proof of the Theorem.

**Proof for the Stokes operator in $R^3$.**

We begin by finding the Green's function for the generalized Stokes problem. Remember that in $R^3$ the fundamental solution $g_\mu(x, y)$ for the Helmholtz equation

$$(-\triangle + \mu)\, g_\mu = \delta(x, y)$$

is, setting $r = |x - y|$,

$$g_\mu = \frac{e^{-\sqrt{\mu} r}}{4\pi r}, \quad \mu \geq 0, \tag{66}$$

and for $\mu > 0$,

$$\|g_\mu\|^2 = \frac{1}{8\pi\sqrt{\mu}} \tag{67}$$

Now let

$$\Phi_\mu = \begin{cases} \dfrac{g_\mu - g_0}{\mu}, & \mu > 0 \\ \dfrac{r}{8\pi}, & \mu = 0. \end{cases} \tag{68}$$

It satisfies

$$\triangle \Phi_\mu = g_\mu. \tag{69}$$

Let $\mathbf{e}^k$ be the unit vector in the $x_k$ direction, $1 \leq k \leq 3$. And let

$$\begin{aligned} \mathbf{U}_\mu &= (\triangle - \nabla\nabla\cdot)\,\Phi_\mu \mathbf{e}^k \\ P_\mu &= (-\triangle + \mu)\,\nabla\nabla \cdot \Phi_\mu \mathbf{e}^k \quad ?? \end{aligned} \tag{70}$$

It is "easy" to verify that they are the fundamental solution of the generalized Stokes system

$$\begin{aligned} (-\triangle + \mu)\,\mathbf{U}_\mu + \nabla P_\mu &= \delta(x, y)\,\mathbf{e}^k \\ \nabla \cdot \mathbf{U}_\mu &= 0. \end{aligned}$$

We won't need them, but the explicit three-dimensional expressions for $\mathbf{U}_\mu$ and $P_\mu$ are

$$\begin{aligned} \mathbf{U}_\mu &= \frac{1}{4\pi r}\left(a\left(\sqrt{\mu} r\right)\mathbf{e}^k + b\left(\sqrt{\mu} r\right)\frac{r_k \mathbf{r}}{r^2}\right) \\ P_\mu &= -\frac{r_k}{4\pi r^3}, \end{aligned}$$

where

$$a(s) = \frac{1 + s + s^2 - e^s}{s^2 e^s} \quad \text{and} \quad b(s) = \frac{3e^s - 3 - 3s - s^2}{s^2 e^s}.$$

**Lemma.** If $\mu > 0$,

$$\left\|\mathbf{U}_\mu\right\|^2 = \frac{1}{12\pi\sqrt{\mu}} \tag{71}$$

and

$$\left\|\nabla\left(\mathbf{U}_\mu - \mathbf{U}_0\right)\right\|^2 = \frac{\sqrt{\mu}}{12\pi}\,. \tag{72}$$

**Proof:** Let $(\cdot,\cdot)$ denote the $L^2$ inner product. From (70) we have

$$\begin{aligned}
\left\|\mathbf{U}_\mu\right\|^2 &= \left\|\triangle\Phi_\mu\mathbf{e}^k - \nabla\frac{\partial}{\partial x_k}\Phi_\mu\right\|^2 \\
&= \left\|\triangle\Phi_\mu\right\|^2 - 2\left(\triangle\Phi_\mu\,,\,\frac{\partial^2}{\partial x_k^2}\Phi_\mu\right) + \left\|\nabla\frac{\partial}{\partial x_k}\Phi_\mu\right\|^2 \\
&= \left\|\triangle\Phi_\mu\right\|^2 - \left(\triangle\Phi_\mu\,,\,\frac{\partial^2}{\partial x_k^2}\Phi_\mu\right)\,,
\end{aligned}$$

and hence, since $\left\|\mathbf{U}_\mu\right\|$ is independent of $k$,

$$3\left\|\mathbf{U}_\mu\right\|^2 = (3-1)\left\|\triangle\Phi_\mu\right\|^2.$$

Therefore, by (69),

$$\left\|\mathbf{U}_{\mu}\right\|^{2}=\frac{(3-1)}{3}\left\|g_{\mu}\right\|^{2},$$

which together with (67) implies (71). By a similar argument,

$$\left\|\nabla\left(\mathbf{U}_{\mu}-\mathbf{U}_{0}\right)\right\|^{2}=\frac{2}{3}\left\|\nabla\left(g_{\mu}-g_{0}\right)\right\|^{2}.$$

Thus we obtain (72) since

$$\left\|\nabla\left(g_{\mu}-g_{0}\right)\right\|^{2}=-\left(g_{\mu}-g_{0}\,,\,\triangle\left(g_{\mu}-g_{0}\right)\right)=\left(g_{\mu}-g_{0}\,,\,\mu g_{\mu}\right)=\frac{\sqrt{\mu}}{8\pi}.$$

The last step here is by straightforward integration. This completes the proof of the lemma.

**Proof of the theorem:** Similarly to (62) we obtain the representation formula

$$\begin{aligned} u_{k}(y) &= \left(\mathbf{U}_{0}\,,\,-\widetilde{\triangle}\mathbf{u}\right)\\ &= \left(\mathbf{U}_{\mu}-\mathbf{U}_{0}\,,\,\widetilde{\triangle}\mathbf{u}\right)-\left(\mathbf{U}_{\mu}\,,\,\widetilde{\triangle}\mathbf{u}\right)\\ &= -\left(\nabla\left(\mathbf{U}_{\mu}-\mathbf{U}_{0}\right)\,,\nabla\mathbf{u}\right)-\left(\mathbf{U}_{\mu}\,,\,\widetilde{\triangle}\mathbf{u}\right). \end{aligned}$$

Repeating the last line,

$$u_k\left(y\right) = -\left(\nabla\left(\mathbf{U}_\mu - \mathbf{U}_0\right), \nabla\mathbf{u}\right) - \left(\mathbf{U}_\mu, \widetilde{\triangle}\mathbf{u}\right),$$

and we can apply the Schwarz inequality to obtain

$$\left|u_k\left(y\right)\right| \leq \left\|\nabla\left(\mathbf{U}_\mu - \mathbf{U}_0\right)\right\| \left\|\nabla\mathbf{u}\right\| + \left\|\mathbf{U}_\mu\right\| \left\|\widetilde{\triangle}\mathbf{u}\right\|.$$

Then, remembering our estimates (71) and (72) for $\left\|\mathbf{U}_\mu\right\|^2$ and $\left\|\nabla\left(\mathbf{U}_\mu - \mathbf{U}_0\right)\right\|^2$, we have

$$\left|u_k\left(y\right)\right| \leq= \sqrt{\frac{\sqrt{\mu}}{12\pi}} \left\|\nabla\mathbf{u}\right\| + \sqrt{\frac{1}{12\pi\sqrt{\mu}}} \left\|\widetilde{\triangle}\mathbf{u}\right\|. \tag{73}$$

The minimum with respect to $\mu$ occurs when

$$\mu = \left\|\widetilde{\triangle}\mathbf{u}\right\|^2 / \left\|\nabla\mathbf{u}\right\|^2. \tag{74}$$

Substituting that value of $\mu$ into (73) finally proves the inequality (52).

$$\left|u_k\left(y\right)\right| \leq \frac{1}{\sqrt{3\pi}} \left\|\nabla\mathbf{u}\right\|^{1/2} \left\|\widetilde{\triangle}\mathbf{u}\right\|^{1/2}, \tag{75}$$

It remains to prove the uniqueness of the optimizing functions (53). The Schwarz inequality for the first integral achieves equality if and only if

$$\mathbf{u} = c\left(\mathbf{U}_\mu - \mathbf{U}_0\right) \tag{76}$$

for some constant $c$. Suppose $\mathbf{u}$ is given in the form of (76) with an arbitrary $\mu > 0$. Then we have

$$\widetilde{\triangle}\mathbf{u} = c\mu\mathbf{U}_\mu\,,$$

so that the Schwarz inequality for the second integral also becomes an equality. Furthermore, (74) is satisfied since by (71) and (72)

$$\frac{\left\|\widetilde{\triangle}\mathbf{u}\right\|^2}{\left\|\nabla\mathbf{u}\right\|^2} = \frac{\left\|\mu\mathbf{U}_\mu\right\|^2}{\left\|\nabla\mathbf{U}_\mu - \nabla\mathbf{U}_0\right\|^2} = \frac{\mu^2\frac{1}{12\pi\sqrt{\mu}}}{\frac{\sqrt{\mu}}{12\pi}} = \mu\,.$$

Therefore equality is indeed achieved by functions in the form (76), or (53), in view of the formulas (70),(68) and (66).

References (for the fourth lecture):

W. Xie, Integral representations and $L^\infty$ bounds for solutions of the Helmholtz equation on arbitrary open sets in $R^2$ and $R^3$, Diff. and Int. Eq., 8 (1995), 689-698.

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