1. Fine properties of Sobolev functions.
2. Properties of solutions to Euler equations

November 4, 2013

Speaking about functions w from Sobolev spaces we always assume that we have chosen the precise representative of w. If $v \in L^1_{loc}(\Omega)$, then the precise representative w^* is defined by

$$w^*(x) = \lim_{r \to 0} \frac{1}{mes(B_r(x))} \int_{B_r(x)} w(z)dz$$

if the the limit exists, and $w^*(x) = 0$ otherwise.

Fine properties of Sobolev functions

Theorem A. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with a Lipschitz boundary,* $w \in W^{1,1}(\Omega)$*. Then there exists a set* $A_w \subset \overline{\Omega}$ *such that*
$\mathfrak{H}^1(A_w) = 0$ *and*
(i)

$$\lim_{r \to 0} \fint_{B_r(x)} w(z)dz = w^*(x)$$

exists for each $x \in \Omega \setminus A_w$*.*
(ii) *for each* $x \in \Omega \setminus A_w$

$$\lim_{r \to 0} \fint_{B_r(x)} |w(z) - w(x)|^2 dz = 0;$$

(iii) *for all* $\varepsilon > 0$ *there exists a set* $U \subset \mathbb{R}^2$ *such that* $\mathfrak{H}^1_\infty(U) < \varepsilon$*,* $A_w \subset U$ *and the function* w *is continuous in* $\overline{\Omega} \setminus U$*.*

Fine properties of Sobolev functions

Theorem B. (J. Bourgain, M.V. Korobkov, J. Kristensen, 2013)
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary, $w \in W^{2,1}(\Omega)$. Then
(i) *for every $\varepsilon > 0$ there exists an open set $V \subset \mathbb{R}$ and a function $g \in C^1(\mathbb{R}^2)$ such that $\mathfrak{H}^1(V) < \varepsilon$, and for $w(x) \in \mathbb{R} \setminus V$ the function w is differentiable in the point x and $w(x) = g(x)$, $\nabla w(x) = \nabla g(x) \neq 0$;*
(ii) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any set $U \subset \overline{\Omega}$ with $\mathfrak{H}^1_\infty(U) < \delta$ holds $\mathfrak{H}^1(w(U)) < \varepsilon$.*

$$\mathfrak{H}^1(F) = \lim_{t \to 0+} \mathfrak{H}^1_t(F),$$

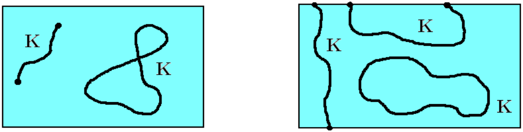
where $\mathfrak{H}^1_t(F) = \inf\{\sum_{i=1}^{\infty} \mathrm{diam} F_i : \mathrm{diam} F_i \leq t, F \subset \bigcup_{i=1}^{\infty} F_i\}$.

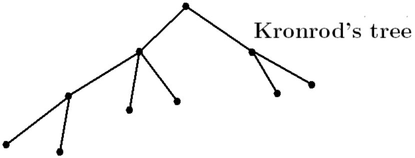
Level sets of functions in $\mathbb{R}^2$. Kronrod's graph

Let $\mathbb{Q} = (-a, a) \times (-a, a)$ and let f be a continuous function defined in $\mathbb{Q}$. Denote by E_t a level set of the function f, i.e.

$$E_t = \{x \in \mathbb{Q} : f(x) = t\}.$$

A component K of the level set E_t containing a point x_0 is a maximal connected subset of E_t containing x_0. A component K of the level set E_t is called *regular*, if it divides the quadrate $\mathbb{Q}$ into two parts.





Level sets of functions in $\mathbb{R}^2$. Kronrod's graph

Theorem C. (Kronrod, 1950) *Let f be a continuous function defined in $\mathbb{Q}$. Then there is at most a countable number of nonregular components of level sets $f^{-1}(f(\overline{\mathbb{Q}}))$.*

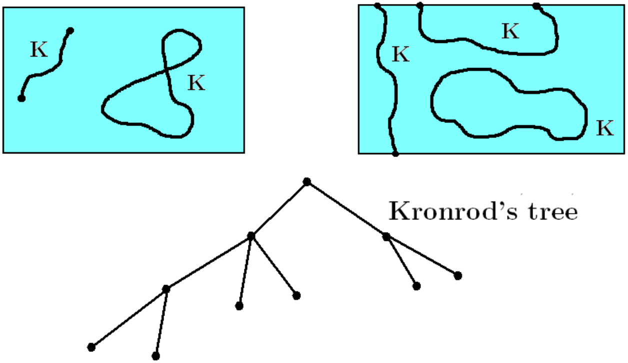
Kronrod's graph. To any continuous on $\mathbb{Q}$ function f corresponds a family T_f of all connected components of its level sets E_t. T_f is a topological space (equipped with a natural topology). This definition induces a natural map

$$\tau_f : \mathbb{Q} \to T_f : \; \tau_f(x) = K_x \quad \forall x \in \mathbb{Q},$$

where K_x is a connected component of the level set $E_{f(x)}$ containing the point x (K is considered as a point in T_f). The map τ_f is continuous.

Theorem D (A. Kronrod) *Let f be a continuous function defined in $\mathbb{Q}$. Then*
(1) T_f *is one–dimensional tree consisting from the set of endpoints plus at most a countable number of simple arcs that mutually intersect at most in one point which is a bifurcation point.*
(2) *Non-regular components of level sets can be classified on the tree T_f as follows:*
(a) *if K divides $\mathbb{Q}$ into $n \geq 3$ connected parts, then on T_f to K corresponds a bifurcation point of the same multiplicity n, and vice versa;*
(b) *if K does not divide $\mathbb{Q}$, then the corresponding point on T_f belongs to the set of endpoints, and vice versa.*

Level sets of functions in $\mathbb{R}^2$. Kronrod's graph



A set is called *an arc* if it is homeomorphic to the unit interval $[0, 1]$.

Lemma1. *If $f \in C(\mathbb{Q})$, then for any two different points $A \in T_f$ and $B \in T_f$, there exists a unique arc $J = J(A, B) \subset T_f$ joining A to B. Moreover, for every inner point C of this arc the points A, B lie in different connected components of the set $T_f \setminus \{C\}$.*

We can reformulate the above Lemma in the following equivalent form.

Lemma 2. *If $f \in C(\mathbb{Q})$, then for any two different points $A, B \in T_f$, there exists an injective function $\varphi : [0,1] \to T_f$ with the properties*
(i) $\varphi(0) = A$, $\varphi(1) = B$;
(ii) *for any $t_0 \in [0,1]$,*

$$\lim_{[0,1]\ni t \to t_0} \sup_{x \in \varphi(t)} (x, \varphi(t_0)) \to 0;$$

(iii) *for any $t \in (0,1)$ the sets A, B lie in different connected components of the set $Q \setminus \varphi(t)$.*

Theorem. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. If $\psi \in W^{2,1}(\Omega)$, then for $\mathfrak{H}^1$–almost all $y \in \psi(\overline{\Omega}) \subset \mathbb{R}$ the preimage $\psi^{-1}(y)$ is a finite disjoint family of C^1-curves S_j, $j = 1, 2, \ldots, N(y)$. Each S_j is either a cycle in Ω (i.e., $S_j \subset \Omega$ is homeomorphic to the unit circle $\mathbb{S}^1$) or a simple arc with endpoints on $\partial\Omega$ (in this case S_j is transversal to $\partial\Omega$).*

This theorem is proved by J. Bourgain, M.V. Korobkov and J. Kristensen: On the Morse–Sard property and level sets of Sobolev and BV functions, *Rev. Mat. Iberoam.* **29**, no. 1 (2013), 1–23.

All results remain valid for level sets of continuous functions $f : \overline{\Omega} \to \mathbb{R}$, where Ω is a multi–connected bounded domain, provided $f \equiv \xi_j =$ on each inner boundary component S_j with $j = 1, \ldots, N$.

Indeed, we can extend f to the whole $\overline{\Omega}_0$ by putting $f(x) = \xi_j$ for $x \in \overline{\Omega}_j$, $j = 1, \ldots, N$. The extended function f will be continuous on the set $\overline{\Omega}_0$ which is homeomorphic to the unit square $\mathbb{Q} = [0, 1]^2$.

Properties of solutions to Euler equations

$$(\mathbf{v}\cdot\nabla)\mathbf{v}+\nabla p=0, \qquad (E)$$

$$div\mathbf{v}=0.$$

$$\mathbf{v}\in H(\Omega),\ \ p\in W^1_s(\Omega),\ s\in[1,2)$$

We already know that that if $\mathbf{v}=0$ on $\partial\Omega$ (in the sense of trace), then the pressure $p(x)$ is constant on $\partial\Omega$ and $p(x)$ could take different constant values $p_j=p(x)\big|_{S_j}$ on different connected components S_j of the boundary $\partial\Omega$.

Some facts from harmonic analysis.

The Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ consists of distributions f such that for some function Φ with $\int \Phi = 1$ the maximal function

$$(M_\Phi f)(x) = \sup_{t>0} |f \star (\Phi_t)(x)|$$

is in $L^1(\mathbb{R}^2)$, where $\Phi_t(x) = t^{-2}\Phi(x/t)$, $\|f\|_{\mathcal{H}^1(\mathbb{R}^2)} = \|M_\Phi f\|_{L^1(\mathbb{R}^2)}$.

$\mathcal{H}^1(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)$.

Lemma H1. *Let* $f \in \mathcal{H}^1(\mathbb{R}^2)$ *and let*

$$J(x) = \int_{\mathbb{R}^2} \log|x-y| f(y)\, dy.$$

Then
(i) $J \in C(\mathbb{R}^2)$*;*
(ii) $\nabla J \in L^2(\mathbb{R}^2)$, $D^\alpha J \in L^1(\mathbb{R}^2)$, $|\alpha| = 2$.

Lemma H2. *Let* $\mathbf{w} \in W^{1,2}(\mathbb{R}^2)$ *and* $div\mathbf{w} = 0$. *Then*

$$div\big[(\mathbf{w}\cdot\nabla)\mathbf{w}\big] = \sum_{i,j=1}^{2} \frac{\partial w_i}{\partial x_j}\frac{\partial w_j}{\partial x_i} \in \mathcal{H}^1(\mathbb{R}^2).$$

Lemma 2 follows from div-curl lemma with two cancelations.

Theorem (continuity of the pressure). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary. Let $(\mathbf{v}, p)$ satisfy the Euler equations for almost all $x \in \Omega$, $\mathbf{v} \in \mathring{W}^{1,2}(\Omega)$, $p \in W^{1,s}(\Omega),\ s \in [1,2)$. Then*

$$p \in C(\overline{\Omega}) \cap W^{1,2}(\Omega).$$

Proof. Multiply (E) by $\varphi = \nabla\xi$, where $\xi \in C_0^\infty(\Omega)$:

$$\int_\Omega \nabla p \cdot \nabla\xi \, dx = -\int\limits_\Omega \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\cdot\nabla\xi \, dx \quad \forall\xi \in C_0^\infty(\Omega).$$

Thus, $p \in W^{1,s}(\Omega)$ is the unique weak solution of the boundary value problem for the Poisson problem:

$$-\Delta p = div\big[\big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\big], \quad p(x)|_{S_i} = p_i, \ \ i = 1, \dots, N.$$

We have $div\big[(\mathbf{v}\cdot\nabla)\mathbf{v}\big]\in\mathcal{H}^1(\mathbb{R}^2)$. Define

$$J_1(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\, div_y\left[\big(\mathbf{u}(y)\cdot\nabla_y\big)\mathbf{u}(y)\right]dy.$$

Then $J_1\in C(\mathbb{R}^2)$, $\nabla J_1\in L^2(\mathbb{R}^2)$, $D^\alpha J_1\in L^1(\mathbb{R}^2), |\alpha|=2$.
Let $J_2(x)=p(x)-J_1(x)$:

$$-\Delta J_2=0,\quad J_2\big|_{\partial\Omega}=j_2-j_1,$$

where $j_1(x)=J_1(x)\big|_{\partial\Omega}, j_2(x)|_{S_i}\equiv p_i$.
$j_1\in W_2^{1/2}(\partial\Omega)\cap C(\partial\Omega)$, $j_2\in C(\partial\Omega)$ and j_2 could be extended to Ω as a function from $W_2^1(\Omega)$. Thus, there exists solution $J_2\in W_2^1(\Omega)$ such that $J_2\in C(\overline{\Omega})$. By uniqueness $p(x)=J_1(x)+J_2(x)$. Hence, $p\in C(\overline{\Omega})\cap W_2^1(\Omega)$. □

Properties of solutions to Euler equations

Assume that $\mathbf{v} \in W^{1,2}(\Omega)$ and $p \in W^{1,s}(\Omega)$, $s \in [1,2)$, satisfy the Euler equations

$$\big(\mathbf{v} \cdot \nabla\big)\mathbf{v} + \nabla p = 0, \quad div\mathbf{v} = 0 \;\; for \;\; almost \;\; all \;\; x \in \Omega \qquad (E)$$

and let $\int_{S_i} \mathbf{v} \cdot \mathbf{n}\, dS = 0, \;\; i = 1, 2, \ldots, N$, where S_i are connected components of the boundary $\partial\Omega$. Then there exists a continuous stream function $\psi \in W^{2,2}(\Omega)$ such that $\nabla\psi = (-v_2, v_1)$. Denote $\Phi = p + \dfrac{|\mathbf{v}|^2}{2}$ the total head pressure. Then $\Phi \in W^{1,s}(\Omega)$ for all $s \in [1,2)$. By direct calculations we get

$$\nabla\Phi \equiv \Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big)\big(v_2, -v_1\big) = (\Delta\psi)\nabla\psi.$$

If all functions are smooth, from this formula the classical Bernoulli law follows immediately:

The total head pressure $\Phi(x)$ is constant along any streamline of the flow.

Properties of solutions to Euler equations. Bernoulli law (M.Korobkov)

Theorem (Bernoulli law). *Let $\Omega \subset \mathbb{R}^2$ be a bounded multiply connected domain with a Lipschitz boundary $\partial\Omega = \cup_{i=1}^N S_i$. Let $\mathbf{u} \in W^{1,2}(\Omega)$ and $p \in W^{1,s}(\Omega)$ satisfy Euler equations (E) for almost all $x \in \Omega$ and let $\int_{S_i} \mathbf{u} \cdot \mathbf{n} dS = 0, \; i = 1, \ldots, N$. Then for any connected set $K \subset \overline{\Omega}$ such that*

$$\psi\big|_K = const,$$

the identity

$$\Phi(x) = const$$

holds $\mathfrak{H}^1$-almost everywhere on K.

In particular, it follows that if $\mathbf{u} = 0$ on $\partial\Omega$, then the pressure $p(x)$ is constant on $\partial\Omega$, i.e. $p(x)\big|_{S_j} = p_j$.

Proof. Let $\psi \in W^{2,2}(\Omega)$ be a stream function.

(i) Fix any $\varepsilon > 0$ and consider a function $g \in C^1(\mathbb{R}^2)$ and an open set $V \subset \mathbb{R}$ with $\mathfrak{H}^1(V) < \varepsilon$ such that $\psi(x) = g(x)$ and $\nabla\psi(x) = \nabla g(x) \neq 0$ for any $x \in F = \overline{\Omega} \setminus \psi^{-1}(V)$. For almost all $y \in \psi(\overline{\Omega}) \setminus V = g(F)$, for any connected component K of $\psi^{-1}(y)$ (i.e. for any streamline) and for any C^1–smooth parametrization $\gamma : [0,1] \to K$ the restriction $\Phi|_K$ is absolutely continuous, and we have the identity

$$[\Phi(\gamma(t))]' = [\Delta\psi(\gamma(t))]\nabla\psi(\gamma(t))\cdot\gamma'(t) = [\Delta\psi(\gamma(t))]\nabla g(\gamma(t))\cdot\gamma'(t) = 0$$

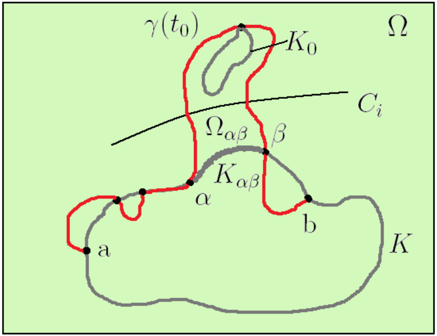
($g|_K = const$ and, hence, $\nabla g(\gamma(t)) \cdot \gamma'(t) = 0$). So, $\Phi|_K = const$.

In view of arbitrariness of $\varepsilon > 0$ for almost all $y \in \psi(\overline{\Omega})$ and for all connected components K of the set $\psi^{-1}(y)$ the equality $K \cap A_v = \emptyset$ holds and $\Phi(x) = const$ on K. The last identity is valid everywhere on K, instead of almost everywhere.

(ii) Take an arbitrary value $y \in \psi(\overline{\Omega})$ and a connected component K of the level set $\psi^{-1}(y)$ and fix them. Take also any pair of points $a, b \in K \setminus A_v$. We shall prove that

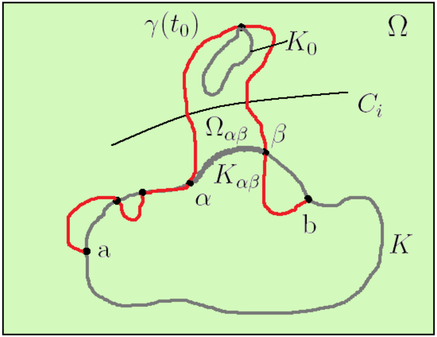
$$\Phi(a) = \Phi(b).$$

Consider a Lipschitz arc $\gamma \subset \overline{\Omega} \setminus A_v$ such that $\gamma(0) = a, \gamma(1) = b$ and Φ is absolutely continuous along γ.

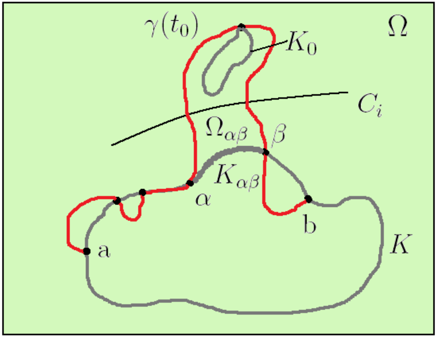


Assume that for any interval (α, β) adjoining the set $\tilde{I} = \gamma^{-1}(K)$ (i.e. $\tilde{I}$ is a compact subset of $[0,1]$, $0, 1 \in \tilde{I}$, and (α, β) is a connected component of the open set $(0,1) \setminus \tilde{I}$) there exists a continuum $K_{\alpha\beta} \subset K$ a simply connected domain $\Omega_{\alpha\beta} \subset \Omega$ such that $\Omega_{\alpha\beta} \cap K = \emptyset, \gamma(\alpha), \gamma(\beta) \in K_{\alpha\beta}$ and $\partial\Omega_{\alpha\beta} = K_{\alpha\beta} \cup \gamma([\alpha, \beta])$.

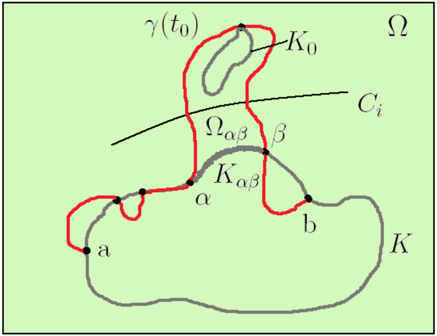
Since Φ is absolutely continuous along almost all segment, we always can chose such Lipschits arc γ.



Take any interval (α, β) adjoining the set $\tilde{I} = \gamma^{-1}(K)$, and consider the corresponding subdomain $\Omega_{\alpha\beta}$. Denote by T the family of all connected components of level sets of the function $\psi_{\alpha\beta} = \psi|_{\overline{\Omega}_{\alpha\beta}}$. According to results of A. Kronrod the space T is a tree.



Let $t_0 \in (\alpha, \beta)$ and let $K_0 \ni \gamma(t_0)$ be a connected component of the level set of $\psi_{\alpha\beta}$. Denote by $J = J(K_{\alpha\beta}, K_0)$ the arc (of the graph T) connecting the points $K_{\alpha\beta}$ and K_0. Take a sequence of regular components $C_i \in J \setminus \{K_{\alpha\beta}, K_0\}$, $C_i \to K_{\alpha\beta}$. For sufficiently large i the level sets C_i intersect the arc γ in two points. Therefore, there exist $t_i \in (\alpha, t_0)$ and $s_i \in (t_0, \beta)$ such that $\gamma(t_i), \gamma(s_i) \in C_i$. Since $C_i \to K_{\alpha\beta}$, we obtain $t_i \to \alpha$, $s_i \to \beta$.



By paragraph (i) we can take C_i such that $\Phi(x) \equiv const$ on C_i. In particular, $g(t_i) = g(s_i)$, where by g we denote the absolutely continuous function $g(t) = \Phi(\gamma(t))$. Since g is continuous, it follows that $g(\alpha) = g(\beta)$ for any interval (α, β) adjoining the set $\tilde{I}$, and hence,

$$\int_\alpha^\beta g'(t)dt = 0.$$

(Absolutely continuous function is differentiable almost everywhere).
Hence, for any interval $(\mu, \nu) \subset (0,\ 1)$, with $\mu, \nu \in \tilde{I}$ and containing only a finite number of points from $\tilde{I}$, we have the equality

$$\int_\mu^\nu g'(t)dt = 0.$$

(iii) Consider now the closed set $I_\infty = \{t \in [0, 1]$: *in any neighborhood of the point* t *there exist infinitely many points from* $\tilde{I}\}$. It follows from (ii) that

$$\int_{[0,1]\setminus I_\infty} g'(t)dt = 0.$$

The function ψ is differentiable in any point $\gamma(t), t \in (0, 1)$. On the other hand, the Lipschitz function $\gamma(t)$ is differentiable for almost all $t \in [0, 1]$. Since the function $\psi(x)$ is equal to a constant on $\psi(I_\infty) \subset K$, we have $\gamma'(t) \cdot \nabla\psi(\gamma(t)) = 0$ for any point $t \in I_\infty$ where the derivatives $\gamma'(t)$ and $\nabla\psi(\gamma(t))$ exist. Then $g'(t) = \frac{d\Phi(\gamma(t))}{dt} = 0$ for almost all $t \in I_\infty$ and

$$\int_{I_\infty} g'(t)dt = 0.$$

Summing the integrals we get $g(1) - g(0) = \int_{[0,1]} g'(t)dt = 0$. □

Properties of solutions to Euler equations

Let $\Omega' \subset \Omega$ and K_x be a connected component of the level set $\{z \in \overline{\Omega} : \psi(z) = \psi(x)\}$ containing the point x. Denote $X = X_{\Omega'} = \{x \in \Omega' : K_x \cap \partial\Omega' = \emptyset\}$. Then for almost all $y \in \psi(X)$ and for any $x \in X \cap \psi^{-1}(y)$ the component $K_x \subset \Omega' \setminus A_v$ is a C^1–smooth curve homeomorphic to the circle and $\nabla\psi \neq 0$ on K_x.

Lemma. *Let $\Omega \subset \mathbb{R}^2$ be a bounded multiply connected domain with Lipschitz boundary. Let $\mathbf{v} \in W^{1,2}(\Omega)$ and $p \in W^{1,s}(\Omega)$ satisfy Euler equations for almost all $x \in \Omega$ and $\int_{\Gamma_i} \mathbf{v} \cdot \mathbf{n} dS = 0,\ i = 1, \ldots, N$. Assume that there exists a sequence of functions $\{\Phi_\mu\}$ such that $\Phi_\mu \in W^{1,s}_{loc}(\Omega)$ and $\Phi_\mu \rightharpoonup \Phi$ in $W^{1,s}_{loc}(\Omega)$ for all $s \in [1,2)$. Then for any subdomain $\Omega' \subset \Omega$ with $X = X_{\Omega'} \neq \emptyset$ the functions $\Phi_\mu|_K$ are continuous on almost all admissible cycles K and the sequence $\{\Phi_\mu|_K\}$ converges to $\Phi|_K$ uniformly: $\Phi_\mu|_K \rightrightarrows \Phi|_K$.*

Proof.

Fix $\varepsilon > 0$ and take a set $V \subset \mathbb{R}$ and a function $g \in C^1(\mathbb{R}^2)$ such that for any $x \in X_g = X_{\Omega'} \setminus \psi^{-1}(V)$ we have $\psi(x) = g(x)$, $\nabla\psi(x) = \nabla g(x) \neq 0$ and K_x coincides with the connected component of the level set $\{z \in \mathbb{R}^2 : g(z) = g(x)\}$ containing the point x. Obviously, the set X_g admits a representation $X_g = \bigcup\limits_{i=1}^{\infty} X_i$ such that for any X_i there exists a C^1-diffeomorphism $G : [0,1] \times \mathbb{S}^1 \to U$ such that $X_i \subset U \Subset \Omega'$ and for any $t \in [0,1]$ the image $\{G(t,\theta) : \theta \in [0,2\pi)\}$ coincides with the connected component of the level set $\{z \in \mathbb{R}^2 : g(z) = g(G(t,0))\}$ containing $G(t,0)$. In particular, for each $x \in X_i$ there exist unique values $t \in [0,1]$, $\phi \in [0,2\pi)$ such that $G(t,\phi) = x$, moreover, $\{G(t,\theta) : \theta \in [0,2\pi)\} = K_x$. Fix i and put $\widetilde{\Phi}(t,\theta) = \Phi(G(t,\theta))$, etc.

Denote

$$z_\mu(t) = \int_0^{2\pi} |\widetilde{\Phi}_\mu(t,\theta) - \widetilde{\Phi}(t,\theta)| \left| \frac{\partial}{\partial\theta}\widetilde{\Phi}_\mu(t,\theta) - \frac{\partial}{\partial\theta}\widetilde{\Phi}(t,\theta) \right| d\theta.$$

Then

$$\int_0^1 z_\mu(t)dt \leq \left(\int_0^1 \int_0^{2\pi} |\widetilde{\Phi}_\mu(t,\theta) - \widetilde{\Phi}(t,\theta)|^q d\theta dt \right)^{\frac{1}{q}} \times$$

$$\times \left(\int_0^1 \int_0^{2\pi} \left| \frac{\partial}{\partial\theta}\widetilde{\Phi}_\mu(t,\theta) - \frac{\partial}{\partial\theta}\widetilde{\Phi}(t,\theta) \right|^s d\theta dt \right)^{\frac{1}{s}}$$

$$\leq c\|\Phi_\mu - \Phi\|_{L^q(U)} \|\nabla(\Phi_\mu - \Phi)\|_{L^s(U)} \leq C\|\Phi_\mu - \Phi\|_{L^q(U)}, \qquad (X)$$

where $\frac{1}{q} + \frac{1}{s} = 1$, $s \in [1,2)$, $U = \overline{U} \subset \Omega$. Since $\Phi_\mu \rightharpoonup \Phi$ in $W^{1,s}_{loc}(\Omega)$ for all $s \in [1,2)$, by Embedding Theorem $\Phi_\mu \to \Phi$ in $L^q(U)$ for all $q \in [1,\infty)$, and it follows from (X) that $z_\mu \to 0$ in $L^1([0,1])$.

Thus, there exists a subsequence (we denote it again by $\{z_\mu\}$) converging to zero almost everywhere on $[0,1]$.
Define

$$H_\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\Phi}_\mu(t,\theta)d\theta, \quad H(t) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\Phi}(t,\theta)d\theta.$$

Since $\Phi_\mu \rightharpoonup \Phi$ in $W^{1,s}(U)$, $s \in [1,2)$, by Embedding Theorem we conclude that $H_\mu \to H$ in $C([0,1])$ as $\mu \to \infty$. Moreover, $\widetilde{\Phi}_\mu, \widetilde{\Phi} \in W^{1,s}([0,1]\times\mathbb{S}^1)$ and, hence, $\widetilde{\Phi}_\mu(t,\cdot), \widetilde{\Phi}(t,\cdot)$ are absolutely continuous functions with respect to θ for almost all $t \in [0,1]$.
Let us fix arbitrary $t_* \in [0,1]$ such that $z_\mu(t_*) \to 0$ and that the functions $\widetilde{\Phi}_\mu(t_*,\cdot), \widetilde{\Phi}(t_*,\cdot)$ are continuous. Let $\theta_\mu \in [0,2\pi]$ be such that

$$\widetilde{\Phi}_\mu(t_*,\theta_\mu) - \widetilde{\Phi}(t_*,\theta_\mu) = H_\mu(t_*) - H(t_*).$$

Then

$$\max_{\theta\in[0,2\pi]} |\widetilde{\Phi}_\mu(t_*,\theta) - \widetilde{\Phi}(t_*,\theta)|^2 \le |\widetilde{\Phi}_\mu(t_*,\theta_\mu) - \widetilde{\Phi}(t_*,\theta_\mu)|^2$$

$$+ \int_0^{2\pi} \left|\frac{\partial}{\partial\theta}\big(\widetilde{\Phi}_\mu(t_*,\theta) - \widetilde{\Phi}(t_*,\theta)\big)^2\right| d\theta$$

$$= |H_\mu(t_*) - H(t_*)|^2 + 2z_\mu(t_*) \to 0$$

as $\mu \to \infty$. Thus, the continuity of $\widetilde{\Phi}_\mu(t,\cdot)$ and the uniform convergence $\widetilde{\Phi}_\mu(t,\cdot) \rightrightarrows \widetilde{\Phi}(t,\cdot)$ is proved for almost all $t \in [0,1]$. So, the claim of the lemma is proved for almost all admissible cycles $K \subset X_i$, and hence, for almost all admissible cycles $K \subset X_{\Omega'} \setminus \psi^{-1}(V)$. Because $\mathfrak{H}^1(V) < \varepsilon$ and $\varepsilon > 0$ is arbitrary, the lemma is proved completely. $\square$