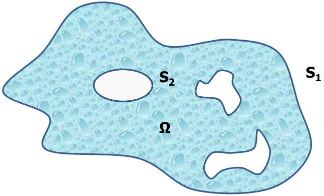
Proof of the existence (by a contradiction)

November 6, 2013

$$\begin{cases} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \\ div\mathbf{v} = 0 \quad \text{in } \Omega, \\ \mathbf{v} = \mathbf{h} \quad \text{on } \partial\Omega, \end{cases} \qquad (NS)$$

$\mathbf{v}$ – velocity of the fluid, p -pressure.
$\Omega \subset \mathbb{R}^n, n = 2, 3,$–multi-connected domain:



Incompressibility of the fluid ($div\mathbf{v} = 0$) implies the necessary compatibility condition for the solvability of problem (NS):

$$\int\limits_{\partial\Omega} \mathbf{h}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N} \int\limits_{\Gamma_j} \mathbf{h}\cdot\mathbf{n}\,dS = \sum_{j=1}^{N} F_j = 0, \qquad (F)$$

$\mathbf{n}$ is a unit vector of the outward normal to $\partial\Omega$.

General Scheme

Let $\mathbf{A} \in W^{1,2}(\Omega)$ be a divergence free extension

$$div\mathbf{A} = 0, \quad \mathbf{A}|_{\partial\Omega} = \mathbf{h}.$$

$\mathbf{u} = \mathbf{v} - \mathbf{A}$ is a weak solution of (NS) if
$\mathbf{u} \in H(\Omega) = \{\mathbf{w} \in \overset{\circ}{W}{}_2^1(\Omega) : div\mathbf{w} = 0\}$ and

$$\nu \int_\Omega \nabla\mathbf{u} \cdot \nabla\boldsymbol{\eta} dx + \int_\Omega ((\mathbf{u} + \mathbf{A}) \cdot \nabla)\mathbf{u} \cdot \boldsymbol{\eta} dx + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{A} \cdot \boldsymbol{\eta} dx =$$

$$= \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta} dx - \nu \int_\Omega \nabla\mathbf{A} \cdot \nabla\boldsymbol{\eta} dx - \int_\Omega (\mathbf{A} \cdot \nabla)\mathbf{A} \cdot \boldsymbol{\eta} dx \quad \forall \boldsymbol{\eta} \in H(\Omega).$$

Integral identity (4) is equivalent to an operator equation:

$$\mathbf{u} = \mathcal{B}\mathbf{u}$$

where $\mathcal{B} : H(\Omega) \to H(\Omega)$ is a compact operator.

In order to apply the Leray–Schauder theorem we have to prove that all possible solutions of the equation

$$\mathbf{u}^{(\lambda)} = \lambda \mathcal{B} \mathbf{u}^{(\lambda)} \quad \lambda \in (0,\ 1]$$

are bounded by a constant independent of λ. Equivalently, we can consider for $\lambda \in [0, 1]$:

$$\int_\Omega \nabla \mathbf{u}^{(\lambda)} \cdot \nabla \boldsymbol{\eta} dx + \frac{\lambda}{\nu} \int_\Omega ((\mathbf{u}^{(\lambda)} + \mathbf{A}) \cdot \nabla) \mathbf{u}^{(\lambda)} \cdot \boldsymbol{\eta} dx + \frac{\lambda}{\nu} \int_\Omega (\mathbf{u}^{(\lambda)} \cdot \nabla) \mathbf{A} \cdot \boldsymbol{\eta} dx$$

$$= \lambda \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta} dx - \lambda \int_\Omega \nabla \mathbf{A} \cdot \nabla \boldsymbol{\eta} dx - \lambda \nu^{-1} \int_\Omega (\mathbf{A} \cdot \nabla) \mathbf{A} \cdot \boldsymbol{\eta} dx \quad \forall \boldsymbol{\eta} \in H(\Omega).$$

Take $\boldsymbol{\eta} = \mathbf{u}^{(\lambda)}$:

$$\int_\Omega |\nabla \mathbf{u}^{(\lambda)}|^2 dx + \frac{\lambda}{\nu} \int_\Omega (\mathbf{u}^{(\lambda)} \cdot \nabla) \mathbf{A} \cdot \mathbf{u}^{(\lambda)} dx$$

$$= \lambda \int_\Omega \mathbf{f} \cdot \mathbf{u}^{(\lambda)} dx - \lambda \int_\Omega \nabla \mathbf{A} \cdot \nabla \mathbf{u}^{(\lambda)} dx - \frac{\lambda}{\nu} \int_\Omega (\mathbf{A} \cdot \nabla) \mathbf{A} \cdot \mathbf{u}^{(\lambda)} dx.$$

Then

$$\int_\Omega |\nabla \mathbf{u}^{(\lambda)}|^2 dx \leq C(\mathbf{A}) + \frac{\lambda}{\nu} \int_\Omega (\mathbf{u}^{(\lambda)} \cdot \nabla)\mathbf{u}^{(\lambda)} \cdot \mathbf{A} dx.$$

We have to prove

$$\int_\Omega |\nabla \mathbf{u}^{(\lambda)}|^2 dx \leq C_1(\mathbf{A}). \tag{$*$}$$

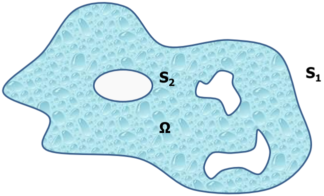
(i) Leray-Hopf construction is impossible.

(ii) We have prove to prove (*) by getting a contradiction.

Main result

Theorem [M. Korobkov, K. Pileckas, R. Russo, 2013] *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2-smooth boundary $\partial\Omega$. If $\mathbf{f} \in W^{1,2}(\Omega)$ and $\mathbf{h} \in W^{3/2,2}(\partial\Omega)$ satisfies condition (F), then problem (NS) admits at least one weak solution.*

$$\int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n} \, dS = \sum_{j=1}^{N} \int_{\Gamma_j} \mathbf{h} \cdot \mathbf{n} \, dS = \sum_{j=1}^{N} F_j = 0, \qquad (F)$$



Take the extension $\mathbf{A} \in W^{1,2}(\Omega)$ as a weak solution of the Stokes problem, i.e., $div\mathbf{A} = 0$, $\mathbf{A}|_{\partial\Omega} = \mathbf{h}$ and

$$\nu \int_\Omega \nabla \mathbf{A} \cdot \nabla \boldsymbol{\eta}\, dx = \int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx \quad \forall\, \boldsymbol{\eta} \in H(\Omega).$$

To prove the solvability it is sufficient to show that all possible solutions $\mathbf{u}^{(\lambda)} \in H(\Omega)$ of the integral identity

$$\nu \int_\Omega \nabla \mathbf{u}^{(\lambda)} \cdot \nabla \boldsymbol{\eta}\, dx + \lambda \int_\Omega \big((\mathbf{u}^{(\lambda)} + \mathbf{A}) \cdot \nabla\big) \mathbf{u}^{(\lambda)} \cdot \boldsymbol{\eta}\, dx + \lambda \int_\Omega \big(\mathbf{u}^{(\lambda)} \cdot \nabla\big) \mathbf{A} \cdot \boldsymbol{\eta}\, dx$$

$$= \lambda \int_\Omega \big(\mathbf{A} \cdot \nabla\big) \boldsymbol{\eta} \cdot \mathbf{A}\, dx \qquad \forall\, \boldsymbol{\eta} \in H(\Omega)$$

are uniformly bounded in $H(\Omega)$ (with respect to $\lambda \in [0,1]$). Assume this is false. Then there exist sequences $\{\lambda_k\} \subset [0,1]$ and $\{\mathbf{u}^{(\lambda_k)}\}_{k\in\mathbb{N}} = \{\mathbf{u}_k\}_{k\in\mathbb{N}} \in H(\Omega)$ such that

$$\lim_{k\to\infty} \lambda_k = \lambda_0 \in (0,1], \qquad \lim_{k\to\infty} J_k = \lim_{k\to\infty} \|\mathbf{u}_k\|_{H(\Omega)} = \infty.$$

The pair $\left(\widehat{\mathbf{u}}_k = \frac{1}{J_k}\mathbf{u}_k,\ \widehat{p}_k = \frac{1}{\lambda_k J_k^2} p_k\right)$ satisfies the following system

$$\begin{cases} -\nu_k \Delta \widehat{\mathbf{u}}_k + \big(\widehat{\mathbf{u}}_k \cdot \nabla\big)\widehat{\mathbf{u}}_k + \nabla \widehat{p}_k &= \mathbf{f}_k \quad \text{in} \quad \Omega, \\ div \widehat{\mathbf{u}}_k &= 0 \quad \text{in} \quad \Omega, \\ \widehat{\mathbf{u}}_k &= \mathbf{h}_k \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\nu_k = (\lambda_k J_k)^{-1}\nu$, $\mathbf{f}_k = \frac{\lambda_k \nu_k^2}{\nu^2}\,\mathbf{f}$, $\mathbf{h}_k = \frac{\lambda_k \nu_k}{\nu}\,\mathbf{h}$.
The norms $\|\widehat{\mathbf{u}}_k\|_{W^{1,2}(\Omega)}$ and $\|\widehat{p}_k\|_{W^{1,q}(\Omega)}$, $q \in [1,2)$, are uniformly bounded, $\widehat{\mathbf{u}}_k \rightharpoonup \mathbf{v}$ in $W^{1,2}(\Omega)$, $\widehat{p}_k \rightharpoonup p$ in $W^{1,q}(\Omega)$. Moreover, $\widehat{\mathbf{u}}_k \in W^{3,2}_{loc}(\Omega)$, $\widehat{p}_k \in W^{2,2}_{loc}(\Omega)$.

(i) Take in the integral identity $\eta = J_k^{-2}\mathbf{u}_k$.
Passing to the limit as $k_l \to \infty$, we get

$$\frac{\nu}{\lambda_0} = \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx. \qquad (C_1)$$

(ii) Let $\varphi \in J_0^\infty(\Omega)$. Take $\eta = J_{k_l}^{-2}\varphi$ and let $k_l \to \infty$:

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\varphi\,dx = 0 \quad \forall\varphi \in J_0^\infty(\Omega).$$

Hence, the pair $(\mathbf{v}, p)$ satisfies for almost all $x \in \Omega$ the Euler equations

$$(\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0, \quad div\,\mathbf{v} = 0,$$

and

$$\mathbf{v}\big|_{\partial\Omega} = 0.$$

Then

$$p_j = p(x)|_{\Gamma_j}.$$

Leray's method

Multiply Euler equations by $\mathbf{A}$ and integrate by parts:

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx = -\int_\Omega \nabla p\cdot\mathbf{A}\,dx = -\int_{\partial\Omega} p\,\mathbf{h}\cdot\mathbf{n}\,dS$$

$$= -\sum_{j=1}^{M} p_j \int_{\Gamma_j} \mathbf{h}\cdot\mathbf{n}\,dS = -\sum_{j=1}^{M} p_j F_j$$

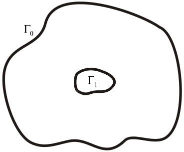
If either all $F_j = 0$ or $p_1 = \cdots = p_M$, we get

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx = 0. \qquad (C_2)$$

This contradicts (C_1):

$$\int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{A}\,dx = \frac{\nu}{\lambda_0} > 0. \qquad (C_1)$$

For simplicity consider the case



There could appear two cases

(a) *The maximum of Φ is attained on the boundary $\partial\Omega$:*

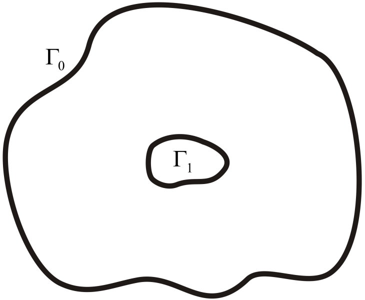
$$\max\{p_0, p_1\} = \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x).$$

(b) *The maximum of Φ is not attained on the boundary $\partial\Omega$:*

$$\max\{p_0, p_1\} < \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x).$$

Consider the case (a). Let $p_0 < 0, p_1 = 0$. We have

$$\Phi(x) \le 0 \quad in \quad \Omega.$$



Remark. $\widehat{\Phi}_k$ satisfies elliptic equation

$$\Delta\widehat{\Phi}_k - \frac{1}{\nu_k} div(\widehat{\Phi}_k \widehat{\mathbf{u}}_k) = \widehat{\omega}_k^2 - \frac{1}{\nu_k}\mathbf{f}_k \cdot \mathbf{u}_k, \quad \widehat{\omega}_k = \partial_2 \widehat{u}_{1k} - \partial_1 \widehat{u}_{2k}.$$

If $\mathbf{f}_k = 0 \Rightarrow \widehat{\Phi}_k$ satisfies maximum principle.

Obtaining a contradiction

T_f - family of all connected components of level sets of f, f is continuous. Then T_f is a tree. Let $K \in T_\psi$ with $diamK > 0$. Take any $x \in K \setminus A$ and put $\Phi(K) = \Phi(x)$. This definition is correct by Bernoulli's Law.

Lemma 1. *Let $A, B \in T_\psi$, $A > 0, B > 0$. Consider the corresponding arc $[A, B] \subset T_\psi$ joining A to B. Then the restriction $\Phi|_{[A,B]}$ is a continuous function.*

Proof. Let $C_i \in (A, B)$ and $C_i \to C_0$ in T_ψ. By construction, each C_i is a connected component of the level set of ψ and the sets A, B lie in different connected components of $\mathbb{R}^2 \setminus C_i$. Therefore,

$$diam(C_i) \geq \min(diam(A), diam(B)) > 0.$$

By the definition of convergence in T_ψ, we have

$$\sup_{x \in C_i} dist(x, C_0) \to 0 \quad \text{as } i \to \infty.$$

Obtaining a contradiction

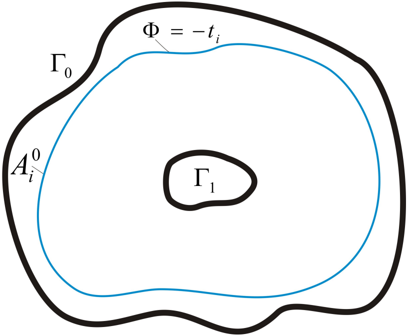
Let $\Phi(x) = c_i$ for a.a. $x \in C_i$, $\Phi(x) = c_0$ for a.a. $x \in C_0$.
Assume $c_i \nrightarrow c_0$. Then $c_i \to c_\infty \neq c_0$. Moreover, components $C_i \to C_0' \subset C_0$ in Hausdorf metric (Blaschke selection theorem).
Let L be a line; I_0 projection of C_0' on L.
Obviously, I_0- interval. For every $z \in I_0$ denote L_z the line s.t. $z \in L_z$ and $L_z \perp L$.
For almost all z the function $\Phi|_{L_z}$ is absolutely continuous. Fix such z. Then $C_i \cap L_z \neq \emptyset$ for sufficiently large i. Let $y_i \in C_i \cap L_z$.
Extract a subsequence $y_{i_l} \to y_0 \in C_0'$.
Then $\Phi(y_{i_l}) \to \Phi(y_0) = c_0$. $\Rrightarrow$ Contradiction. $\square$

Let $B_0, B_1 \in T_\psi$, $B_0 \supset \Gamma_0$, $B_1 \supset \Gamma_1$. Set

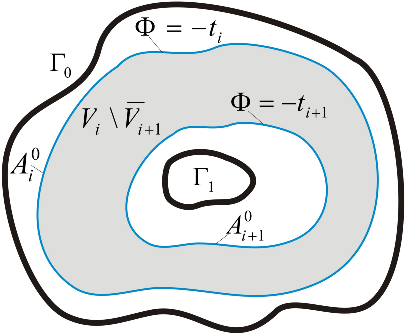
$$\alpha = \min_{C \in [B_0, B_1]} \Phi(C) < 0.$$

Let $t_i \in (0, -\alpha), \quad t_{i+1} = \frac{1}{2} t_i$ and such that

$$\Phi(C) = -t_i \quad \Rightarrow \quad C \in (B_0, B_1) \;\; is \;\; regular$$



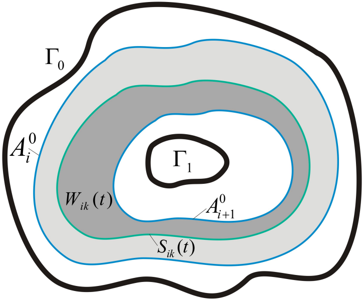
A_i^0 is an element from the set $\{C \in [B_0, B_1] : \Phi(C) = -t_i\}$ which is closest to Γ_1. A_i^0 is regular cycle.



V_i is the connected component of the set $\Omega \setminus A_i^0$ such that $\Gamma_1 \subset \partial V_i$. Obviously, $V_i \supset V_{i+1}$, $\partial V_i = A_i^0 \cup \Gamma_1$. Remind that $t_{i+1} = \frac{1}{2} t_i$.

A_i^0 are regular cycles. Therefore, $\widehat{\Phi}_k|_{A_i^0} \rightrightarrows \Phi|_{A_i^0} = -t_i$, and for sufficiently large k holds

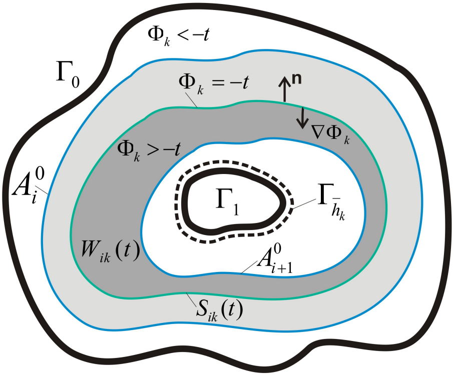
$$\widehat{\Phi}_k|_{A_i^0} < -\frac{7}{8} t_i, \quad \widehat{\Phi}_k|_{A_{i+1}^0} > -\frac{5}{8} t_i, \quad \forall k \geq k_i.$$



Take $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$. $W_{ik}(t)$ is the connected component of the set $\{x \in V_i \setminus \overline{V}_{i+1} : \widehat{\Phi}_k(x) > -t\}$ such that $\partial W_{ik}(t) \supset A_{i+1}^0$. Put $S_{ik}(t) = (\partial W_{ik}(t)) \cap V_i \setminus \overline{V}_{i+1}$. Then $\widehat{\Phi}_k|_{S_{ik}(t)} = -t$, $\partial W_{ik}(t) = S_{ik}(t) \cup A_{i+1}^0$.

Since $\widehat{\Phi}_k \in W^2_{2,loc}(\Omega)$, by the Morse-Sard theorem for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$ the level set $S_{ik}(t)$ consists of a finite number of C^1-cycles; moreover, $\widehat{\Phi}_k$ is differentiable at every point $x \in S_{ik}(t)$ and $\nabla\widehat{\Phi}_k(x) \neq 0$. Such t we call (k, i)-regular.

Geometrical construction

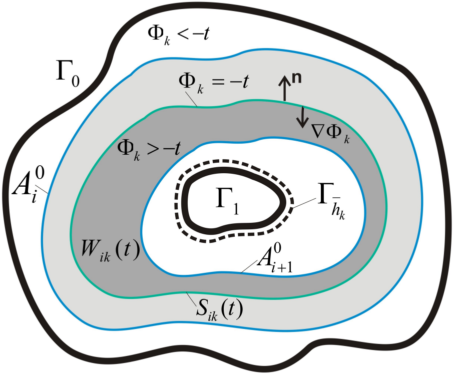


By construction

$$\int_{S_{ik}(t)} \nabla\widehat{\Phi}_k \cdot \mathbf{n} dS = -\int_{S_{ik}(t)} |\nabla\widehat{\Phi}_k| dS < 0,$$

where $\mathbf{n}$ outward with respect to $W_{ik}(t)$ normal to $\partial W_{ik}(t)$.

Geometrical construction



$\Gamma_h = \{x \in \Omega : dist(x, \Gamma_1) = h\}, \;\; \Omega_h = \{x \in \Omega : dist(x, \Gamma_1) < h\}.$
Γ_h is C^1-smooth and

$$\mathfrak{H}(\Gamma_h) \leq C_0 \;\; \forall h \in (0, \delta_0],$$

$C_0 = 3\mathfrak{H}(\Gamma_1).$

Obtaining a contradiction

Since $\Phi(x) \neq const$ on V_i, i.e., $\nabla\Phi(x) \neq 0$, from the identity $\nabla\Phi = \omega\nabla\psi$ it follows that

$$\int_{V_i} \omega^2 dx > 0.$$

From the weak convergence $\widehat{\omega}_k \rightharpoonup \omega$ in $L_2(\Omega)$ we get the following

Lemma 2. *For any $i \in \mathbb{N}$ there exists $\varepsilon_i > 0$, $\delta_i \in (0, \delta_0)$ and $k_i' \in \mathbb{N}$ such that*

$$\int_{V_{i+1} \setminus \Omega_{\delta_i}} \widehat{\omega}_k^2 dx > \varepsilon_i \quad \forall k \geq k_i'.$$

Obtaining a contradiction

The key step is the following estimate

Lemma 3. *For any $i \in \mathbb{N}$ there exists $k(i) \in \mathbb{N}$ such that there holds inequality*

$$\int_{S_{ik}(t)} |\nabla \widehat{\Phi}_k(x)| dS < \mathcal{F}t \ \ \forall k \geq k(i), \ \ \textit{for almost all } t \in [\frac{5}{8}t_i, \frac{7}{8}t_i].$$

The constant $\mathcal{F}$ is independents of t, k and i.

Obtaining a contradiction

We receive the required contradiction using the Coarea formula.
For $i \in$ and $k \geq k(i)$ put

$$E_i = \bigcup_{t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]} S_{ik}(t).$$

By the Coarea formula for any integrable function $g : E_i \to \mathbb{R}$ holds the equality

$$\int_{E_i} g|\nabla \Phi_k|\, dx = \int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \int_{S_{ik}(t)} g(x)\, d\mathfrak{H}^1(x)\, dt.$$

In particular, taking $g = |\nabla \Phi_k|$ and using Lemma 3, we obtain

$$\int_{E_i} |\nabla \Phi_k|^2\, dx = \int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \int_{S_{ik}(t)} |\nabla \Phi_k|\, d\mathfrak{H}^1(x) dt \leq \int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathcal{F} t\, dt = \mathcal{F}' t_i^2.$$

Obtaining a contradiction

Now, taking $g = 1$ in the Coarea formula and using the Hölder inequality we get

$$\int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathfrak{H}^1\big(S_{ik}(t)\big)dt = \int_{E_i} |\nabla\Phi_k|\, dx$$

$$\leq \left(\int_{E_i} |\nabla\Phi_k|^2\, dx\right)^{\frac{1}{2}} \big(meas(E_i)\big)^{\frac{1}{2}} \leq \sqrt{\mathcal{F}'}t_i\big(meas(E_i)\big)^{\frac{1}{2}}.$$

By construction, for almost all $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$ the set $S_{ik}(t)$ is a smooth cycle and $S_{ik}(t)$ separates A_i^0 from A_{i+1}^0. Thus, each set $S_{ik}(t)$ separates Γ_0 from Γ_1. In particular, $\mathfrak{H}^1(S_{ik}(t)) \geq \mathfrak{H}^1(\Gamma_1)$. Hence,

$$\int_{\frac{5}{8}t_i}^{\frac{7}{8}t_i} \mathfrak{H}^1\big(S_{ik}(t)\big)dt \geq \frac{1}{4}\mathfrak{H}^1(\Gamma_1)t_i.$$

So, it holds

$$\frac{1}{4}\mathfrak{H}^1(\Gamma_1)t_i \leq \sqrt{\mathcal{F}'}t_i\big(meas(E_i)\big)^{\frac{1}{2}},$$

Obtaining a contradiction

or

$$\frac{1}{4}\mathfrak{H}^1(\Gamma_1) \leq \sqrt{\mathcal{F}'}\big(meas(E_i)\big)^{\frac{1}{2}}.$$

Since $meas(E_i) \leq meas\big(V_i \setminus V_{i+1}\big) \to 0$ as $i \to \infty$, we obtain a contradiction!!!.

Lemma 3. *For any $i \in \mathbb{N}$ there exists $k(i) \in \mathbb{N}$ such that there holds inequality*

$$\int_{S_{ik}(t)} |\nabla \widehat{\Phi}_k(x)| dS < \mathcal{F} t \ \ \forall k \geq k(i) \ \ \textit{for almost all } t \in [\frac{5}{8} t_i, \frac{7}{8} t_i].$$

The constant $\mathcal{F}$ is independents of t, k and i.

Proof. Fix $i \in \mathbb{N}$ and assume $k \geq k_i$:

$$\widehat{\Phi}_k|_{A_i^0} < -\frac{7}{8} t_i, \quad \widehat{\Phi}_k|_{A_{i+1}^0} > -\frac{5}{8} t_i, \ \ \forall k \geq k_i.$$

Take a sufficiently small $\sigma > 0$ and choose the parameter $\delta_\sigma \in (0, \delta_i]$ small enough to satisfy the following conditions:

$$\Omega_{\delta_\sigma} \cap A_i^0 = \Omega_{\delta_\sigma} \cap A_{i+1}^0 = \emptyset,$$

$$\int_{\Gamma_h} \Phi^2 \, ds < \frac{1}{3} \sigma^2 \quad \forall h \in (0, \delta_\sigma], \tag{1}$$

Proof of Lemma 3

$$-\frac{1}{3}\sigma^2 < \int_{\Gamma_{h'}} \widehat{\Phi}_k^2\, ds - \int_{\Gamma_{h''}} \widehat{\Phi}_k^2\, ds < \frac{1}{3}\sigma^2 \qquad (2)$$

$\forall h', h'' \in (0, \delta_\sigma] \;\; \forall k \in \mathbb{N}$.
This estimate follows from the fact that for any $q \in (1, 2)$ the norms $\|\widehat{\Phi}_k\|_{W^{1,q}(\Omega)}$ are uniformly bounded. Hence, the norms $\|\widehat{\Phi}_k \nabla \widehat{\Phi}_k\|_{L^{\frac{6}{5}}(\Omega)}$ are uniformly bounded and

$$\left| \int_{\Gamma_{h'}} \widehat{\Phi}_k^2\, ds - \int_{\Gamma_{h''}} \widehat{\Phi}_k^2\, ds \right| \leq 2 \int_{\Omega_{h''} \setminus \Omega_{h'}} |\widehat{\Phi}_k| \cdot |\nabla \widehat{\Phi}_k|\, dx$$

$$\leq 2 \left(\int_{\Omega_{h''} \setminus \Omega_{h'}} |\widehat{\Phi}_k \nabla \widehat{\Phi}_k|^{6/5}\, dx \right)^{\frac{5}{6}} meas(\Omega_{h''} \setminus \Omega_{h'})^{\frac{1}{6}} \to 0 \quad \text{as } h', h'' \to 0.$$

Proof of Lemma 3

From the weak convergence $\widehat{\Phi}_k \rightharpoonup \Phi$ in the space $W^{1,q}(\Omega)$, $q \in (1,2)$, it follows that $\widehat{\Phi}_k|_{\Gamma_h} \rightrightarrows \Phi|_{\Gamma_h}$ as $k \to \infty$ for almost all $h \in (0, \delta_\sigma)$. Therefore, from (1)–(2) we see that there exists $k' \in \mathbb{N}$ such that

$$\int\limits_{\Gamma_h} \widehat{\Phi}_k^2 \, ds < \sigma^2 \quad \forall h \in (0, \delta_\sigma] \ \ \forall k \geq k'. \tag{3}$$

For a function $g \in W^{2,2}(\Omega)$ and for an arbitrary C^1-cycle $S \subset \Omega$ we have by the Stokes theorem

$$\int_S \nabla^\perp g \cdot \mathbf{n} \, ds = \int_S \nabla g \cdot \mathbf{l} \, ds = 0,$$

where $\mathbf{l}$ is the tangent vector to S. Since

$$\nabla \widehat{\Phi}_k = -\nu_k \nabla^\perp \widehat{\omega}_k + \widehat{\omega}_k \widehat{\mathbf{u}}_k^\perp + \mathbf{f}_k,$$

we have

$$\int_S \nabla \widehat{\Phi}_k \cdot \mathbf{n} \, ds = \int_S \widehat{\omega}_k \widehat{\mathbf{u}}_k^\perp \cdot \mathbf{n} \, ds.$$

Proof of Lemma 3

Fix a sufficiently small $\varepsilon > 0$. For a given sufficiently large $k \geq k'$ we make a special procedure to find a number $\bar{h}_k \in (0, \delta_\sigma)$ such that

$$\left| \int_{\Gamma_{\bar{h}_k}} \nabla \widehat{\Phi}_k \cdot \mathbf{n} \, ds \right| = \left| \int_{\Gamma_{\bar{h}_k}} \widehat{\omega}_k \widehat{\mathbf{u}}_k^{\perp} \cdot \mathbf{n} \, ds \right| < \varepsilon, \tag{4}$$

$$\int_{\Gamma_{\bar{h}_k}} |\widehat{\mathbf{u}}_k|^2 \, ds < C(\varepsilon) \nu_k^2, \tag{5}$$

where the constant $C(\varepsilon)$ **is independent of** k **and** σ.

Define a sequence of numbers $0 = h_0 < h_1 < h_2 < \dots$ by the recurrent formulas

$$\int_{U_j} |\nabla \widehat{\mathbf{u}}_k| |\widehat{\mathbf{u}}_k| \, dx = \nu_k^2, \tag{6}$$

where $U_j = \{x \in \Omega : dist(x, \Gamma_1) \in (h_{j-1}, h_j)\}$.

Proof of Lemma 3

Since $\int\limits_{\partial\Omega} |\widehat{\mathbf{u}}_k|^2 \, ds = \frac{(\lambda_k \nu_k)^2}{\nu^2} \|\mathbf{h}\|^2_{L_2(\partial\Omega)}$, where $\lambda_k \in (0,1]$, from (6) we get by induction that

$$\int_{\Gamma_h} |\widehat{\mathbf{u}}_k|^2 \, ds \le Cj\nu_k^2 \quad \forall h \in (h_{j-1}, h_j), \tag{7}$$

where C is independent of k, j, σ. Therefore,

$$\int_{U_j} |\widehat{\mathbf{u}}_k|^2 \, dx \le (h_j - h_{j-1}) Cj\nu_k^2. \tag{8}$$

Then

$$\nu_k^2 = \int_{U_j} |\nabla \widehat{\mathbf{u}}_k| \cdot |\widehat{\mathbf{u}}_k| \, dx \le \sqrt{(h_j - h_{j-1}) Cj\nu_k^2} \left(\int_{U_j} |\nabla \widehat{\mathbf{u}}_k|^2 \, dx \right)^{\frac{1}{2}}.$$

Thus, we have

$$\frac{\nu_k^2}{h_j - h_{j-1}} \le Cj \int_{U_j} |\nabla \widehat{\mathbf{u}}_k|^2 \, dx. \tag{9}$$

Proof of Lemma 3

Define h_j for $j = 1, \ldots, j_{max}$, where $j_{\max}$ is the first index which satisfies one of the following two conditions:

STOP CASE 1. $h_{j_{\max}-1} < \delta_\sigma$, $h_{j_{\max}} \geq \delta_\sigma$.

STOP CASE 2. $Cj_{\max} \int_{U_{j_{\max}}} |\nabla \widehat{\mathbf{u}}_k|^2 \, dx < \varepsilon$.

By construction, $\int_{U_j} |\nabla \widehat{\mathbf{u}}_k|^2 \, dx \geq \frac{1}{Cj} \varepsilon$ for every $j < j_{\max}$.

Hence,

$$2 \geq \int\limits_{U_1 \cup \cdots \cup U_{j_{\max}-1}} |\nabla \widehat{\mathbf{u}}_k|^2 \, dx \geq \frac{\varepsilon}{C} \left(1 + \frac{1}{2} + \cdots + \frac{1}{j_{\max} - 1}\right) > C' \varepsilon \ln(j_{\max} - 1).$$

So, for **both** cases we have the following uniform estimate

$$j_{\max} \leq 1 + \exp\left(\frac{1}{C' \varepsilon}\right) \qquad (10)$$

with C' independent of k and σ.

Proof of Lemma 3

Assume that Stop case 1 take place. Then

$$\Omega_{\delta_\sigma} \subset U_1 \cup \cdots \cup U_{j_{max}}$$

and by construction (see (6): $\int_{U_j} |\nabla \widehat{\mathbf{u}}_k||\widehat{\mathbf{u}}_k|\, dx = \nu_k^2$, and (7): $\int_{\Gamma_h} |\widehat{\mathbf{u}}_k|^2\, ds \leq Cj\nu_k^2 \quad \forall h \in (h_{j-1}, h_j)$) we have

$$\int\limits_{\Omega_{\delta_\sigma}} |\nabla \widehat{\mathbf{u}}_k| \cdot |\widehat{\mathbf{u}}_k|\, dx \leq j_{\max}\nu_k^2, \tag{11}$$

$$\int_{\Gamma_h} |\widehat{\mathbf{u}}_k|^2\, ds \leq Cj_{\max}\nu_k^2 \quad \forall h \in (0, \delta_\sigma]. \tag{12}$$

From (11) it follows that there exists $\bar{h}_k \in (0, \delta_\sigma)$ such that

$$\int_{\Gamma_{\bar{h}_k}} |\nabla \widehat{\mathbf{u}}_k| \cdot |\widehat{\mathbf{u}}_k|\, ds \leq \frac{1}{\delta_\sigma} j_{\max}\nu_k^2.$$

Proof of Lemma 3

Then, taking into account that $j_{\max}$ does not depend on σ and k (see (10) $j_{\max} \leq 1 + \exp(\frac{1}{C'\varepsilon})$), and that $\nu_k \to 0$ as $k \to \infty$, we obtain the required estimates (4)–(5) for sufficiently large k:

$$\left| \int_{\Gamma_{\bar{h}_k}} \nabla \widehat{\Phi}_k \cdot \mathbf{n} \, ds \right| = \left| \int_{\Gamma_{\bar{h}_k}} \widehat{\omega}_k \widehat{\mathbf{u}}_k^{\perp} \cdot \mathbf{n} \, ds \right| < \varepsilon, \tag{4}$$

$$\int_{\Gamma_{\bar{h}_k}} |\widehat{\mathbf{u}}_k|^2 \, ds < C(\varepsilon) \nu_k^2, \tag{5}$$

Now, let Stop case 2 arises. By definition of this case and by (9) ($\frac{\nu_k^2}{h_j - h_{j-1}} \leq C\, j \int_{U_j} |\nabla \widehat{\mathbf{u}}_k|^2\, dx$), we obtain

$$\frac{1}{h_{j_{\max}} - h_{j_{\max}-1}} \int_{U_{j_{\max}}} |\nabla \widehat{\mathbf{u}}_k| \cdot |\widehat{\mathbf{u}}_k|\, dx = \frac{\nu_k^2}{h_{j_{\max}} - h_{j_{\max}-1}}$$

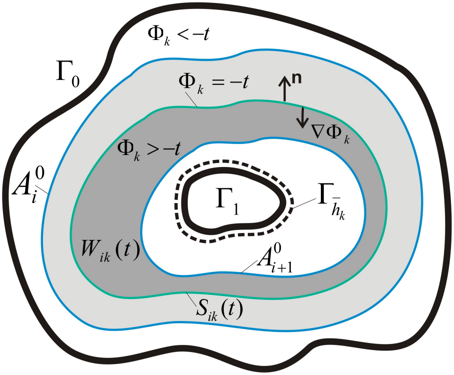
$$\leq C j_{max} \int_{U_{j_{\max}}} |\nabla \widehat{\mathbf{u}}_k|^2 dx < \varepsilon.$$

Therefore, there exists $\bar{h}_k \in (h_{j_{\max}-1}, h_{j_{\max}})$ such that (4) holds. Estimate (5) follows again from (7) and the fact that $j_{\max}$ depends on ε only. So, for any sufficiently large k we have proved the existence of $\bar{h}_k \in (0, \delta_\sigma)$ such that (4)–(5) hold.

Now, for (k,i)-regular value $t \in [\frac{5}{8}t_i, \frac{7}{8}t_i]$ consider the domain

$$\Omega_{i\bar{h}_k}(t) = W_{ik}(t) \cup \overline{V}_{i+1} \setminus \overline{\Omega}_{\bar{h}_k}.$$

By construction, $\partial\Omega_{i\bar{h}_k}(t) = \Gamma_{\bar{h}_k} \cup S_{ik}(t)$.



Proof of Lemma 3

Integrating the equation

$$\Delta\widehat{\Phi}_k = \widehat{\omega}_k^2 + \frac{1}{\nu_k}div(\widehat{\Phi}_k\widehat{\mathbf{u}}_k) - \frac{1}{\nu_k}\mathbf{f}_k\cdot\mathbf{u}_k$$

over the domain $\Omega_{i\bar{h}_k}(t)$, we have

$$\int_{S_{ik}(t)}\nabla\widehat{\Phi}_k\cdot\mathbf{n}\,ds + \int_{\Gamma_{\bar{h}_k}}\nabla\widehat{\Phi}_k\cdot\mathbf{n}\,ds = \int_{\Omega_{i\bar{h}_k}(t)}\widehat{\omega}_k^2\,dx$$

$$+\frac{1}{\nu_k}\int_{S_{ik}(t)}\widehat{\Phi}_k\widehat{\mathbf{u}}_k\cdot\mathbf{n}\,ds + \frac{1}{\nu_k}\int_{\Gamma_{\bar{h}_k}}\widehat{\Phi}_k\widehat{\mathbf{u}}_k\cdot\mathbf{n}\,ds - \frac{1}{\nu_k}\int_{\Omega_{i\bar{h}_k}(t)}\mathbf{f}_k\cdot\mathbf{u}_k dx$$

$$= \int_{\Omega_{i\bar{h}_k}(t)}\widehat{\omega}_k^2\,dx - t\lambda_k\bar{\mathcal{F}} + \frac{1}{\nu_k}\int_{\Gamma_{\bar{h}_k}}\widehat{\Phi}_k\widehat{\mathbf{u}}_k\cdot\mathbf{n}\,ds - \frac{1}{\nu_k}\int_{\Omega_{i\bar{h}_k}(t)}\mathbf{f}_k\cdot\mathbf{u}_k dx, \quad (13)$$

where $\bar{\mathcal{F}} = \frac{1}{\nu}F_0$.

Proof of Lemma 3

Since

$$\int_{S_{ik}(t)} \nabla\widehat{\Phi}_k \cdot \mathbf{n} dS = -\int_{S_{ik}(t)} |\nabla\widehat{\Phi}_k| dS$$

and

$$\left|\int_{\Gamma_{\bar{h}_k}} \nabla\widehat{\Phi}_k \cdot \mathbf{n} dS\right| \le \varepsilon$$

(see (4)), we get

$$\int_{S_{ik}(t)} |\nabla\widehat{\Phi}_k|\, ds \le t\mathcal{F} + \varepsilon - \int_{\Omega_{i\bar{h}_k}(t)} \widehat{\omega}_k^2\, dx$$

$$+\frac{1}{\nu_k}\Big(\int_{\Gamma_{\bar{h}_k}} \widehat{\Phi}_k^2\, ds\Big)^{\frac{1}{2}}\Big(\int_{\Gamma_{\bar{h}_k}} |\widehat{\mathbf{u}}_k|^2\, ds\Big)^{\frac{1}{2}} - \frac{1}{\nu_k}\int_{\Omega_{i\bar{h}_k}(t)} \mathbf{f}_k \cdot \mathbf{u}_k dx$$

with $\mathcal{F} = |\bar{\mathcal{F}}|$.

Proof of Lemma 3

By definition $\frac{1}{\nu_k}\|\mathbf{f}_k\|_{L^2(\Omega)} = \frac{\lambda_k \nu_k}{\nu^2}\|\mathbf{f}\|_{L^2(\Omega)} \to 0$ as $k \to \infty$. Therefore,

$$\Big|\frac{1}{\nu_k}\int_{\Omega_{i\bar{h}_k}(t)} \mathbf{f}_k \cdot \mathbf{u}_k dx\Big| \le \varepsilon \ \textit{for sufficiently large } k.$$

Using inequalities

$$\int_{\Gamma_h} |\widehat{\Phi}_k|^2 ds \le \sigma^2, \quad h \in (0, \delta_\sigma], \quad k \ge k',$$

and

$$\int_{\Gamma_{\bar{h}_k}} |\widehat{\mathbf{u}}_k|^2 ds \le C(\varepsilon)\nu_k^2,$$

(see (3), (5)), we obtain

$$\int_{S_{ik}(t)} |\nabla \widehat{\Phi}_k| \, ds \leq t\mathcal{F} + 2 + \sigma\sqrt{C(\varepsilon)} - \int_{\Omega_{i\bar{h}_k}(t)} \widehat{\omega}_k^2 \, dx$$
$$\leq t\mathcal{F} + 2\varepsilon + \sigma\sqrt{C(\varepsilon)} - \int_{V_{i+1}\setminus\Omega_{\delta_i}} \widehat{\omega}_k^2 \, dx,$$

where $C(\varepsilon)$ is independent of k and σ.
By Lemma 2,

$$\int_{V_{i+1}\setminus\Omega_{\delta_i}} \widehat{\omega}_k^2 dx > \varepsilon_i \quad \forall k \geq k_i', \quad \delta_i \in (0, \delta_0).$$

Choosing $\varepsilon = \frac{1}{6}\varepsilon_i$, $\sigma = \frac{1}{3\sqrt{C(\varepsilon)}}\varepsilon_i$, and a sufficiently large k, we obtain $2\varepsilon + \sigma\sqrt{C(\varepsilon)} - \int_{V_{i+1}\setminus\Omega_{\delta_i}} \omega_k^2 \, dx \leq 0$.
Therefore,

$$\int_{S_{ik}(t)} |\nabla \widehat{\Phi}_k| \, ds \leq t\mathcal{F}.$$

□

The case (b).

(b) *The maximum of* Φ *is not attained on the boundary* $\partial\Omega$:

$$\max\{p_0, p_1\} < \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x).$$

We do not exclude the case $\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = +\infty$.

We can assume that

$$\max\{p_0, p_1\} < 0 < \operatorname*{ess\,sup}_{x\in\Omega} \Phi(x).$$

Denote $\sigma = \max\{p_0, p_1\} < 0$.

Lemma 4. *There exists* $F \in T_\psi$ *such that* $diamF > 0$*,* $F \cap \partial\Omega = \emptyset$ *and* $\Phi(F) > \sigma$*.*

Fix F and consider the behavior of Φ on the arcs $[B_0, F]$ and $[B_1, F]$. All other considerations are similar to above. The role of B_1 is played now by F.

Axially symmetric 3D - case