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Galerkin-Characteristics Finite Element Methods for Flow Problems, I

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- Convergence analysis of

① the scheme for the convection-diffusion equation
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④ 2^{nd} order schemes in time and a stabilized scheme for the Navier-Stokes equations

Convection-dominated phenomenon

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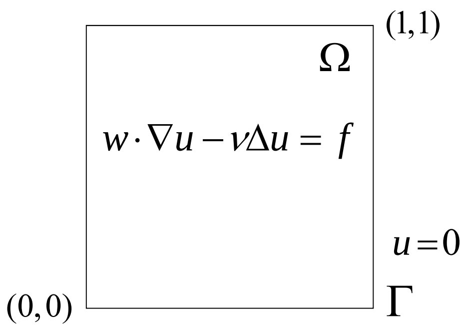
An example of convection-diffusion problem

Find $u : \Omega \to \mathbf{R}$ such that



$w : \Omega \to \Re^2$:given function

$\nabla \cdot w = 0$

$a(u,v) \equiv \int_\Omega \left(w \cdot \nabla u + \nabla u \cdot \nabla v \right) dx$

where $w = (1,0), \; f = 1$ i.e., $\frac{\partial u}{\partial x_1} - \nu \Delta u = 1$, i.e., $u(x_1, x_2) \approx x_1$

$\nu = 0.1, \; 0.01, \; 0.001 \quad \mathrm{Pe} = 10, \; 100, \; 1000$

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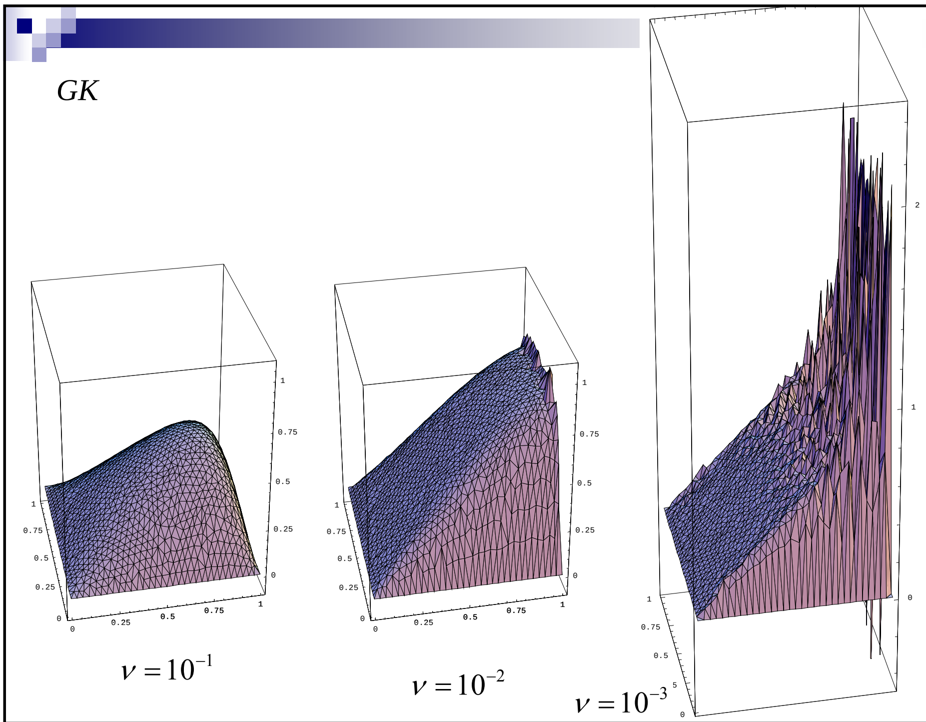
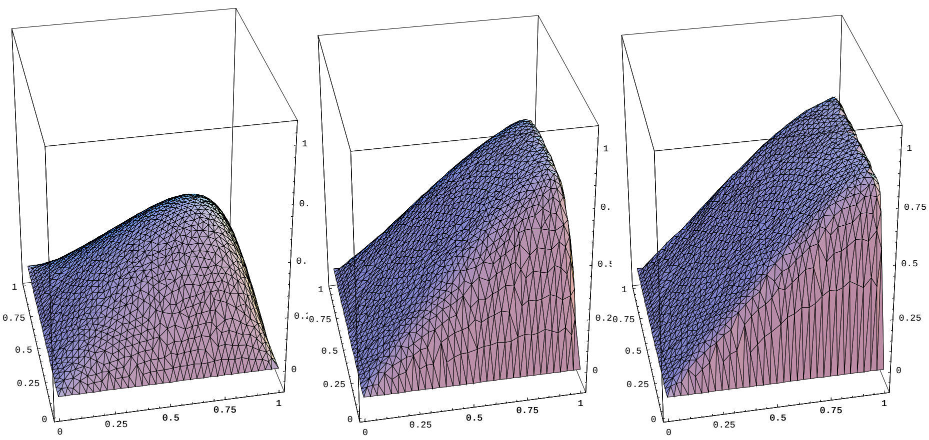


$\nu = 10^{-1}$ $\nu = 10^{-2}$ $\nu = 10^{-3}$

T[1977], Pironneau-T[2010]



$\nu = 10^{-1}$ $\nu = 10^{-2}$ $\nu = 10^{-3}$

Finite Element Schemes for Flow Problems

Upwind approximation:

upwind element choice approximation, T[1977]

mass-conservative upwind approximation, Baba-T[1981]

streamline upwind Petrov/Galerkin approximation

Brooks-Hughes[1982], Johnson, Franca,...

dicscontinuous Galerkin method

Lesaint-Raviart[1974], Johson, Brezzi-Marini-Sulli,...

Characteristic method:

first order in Δt

Pironneau[1982], Douglas-Russel[1982], Suli[1988]

monotone: Pironneau-T[2010], stabilized: Notsu-T[2013]

second order in Δt

multi-step method: Ewing-Russel[1981]

Boukir-Maday-Metivet-Razafindrakoto[1997]

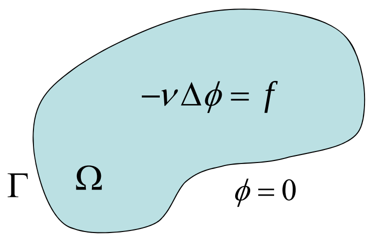
single-step method: Rui-T[2002], Notsu-T[2009]

Galerkin-Characteristics FEM Method

Galerkin Method

Find $\phi : \Omega \to \Re$ s.t.

$$-\nu \Delta \phi = f \quad (x \in \Omega)$$
$$\phi = 0 \quad (x \in \partial\Omega)$$



$$V_h \equiv \{\phi_j ; j = 1, \cdots, N(h)\} \subset V \equiv H_0^1(\Omega)$$

Find $\phi_h \in V_h$ s.t.

$$\nu \int_\Omega \nabla \phi_h \cdot \nabla \psi_h \, dx = \int_\Omega f \psi_h \, dx \quad (\forall \psi_h \in V_h)$$

Note. Ritz-Galerkin method, Petrov-Galerkin method

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Method of Characteristics

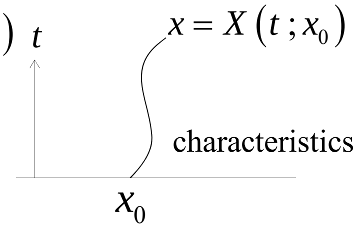
Find $\phi : \Re^d \times (0,T) \to \Re$,

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = f \quad (x \in \Re^d, t > 0), \quad \phi(\cdot, 0) = \phi^0$$

where $u : \Re^d \times (0,T) \to \Re^d$, $f : \Re^d \times (0,T) \to \Re$, $\phi^0 : \Re^d \to \Re$.

Find $(X, \Phi) : (0,T) \to \Re^d \times \Re$,

$$\frac{dX}{dt} = u(X,t), \quad \frac{d\Phi}{dt} = f(X,t) \quad (t > 0)$$
$$X(0) = x_0, \quad \Phi(0) = \phi^0(x_0)$$



$$\Rightarrow \quad \phi(X(t;x_0), t) = \Phi(t; x_0)$$

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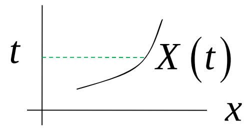
Approximation of the Material Derivative

Material derivative: $\dfrac{D\phi}{Dt} \equiv \dfrac{\partial \phi}{\partial t} + u \cdot \nabla \phi$

Position of a fluid particle: $X : (0,T) \to \mathbf{R}^d$

$$\frac{dX}{dt}(t) = u(X(t),t) \ , \quad t \in (0,T)$$



$$\Rightarrow \qquad \frac{D\phi}{Dt}\left(X(t),t\right) = \frac{d}{dt}\phi(X(t),t)$$

- Approximation of material derivative:

$$\frac{D\phi}{Dt}(X(t),t) \cong \frac{\phi(X(t),t) - \phi(X(t-\Delta t),t-\Delta t)}{\Delta t}$$

Idea of the Galerkin-Characteristics FEM

$$\frac{D\phi}{Dt} \equiv \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi$$

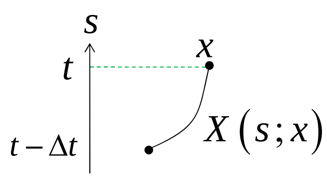
$\Rightarrow$

$$\left(\frac{D\phi}{Dt}, \psi\right) \cong \left(\frac{\phi - \phi(X(t-\Delta t;x),t-\Delta t)}{\Delta t}, \psi\right) + \nu\left(\nabla \phi, \nabla \psi\right)$$

where

$$\frac{dX}{ds} = u\left(X, s\right) \quad \left(s < t\right),$$

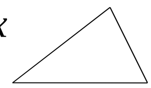
$$X(t;x) = x$$



The pros and cons

- The matrices are symmetric and independent of time step for convection-diffusion, Oseen, and Navier-Stokes problems; positive definite for CD.
- The schemes are unconditionally stable and convergent for CD and Os, conditionally stable and convergent for NS.
- The schemes are robust for convection-dominated problems. c of L^2-estimate is independent of ν for CD.
- The attention should be paid for the computation of composite function terms, $\int_\Omega \phi_h^{n-1} \circ X_1^n \, \psi_{hi} \, dx$

$\phi_h^{n-1}\left(x - \Delta t \, u^n(x)\right)\psi_{hi}(x)$ is not smooth on K!



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I. Convection-diffusion equation

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Convection-Diffusion Equation

$\Omega \subset \mathbf{R}^d \ (d = 2,3)$, bounded

$T > 0$

Find $\phi : \Omega \times (0,T) \to \mathbf{R}$ such that

$$\begin{aligned} &\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi = f && \text{in } \Omega \times (0,T) \\ &\phi = 0 && \text{on } \Gamma \times (0,T) \\ &\phi = \phi^0 && \text{at } t = 0 \ \text{ in } \ \Omega \end{aligned}$$

where

$$u \in W_0^{1,\infty}(\Omega)^d, \ f \in L^2(\Omega), \ \phi^0 \in L^2(\Omega)$$

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Weak Formulation

$V \equiv H_0^1(\Omega)$

Find $\phi : (0,T) \to V$ such that

$$\begin{aligned} &\left(\frac{D\phi}{Dt}, \psi \right) + a(\phi, \psi) = (f, \psi), \quad \forall \psi \in V \\ &\phi(0) = \phi^0 \end{aligned}$$

where

$$\frac{D\phi}{Dt} \equiv \frac{D^{(u)}\phi}{Dt} \equiv \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi \ : \text{material derivative}$$

$$a(\phi, \psi) \equiv \nu (\nabla \phi, \nabla \psi)$$

$$(f, g) \equiv \int_\Omega f g \, dx$$

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1^{st} order scheme in time

Δt : time increment, $N_T \equiv \lfloor T/\Delta t \rfloor$

$V_h \subset V$: FE-space

Find $\phi_h^n \in V_h, n = 1, \ldots, N_T$, such that

$$\left(\frac{\phi_h^n - \phi_h^{n-1} \circ X^n}{\Delta t}, \psi_h \right) + a(\phi_h^n, \psi_h) = (f_h^n, \psi_h), \quad \forall \psi_h \in V_h$$

$$\phi_h^0 = \Pi_h^P \phi^0$$

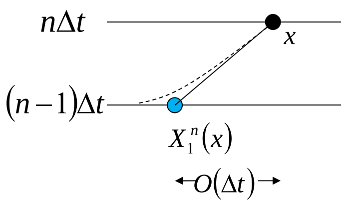
where $X^n(x) \equiv x - u^n(x)\Delta t$

Note. Backward Euler method

Note. $\left(\phi_h^{n-1} \circ X_1^n\right)(x) = \phi_h^{n-1}\left(x - u^n(x)\Delta t\right)$

composite function

Note. Π_h^P : Poisson projection



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1^{st} order scheme in time (cont.)

<u>Theorem</u> V_h : Poisson projection of order k

$(P_k\text{-element})$

$$\Delta t < \frac{1}{\|u\|_{W^{1,\infty}}}$$

$\Rightarrow$

$\exists c(\phi, u, T) > 0$

$$\|\phi_h - \phi\|_{\ell^\infty(L^2)}, \ \left\|\sqrt{\nu}\nabla(\phi_h - \phi)\right\|_{\ell^2(L^2)} \le c\left(\Delta t + h^k\right)$$

where

$$\|\phi_h\|_{\ell^\infty(X)} = \max\left\{ \|\phi_h^n\|_X ; n = 0, \ldots, N_T \right\},$$

$$\|\phi_h\|_{\ell^2(X)} = \left\{ \Delta t \sum_{n=1}^{N_T} \|\phi_h^n\|_X^2 \right\}^{1/2}$$

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Framework for the convection-diffusion equation

Hilbert space.

$$V = H_0^1(\Omega),\quad \Omega: \text{bdd.} \subset \Re^d,\ d = 2,3$$

FEM space. $V_h \subset V,\ Q_h \subset Q$

$\{\mathcal{T}_h\}_{h\downarrow 0}$: regular, inverse ineq.

$$\exists \sigma > 0,\ \forall h, \forall K \in \mathcal{T}_h,\ \text{diam}(K) \le \sigma\rho(K)$$

$\rho(K)$: radius of the inscribe ball

$$\exists c_2, c_2 > 0,\ \forall h, \forall K \in \mathcal{T}_h,\ c_1 h \le \text{diam}(K) \le c_2 h$$

Bilinear form. $a : V \times V \to \Re$,

$$a(\phi,\psi) = \nu \int_\Omega \nabla\phi \cdot \nabla\psi\, dx$$

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Poisson projection

$$\Pi_h^P : V \to V_h,\quad \Pi_h^P \phi \equiv \hat{\phi}_h,$$

$$a(\hat{\phi}_h, \psi_h) = a(\phi, \psi_h),\ \forall \psi_h \in V_h$$

$$\exists c, k > 0,\quad \left\| \hat{\phi}_h - \phi \right\|_V \le c \|\phi\|_{H^{k+1}} h^k$$

$$i.e.,\ \left\| I - \Pi_h^P \right\|_{\mathcal{L}(H^{k+1}, V)} \le ch^k$$

Ex. $V_h : P_k$-element, $k \ge 1$

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Plan of the proof

- Poisson projection
- Transformation $X^n : \Omega \to \Omega$
- Error equation in $e_h \equiv \phi_h - \Pi_h^P \phi$

$$\phi_h - \phi = e_h + \left(\Pi_h^P - I\right)\phi$$

- Discrete Gronwall's inequality

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Transformation $X^n : \Omega \to \Omega$

$$X^n(x) = x - u^n(x)\Delta t$$

<u>Lemma</u>

$$w \in W_0^{1,\infty}(\Omega)^d$$

$$X(x) = x - w(x)\Delta t$$

(1) $\|w\|_{W^{1,\infty}} \Delta t < 1 \quad \Rightarrow X : \Omega \to \Omega$, 1 to 1, onto

(2) $\forall \varepsilon, \exists \delta \in (0,1),\ \|w\|_{W^{1,\infty}} \Delta t \le \delta$

$$\Rightarrow \frac{1}{1+\varepsilon} \le \left|\frac{\partial X}{\partial x}\right| \le 1+\varepsilon$$

Ref. Rui-T[Prop. 1, 2002]

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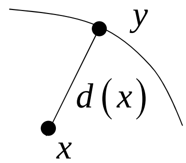
$\because$ (1) $\|w\|_{W^{1,\infty}} \Delta t < 1 \quad \Rightarrow X : \Omega \to \Omega$, 1 to 1, onto

$\forall x \in \Omega, \ \ d(x) \equiv \mathrm{dist}(x, \partial\Omega)$



$$X(x) - x = -w(x)\Delta t$$
$$= -\left(w(x) - w(y)\right)\Delta t$$

$$|X(x) - x| \le |w(x) - w(y)|\Delta t$$
$$\le |x-y||\nabla w(\xi)|\Delta t$$
$$\le |x-y|\|w\|_{W^{1,\infty}}\Delta t < d(x)$$

$$\therefore \quad X(x) \in \Omega$$

$\forall x \in \partial\Omega, \ \ X(x) \in \partial\Omega$

(2) $X_i(x) = x_i - w_i(x)\Delta t$

$$X_{i,j} = \delta_{ij} - w_{i,j}(x)\Delta t$$

$$\left|\frac{\partial X}{\partial x}\right| = \det\left(X_{i,j}\right) = \det\left(\delta_{ij} - w_{i,j}(x)\Delta t\right)$$

■

A tool for the proof

Lemma

$w \in W_0^{1,\infty}(\Omega)^d, \quad X(x) = x - w(x)\Delta t$

$\exists \delta \in (0,1), \ \|w\|_{W^{1,\infty}}\Delta t < \delta$

$\Rightarrow$

$\exists c(\delta) > 0, \ \ \|\phi \circ X^n\|_{L^2} \le (1 + c\Delta t)\|\phi\|_{L^2}$

$\because$

$$\|\phi \circ X^n\|_{L^2}^2 = \int_\Omega \left|\phi\left(X^n(x)\right)\right|^2 dx$$
$$= \int_\Omega |\phi(y)|^2 \left|\frac{\partial x}{\partial y}\right| dy \qquad \left(y = X^n(x)\right)$$
$$\le (1 + c\Delta t)\int_\Omega |\phi(y)|^2 dy$$
$$\le (1 + c\Delta t)\|\phi\|_{L^2}^2$$

■

Error equation in $e_h \equiv \phi_h - \hat{\phi}_h,\ \hat{\phi}_h \equiv \Pi_h^P \phi$

<u>Lemma</u>

$\left\{e_h^n\right\}_{n=1}^{N_T}$ satisfies

$$\left(\frac{e_h^n - e_h^{n-1}\circ X^n}{\Delta t}, \psi_h\right) + a\left(e_h^n, \psi_h\right) = \left(R_h^n, \psi_h\right),\quad \forall \psi_h \in V_h$$

$$e_h^0 = 0$$

where

$$R_h^n \equiv R_{h1}^n + R_{h2}^n$$

$$R_{h1}^n \equiv \frac{D\phi^n}{Dt} - \frac{\phi^n - \phi^{n-1}\circ X^n}{\Delta t},\quad R_{h2}^n \equiv \frac{\eta_h^n - \eta_h^{n-1}\circ X^n}{\Delta t},$$

$$\eta_h \equiv \phi - \hat{\phi}_h : \Omega\times[0,T] \to \Re$$

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$$\because\ \left(\frac{\phi_h^n - \phi_h^{n-1}\circ X^n}{\Delta t}, \psi_h\right) + a\left(\phi_h^n, \psi_h\right) = \left(f^n, \psi_h\right)$$

$$\left(\frac{D\phi^n}{Dt}, \psi_h\right) + a\left(\hat{\phi}^n, \psi_h\right) = \left(f^n, \psi_h\right)$$

$(\phi$: solution of CD, Poisson projection$)$

$$\left(\frac{e_h^n - e_h^{n-1}\circ X^n}{\Delta t}, \psi_h\right) + a\left(e_h^n, \psi_h\right) = \left(R_h^n, \psi_h\right)$$

$$R_h^n = \frac{D\phi^n}{Dt} - \frac{\hat{\phi}^n - \hat{\phi}^{n-1}\circ X^n}{\Delta t}$$

$$= \frac{D\phi^n}{Dt} - \frac{\phi^n - \phi^{n-1}\circ X^n}{\Delta t} \quad \left(R_{h1}^n : \text{truncation error}\right)$$

$$+ \frac{\eta_h^n - \eta_h^{n-1}\circ X^n}{\Delta t} \quad \left(R_{h2}^n : \text{projection error}\right)$$

$$e_h^0 = \phi_h^0 - \hat{\phi}_h^0 = \Pi_h^P \phi^0 - \hat{\phi}_h^0 = 0$$

■

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Estimates of the remainders

Lemma

(1) $\left\| R_{h1}^{n} \right\|_0 \le c(\phi,u)\Delta t$

(2) $\left\| R_{h2}^{n} \right\|_0 \le c(\phi,u) h^k$

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$\because$ $$R_{h1}^{n}(x) = \frac{D\phi^n}{Dt}(x) - \frac{\phi^n(x) - \phi^{n-1}\left(X^n(x)\right)}{\Delta t}$$

$$\frac{1}{\Delta t}\left\{\phi^n(x) - \phi^{n-1}\left(X^n(x)\right)\right\} = \frac{1}{\Delta t}\Big[\phi\big(x-(1-s)u^n(x)\Delta t, t_n - (1-s)\Delta t\big)\Big]_{s=0}^{1}$$

$$\left(\because X^n(x) \equiv x - u^n(x)\Delta t\right)$$

$$= \int_{s=0}^{1}\left\{u^n(x)\cdot\nabla\phi + \partial_t\phi\right\}\big(x-(1-s)u^n(x)\Delta t, t_n-(1-s)\Delta t\big)ds$$

$$R_{h1}^{n}(x) = \left(u\cdot\nabla\phi + \partial_t\phi\right)(x,t_n) - \int_{s=0}^{1}\left\{u^n(x)\cdot\nabla\phi + \partial_t\phi\right\}\big(x-(1-s)u^n(x)\Delta t, t_n-(1-s)\Delta t\big)ds$$

$$= \int_{s=0}^{1}\Big[\left\{u^n(x)\cdot\nabla\phi + \partial_t\phi\right\}\big(x-(1-s_1)(1-s)u^n(x)\Delta t, t_n-(1-s_1)(1-s)\Delta t\big)\Big]_{s_1=0}^{1}ds$$

$$= \Delta t\int_{s=0}^{1}(1-s)\,ds\int_{s_1=0}^{1}\left\{u_j^n(x)u_k^n(x)\phi_{,jk} + 2u_j^n(x)\phi_{,jt} + \phi_{,tt}\right\}\big(x-(1-s_1)(1-s)u^n(x)\Delta t, t_n-(1-s_1)(1-s)\Delta t\big)ds_1$$

$$\left\| R_{h1}^{n} \right\|_0 \le c(\phi,u)\Delta t$$

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$$\because \quad R_{h2}^n = \frac{\eta_h^n - \eta_h^{n-1} \circ X^n}{\Delta t}$$

$$R_{h2}^n(x) = \frac{1}{\Delta t}\Big[\eta_h\big(x-(1-s)u^n(x)\Delta t, t_n-(1-s)\Delta t\big)\Big]_{s=0}^1$$

$$= \int_{s=0}^1 \{u^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\}\big(x-(1-s)u^n(x)\Delta t, t_n-(1-s)\Delta t\big)ds$$

$$\left|R_{h2}^n(x)\right|^2 \le \int_{s=0}^1 \left|u^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right|^2\big(x-(1-s)u^n(x)\Delta t, t_n-(1-s)\Delta t\big)ds$$

$$\left\|R_{h2}^n\right\|_0 \le \left\{\int_\Omega dx\int_{s=0}^1 \left|u^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right|^2\big(x-(1-s)u^n(x)\Delta t, t_n-(1-s)\Delta t\big)ds\right\}^{1/2}$$

$$= \left\{\int_{s=0}^1 ds\int_\Omega \left|u^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right|^2\big(x-(1-s)u^n(x)\Delta t, t_n-(1-s)\Delta t\big)dx\right\}^{1/2}$$

$$= \left\{\int_{s=0}^1 ds\int_\Omega \left|u^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right|^2\big(y, t_n-(1-s)\Delta t\big)\left|\frac{\partial x}{\partial y}\right|dy\right\}^{1/2}$$

$$= c(u)\left\{\int_{t_{n-1}}^{t_n}\left(\|\nabla\eta_h\|_0^2 + \|\partial_t\eta_h\|_0^2\right)dt\right\}^{1/2} \qquad \begin{pmatrix}\because y = x-(1-s)u^n(x)\Delta t\\ \because t = t_n-(1-s)\Delta t\end{pmatrix}$$

$$\le c(\phi,u)h^k$$

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Discrete energy inequality in $\left\|e_h^n\right\|_0$

<u>Lemma</u>

$$\bar{D}_{\Delta t}\left\|e_h^n\right\|_0^2 + \nu\left\|\nabla e_h^n\right\|_0^2 \le \varepsilon\left\|e_h^n\right\|_0^2 + c_2(u)\left\|e_h^{n-1}\right\|_0^2 + c_2(\phi,u,\varepsilon)\left(h^{2k}+\Delta t^2\right)$$

$$e_h^0 = 0 \qquad\qquad (n=1,\cdots,N_T)$$

Proof.

$$\left(\frac{e_h^n - e_h^{n-1}\circ X^n}{\Delta t}, \psi_h\right) + a\left(e_h^n,\psi_h\right) = \left(R_h^n,\psi_h\right), \quad \forall\psi_h\in V_h$$

Substitute e_h^n into ψ_h.

$$\left(\frac{e_h^n - e_h^{n-1}\circ X^n}{\Delta t}, e_h^n\right) + a\left(e_h^n,e_h^n\right) = \left(R_h^n,e_h^n\right), \quad \forall n=1,\cdots,N_T$$

$$\frac{1}{2\Delta t}\left(\left\|e_h^n\right\|_0^2 - \left\|e_h^{n-1}\circ X^n\right\|_0^2\right) + \nu\left\|\nabla e_h^n\right\|_0^2 \le \left(\left\|R_{h1}^n\right\|_0 + \left\|R_{h2}^n\right\|_0\right)\left\|e_h^n\right\|_0$$

$$\bar{D}_{\Delta t}\left\|e_h^n\right\|_0^2 + 2\nu\left\|\nabla e_h^n\right\|_0^2 \le \varepsilon\left\|e_h^n\right\|_0^2 + c_1(u)\left\|e_h^{n-1}\right\|_0^2 + c_2(\phi,u,\varepsilon)\left(h^{2k}+\Delta t^2\right)$$

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Discrete Gronwall's inequality

$$\bar{D}_{\Delta t} x_n \equiv \frac{x_n - x_{n-1}}{\Delta t} : \text{backward difference}$$

<u>Lemma</u>

$$\{x_n\}_{n=0}^N, \{y_n\}_{n=1}^N, \{b_n\}_{n=1}^N, a_0, a_1, x_n, y_n, b_n \ge 0$$

$$\bar{D}_{\Delta t} x_n + y_n \le a_0 x_n + a_1 x_{n-1} + b_n, \quad n = 1, \cdots, N$$

$$\Delta t \in (0, \frac{1}{2a_0}]$$

$$\Rightarrow x_n + \Delta t \sum_{j=1}^n y_j \le e^{(2a_0 + a_1) n \Delta t} \left(x_0 + \Delta t \sum_{j=1}^n b_j \right), \quad n = 1, \cdots, N$$

Proof. $$\frac{x_n - x_{n-1}}{\Delta t} + y_n \le a_0 x_n + a_1 x_{n-1} + b_n, \quad n = 1, \cdots, N$$

$$(1 - a_0 \Delta t) x_n + y_n \Delta t \le (1 + a_1 \Delta t) x_{n-1} + b_n \Delta t, \quad n = 1, \cdots, N$$

The result is proved by induction. ■

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Proof of Theorem

Discrte energy inequality :

$$\bar{D}_{\Delta t} \|e_h^n\|_0^2 + \nu \|\nabla e_h^n\|_0^2 \le \varepsilon \|e_h^n\|_0^2 + c_2(u) \|e_h^{n-1}\|_0^2 + c_2(\phi, u, \varepsilon)(h^{2k} + \Delta t^2)$$

$$e_h^0 = 0 \qquad (n = 1, \cdots, N_T)$$

$$\forall \varepsilon > 0$$

Apply the discrte Gronwall's inequality for

$$\Delta t < \frac{1}{\|u\|_{W^{1,\infty}}} \le \frac{1}{2\varepsilon}$$

$$\|e_h\|_{l^\infty(L^2)}^2 + \nu \|\nabla e_h^n\|_{l^2(L^2)}^2 \le c(\phi, u, T)(h^{2k} + \Delta t^2)$$

$$\|e_h\|_{l^\infty(L^2)} + \sqrt{\nu} \|\nabla e_h^n\|_{l^2(L^2)} \le c(\phi, u, T)(h^k + \Delta t)$$

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Galerkin-Characteristics Finite Element Methods for Flow Problems, II

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II. Oseen Equations

Oseen Equations

$\Omega \subset \mathbf{R}^d \ (d = 2,3)$, bounded

$\Gamma = \partial\Omega \qquad T > 0$

Find $(u, p) : \Omega \times (0,T) \to \mathbf{R}^d \times \mathbf{R}$ such that

$$\frac{\partial u}{\partial t} + (w \cdot \nabla) u - \nu \Delta u + \nabla p = f \quad \text{in } \Omega \times (0,T)$$
$$\nabla \cdot u = 0 \quad \text{in } \Omega \times (0,T)$$
$$u = 0 \quad \text{on } \Gamma \times (0,T)$$
$$u = u^0 \quad \text{at } t = 0 \text{ in } \Omega$$

where

$$w \in W_0^{1,\infty}(\Omega)^d,\ f \in L^2(\Omega)^d,\ u^0 \in H_0^1(\Omega)^d, \nabla \cdot u^0 = 0$$

Weak formulation

$$V \equiv H_0^1(\Omega)^d,\ Q = L_0^2(\Omega)$$

Find $(u, p) : (0,T) \to V \times Q$ such that

$$\left(\frac{Du}{Dt}, v\right) + a(u,v) + b(v,p) = (f, v), \quad \forall v \in V$$
$$b(u,q) = 0, \quad \forall q \in Q$$
$$u(0) = u^0$$

where $\displaystyle \frac{Du}{Dt} \equiv \frac{D^{(w)}u}{Dt} \equiv \frac{\partial u}{\partial t} + (w \cdot \nabla) u$

$$a(u,v) \equiv 2\nu (D(u), D(v)),\ D_{ij}(v) \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$
$$b(u,q) \equiv -(\nabla \cdot u, q)$$

1^{st} order scheme in time

Δt : time increment, $N_T \equiv \lfloor T / \Delta t \rfloor$

$V_h \subset V,\ Q_h \subset Q$: FE-space

Find $\left(u_h^n, p_h^n\right) \in V_h \times Q_h,\ n = 1, \ldots, N_T$, such that

$$\left(\frac{u_h^n - u_h^{n-1} \circ X^n}{\Delta t}, v_h\right) + a\left(u_h^n, v_h\right) + b\left(v_h, p_h^n\right) = \left(f_h^n, v_h\right), \quad \forall v_h \in V_h$$

$$b\left(u_h^n, q_h\right) = 0, \qquad \forall q_h \in Q_h$$

$$u_h^0 = \left(\Pi_h^S\left(u^0, 0\right)\right)_1$$

where

$$X^n(x) \equiv x - w^n(x)\Delta t$$

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1^{st} order scheme in time (cont.)

<u>Theorem</u> V_h / Q_h : Stokes projection of order k

$$\Delta t < \frac{1}{\|w\|_{W^{1,\infty}}}$$

$\Rightarrow$

$\exists c\left(\nu, T, w, u, p\right) > 0,$

$$\left\|u_h - u\right\|_{\ell^\infty\left(H^1\right)}, \left\|p_h - p\right\|_{\ell^2\left(L^2\right)} \leq c\left(\Delta t + h^k\right)$$

where

$$\left\|v_h\right\|_{\ell^\infty(X)} = \max\left\{ \left\|v_h^n\right\|_X ; n = 0, \ldots, N_T \right\},$$

$$\left\|v_h\right\|_{\ell^2(X)} = \left\{\Delta t \sum_{n=1}^{N_T} \left\|v_h^n\right\|_X^2\right\}^{1/2}$$

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Framework for the Oseen equations

Hilbert spaces.

$$V = H_0^1(\Omega)^d,\ d = 2,3,\quad \Omega : \text{bdd.} \subset \Re^d$$
$$Q = L_0(\Omega) \equiv \left\{ q \in L^2(\Omega); \int_\Omega q(x)\,dx = 0 \right\}$$

FEM spaces. $V_h \subset V,\ Q_h \subset Q$

$\{\mathcal{T}_h\}_{h\downarrow 0}$: regular, inverse ineq.

$$\exists \sigma > 0,\ \forall h, \forall K \in \mathcal{T}_h,\ \mathrm{diam}(K) \le \sigma \rho(K)$$

$\rho(K)$: radius of the inscribe ball

$$\exists c_2, c_2 > 0,\ \forall h, \forall K \in \mathcal{T}_h,\ c_1 h \le \mathrm{diam}(K) \le c_2 h$$

Bilinear forms. $a : V \times V \to \Re,\ b : V \times Q \to \Re$

$$a(u,v) = 2\nu \int_\Omega D(u) : D(v)\,dx\ ,\ D_{ij}(v) = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$
$$b(v,q) = -\int_\Omega q \nabla\cdot v\ dx$$

(strain rate tensor)

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Stokes projection

$$\Pi_h^S : V \times Q \to V_h \times Q_h,\quad \Pi_h^S(u,p) \equiv (\hat{u}_h, \hat{p}_h)$$

$$a(\hat{u}_h, v_h) + b(v_h, \hat{p}_h) = a(u, v_h) + b(v_h, p),\ \forall v_h \in V_h$$
$$b(\hat{u}_h, q_h) = b(u, q_h),\ \forall q_h \in Q_h$$

$$\exists c, k > 0,\quad \left\| (\hat{u}_h - u, \hat{p}_h - p) \right\|_{V\times Q} \le c \left\| (u,p) \right\|_{H^{k+1}\times H^k} h^k$$
$$i.e.,\quad \left\| I - \Pi_h^S \right\|_{\mathcal{L}(H^{k+1}\times H^k, V\times Q)} \le c h^k$$

Ex. 1 $V_h / Q_h : P2/P1$, Taylor-Hood element, $k = 2$
Ex. 2 $V_h / Q_h : P1 + \text{bubble}/P1$, MINI element, $k = 1$
Note. $V_h / Q_h : P1/P0$ does not work.

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Stokes projection (cont.)

Key condition for the proof: Inf-sup condition.

$$\exists \beta_0 > 0, \quad \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|q_h\|_Q \|v_h\|_V} \geq \beta_0$$

Taylor-Hood element satisfies this condition.
MINI element satisfies this condition.

Reference.
[1] Brezzi, F., Fortin, M., Mixed and hypbrid finite element methods, Springer, New York, 1991.
[2] Tabata, M., Numerical analyis of partial differential equations, Iwanami, Tokyo, 2010.

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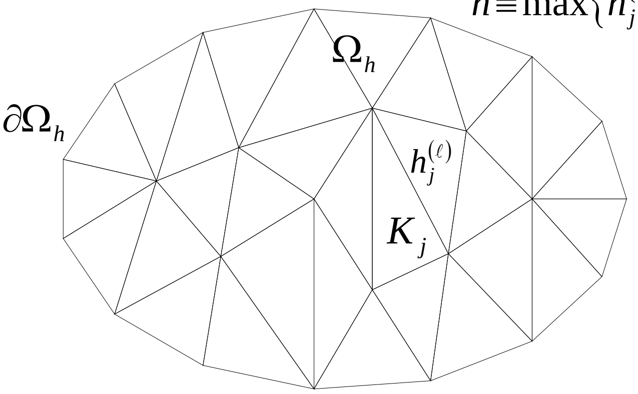
Stokes problem (d=2)

Find $(u_h, p_h) : \Omega_h \to \Re^2 \times \Re$

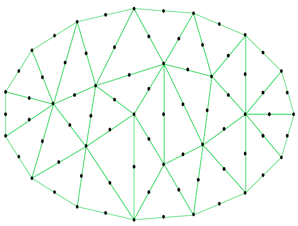
$$h \equiv \max\left\{ h_j^{(\ell)} ; \ell = 1,2,3, j \right\}$$



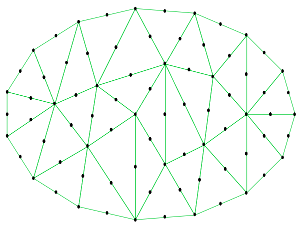
$(u_h, p_h) \in V_h \times Q_h$: finite element spaces

P2/P1 space $(d=2)$

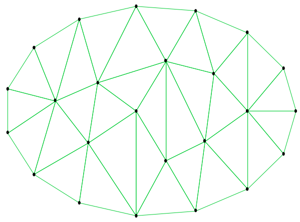
$((u_{h1}, u_{h2}), p_h) \in V_h \times Q_h$: finite element space



u_{h1}



u_{h2}



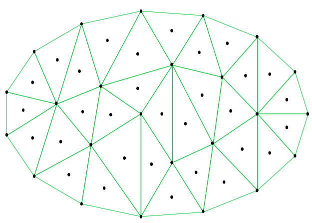
p_h

$u_{h1|K_j}, u_{h2|K_j} \in [1, x_1, x_2, x_1^2, x_1 x_2, x_2^2]$ $\quad p_{h|K_j} \in [1, x_1, x_2]$

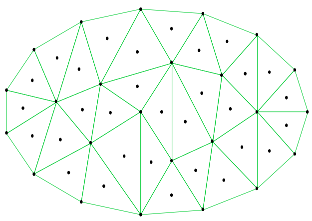
Note. Taylor-Hood element, $k=2$

P1+/P1 space $(d=2)$

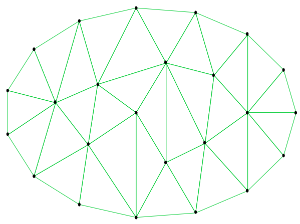
$((u_{h1}, u_{h2}), p_h) \in V_h \times Q_h$: finite element space



u_{h1}



u_{h2}



p_h

$u_{h1|K_j}, u_{h2|K_j} \in [1, x_1, x_2, \lambda_1 \lambda_2 \lambda_3]$ $\quad p_{h|K_j} \in [1, x_1, x_2]$

Note. MINI element, $k=1$; $\lambda_1 \lambda_2 \lambda_3$: bubble function

Plan of the proof

- Stokes projection
- Transformation $X^n : \Omega \to \Omega$
- Error equation in $(e_h, \varepsilon_h) \equiv (u_h, p_h) - \Pi_h^S (u, p)$

$$(u_h - u, p_h - p) = (e_h, \varepsilon_h) + (\Pi_h^S - I)(u, p)$$

- Discrete Gronwall's inequality

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Transformation $X^n : \Omega \to \Omega$

$$X^n(x) = x - w^n(x)\Delta t$$

<u>Lemma</u>

$$w \in W_0^{1,\infty}(\Omega)^d, \quad X(x) = x - w(x)\Delta t$$

(1) $\|w\|_{W^{1,\infty}} \Delta t < 1 \quad \Rightarrow X : \Omega \to \Omega$, 1 to 1, onto

(2) $\forall \varepsilon, \exists \delta \in (0,1), \ \|w\|_{W^{1,\infty}} \Delta t < \delta$

$$\Rightarrow \frac{1}{1+\varepsilon} \le \left| \frac{\partial X}{\partial x} \right| \le 1 + \varepsilon$$

$$\|\phi - \phi \circ X\|_0 \le (1+\varepsilon)\Delta t \|w\|_{L^\infty} \|\nabla \phi\|_0, \ \forall \phi \in H^1(\Omega)$$

$$\|\phi - \phi \circ X\|_0 \le \Delta t \|w\|_0 \|\nabla \phi\|_{L^\infty}, \ \forall \phi \in W^{1,\infty}(\Omega)$$

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Tools for the proof

- Korn's inequality

$$\exists c>0,\ \left\|D(v)\right\|_0+\left\|v\right\|_0\geq c\left\|v\right\|_1,\quad \forall v\in H^1(\Omega)^d$$

- Poincare's inequality

$$\exists c>0,\ \left\|v\right\|_0\leq c\left\|\nabla v\right\|_0,\quad \forall v\in H_0^1(\Omega)$$

$$\exists c_1,c_2>0,\quad c_1\left\|v\right\|_1\leq\left\|D(v)\right\|_0\leq c_2\left\|v\right\|_1,\quad \forall v\in H_0^1(\Omega)^d$$

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Error equation 1 in (e_h,ε_h)

$(e_h,\varepsilon_h)\equiv(u_h-\hat{u}_h,p_h-\hat{p}_h),\ (\hat{u}_h,\hat{p}_h)\equiv\Pi_h^S(u,p),$ satisfies

$$\left(\frac{e_h^n-e_h^{n-1}\circ X^n}{\Delta t},v_h\right)+a\left(e_h^n,v_h\right)+b\left(v_h,\varepsilon_h^n\right)=\left(R_h^n,v_h\right),\quad \forall v_h\in V_h$$

$$b\left(e_h^n,q_h\right)=0,\qquad \forall q_h\in Q_h,\quad n=1,\cdots,N_T$$

$$e_h^0=\left(\Pi_h^S\left(0,-p^0\right)\right)_1$$

where

$$R_h^n=R_{h1}^n+R_{h2}^n,$$

$$R_{h1}^n\equiv\frac{Du^n}{Dt}-\frac{u^n-u^{n-1}\circ X^n}{\Delta t},\qquad R_{h2}^n\equiv\frac{\eta_h^n-\eta_h^{n-1}\circ X^n}{\Delta t},$$

$$\eta_h\equiv u-\hat{u}_h:\Omega\times[0,T]\to\Re^d,\qquad X^n(x)\equiv x-\Delta t\ w^n(x)$$

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$$\because \quad \left(\frac{u_h^n - u_h^{n-1}\circ X^n}{\Delta t}, v_h\right) + a\left(u_h^n, v_h\right) + b\left(v_h, p_h^n\right) = \left(f^n, v_h\right)$$

$$\left(\frac{Du^n}{Dt}, v_h\right) + a\left(\hat{u}^n, v_h\right) + b\left(v_h, \hat{p}^n\right) = \left(f^n, v_h\right)$$

$\big((u,p)$: solution of NS, Stokes projection$\big)$

$$R_h^n = \frac{Du^n}{Dt} - \frac{\hat{u}^n - \hat{u}^{n-1}\circ X^n}{\Delta t}$$

$$= \frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X^n}{\Delta t} \quad \left(R_{h1}^n : \text{truncation error}\right)$$

$$+ \frac{\eta_h^n - \eta_h^{n-1}\circ X^n}{\Delta t} \quad \left(R_{h2}^n : \text{projection error}\right)$$

$$b\left(e_h^n, q_h\right) = b\left(u_h^n, q_h\right) - b\left(\hat{u}_h^n, q_h\right) = 0 - 0 = 0.$$

$$e_h^0 = u_h^0 - \hat{u}_h^0 = \left(\Pi_h^S\left(u^0, 0\right)\right)_1 - \left(\Pi_h^S\left(u^0, p^0\right)\right)_1 = \left(\Pi_h^S\left(0, -p^0\right)\right)_1 \quad \blacksquare$$

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Error equation 2 in $\left(e_h, \varepsilon_h\right)$

$\left(e_h, \varepsilon_h\right) \equiv \left(u_h - \hat{u}_h, p_h - \hat{p}_h\right)$, $\left(\hat{u}_h, \hat{p}_h\right) \equiv \Pi_h^S\left(u, p\right)$, satisfies

$$\left(\frac{e_h^n - e_h^{n-1}}{\Delta t}, v_h\right) + a\left(e_h^n, v_h\right) + b\left(v_h, \varepsilon_h^n\right) = \left(R_h^n + R_{h3}^n, v_h\right), \quad \forall v_h \in V_h$$

$$b\left(e_h^n, q_h\right) = 0, \qquad \forall q_h \in Q_h, \quad n = 1, \cdots, N_T$$

$$e_h^0 = \left(\Pi_h^S\left(0, -p^0\right)\right)_1$$

where

$$R_h^n = R_{h1}^n + R_{h2}^n, \qquad R_{h1}^n \equiv \frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X^n}{\Delta t},$$

$$R_{h2}^n \equiv \frac{\eta_h^n - \eta_h^{n-1}\circ X^n}{\Delta t}, \qquad R_{h3}^n \equiv -\frac{e_h^{n-1} - e_h^{n-1}\circ X^n}{\Delta t}$$

$$\eta_h \equiv u - \hat{u}_h : \Omega \times [0,T] \to \Re^d, \qquad X^n(x) \equiv x - \Delta t\, w^n(x)$$

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Estimates of the remainders

<u>Lemma</u>

$$\left\| R_{h1}^n \right\|_0 \le c(w,u)\Delta t$$

$$\left\| R_{h2}^n \right\|_0 \le c(\nu,w,u,p)h^k$$

$$\left\| R_{h3}^n \right\|_0 \le c(w)\left\| \nabla e_h^{n-1} \right\|_0$$

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$$\because \quad R_{h1}^n(x) = \frac{Du^n}{Dt}(x) - \frac{u^n(x) - u^{n-1}\left(X^n(x)\right)}{\Delta t}$$

$$\frac{1}{\Delta t}\left\{u^n(x) - u^{n-1}\left(X^n(x)\right)\right\} = \frac{1}{\Delta t}\Big[u\big(x-(1-s)w^n(x)\Delta t, t_n-(1-s)\Delta t\big)\Big]_{s=0}^1$$

$$\left(\because X^n(x) \equiv x - w^n(x)\Delta t\right)$$

$$= \int_{s=0}^1 \left\{w^n(x)\cdot\nabla u + \partial_t u\right\}\big(x-(1-s)w^n(x)\Delta t, t_n-(1-s)\Delta t\big)ds$$

$$R_{h1}^n(x) = \left(w\cdot\nabla u + \partial_t u\right)(x,t_n)$$

$$-\int_{s=0}^1 \left\{w^n(x)\cdot\nabla u + \partial_t u\right\}\big(x-(1-s)w^n(x)\Delta t, t_n-(1-s)\Delta t\big)ds$$

$$= \int_{s=0}^1 \Big[\left\{w^n(x)\cdot\nabla u + \partial_t u\right\}\big(x-(1-s_1)(1-s)w^n(x)\Delta t, t_n-(1-s_1)(1-s)\Delta t\big)\Big]_{s_1=0}^1 ds$$

$$= \Delta t\int_{s=0}^1 (1-s)ds\int_{s_1=0}^1 \left\{w_j^n(x)w_k^n(x)u_{,jk} + 2w_j^n(x)u_{,jt} + u_{,tt}\right\}$$
$$\big(x-(1-s_1)(1-s)w^n(x)\Delta t, t_n-(1-s_1)(1-s)\Delta t\big)ds_1$$

$$\left\| R_{h1}^n \right\|_0 \le c(w,u)\Delta t$$

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$$\because \quad R_{h2}^n = \frac{\eta_h^n - \eta_h^{n-1} \circ X^n}{\Delta t} \quad \left(R_{h2}^n : \text{projection error}\right)$$

$$R_{h2}^n(x) = \frac{1}{\Delta t}\Big[\eta_h\big(x-(1-s)w^n(x)\Delta t, t_n-(1-s)\Delta t\big)\Big]_{s=0}^1$$

$$= \int_{s=0}^1 \left\{w^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right\}\big(x-(1-s)w^n(x)\Delta t, t_n-(1-s)\Delta t\big)\,ds$$

$$\left|R_{h2}^n(x)\right|^2 \le \int_{s=0}^1 \left|w^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right|^2\big(x-(1-s)w^n(x)\Delta t, t_n-(1-s)\Delta t\big)\,ds$$

$$\left\|R_{h2}^n\right\|_0 \le \left\{\int_\Omega dx \int_{s=0}^1 \left|w^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right|^2\big(x-(1-s)w^n(x)\Delta t, t_n-(1-s)\Delta t\big)\,ds\right\}^{1/2}$$

$$= \left\{\int_{s=0}^1 ds \int_\Omega \left|w^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right|^2\big(x-(1-s)w^n(x)\Delta t, t_n-(1-s)\Delta t\big)\,dx\right\}^{1/2}$$

$$= \left\{\int_{s=0}^1 ds \int_\Omega \left|w^n(x)\cdot\nabla\eta_h + \partial_t\eta_h\right|^2\big(y, t_n-(1-s)\Delta t\big)\left|\frac{\partial x}{\partial y}\right| dy\right\}^{1/2}$$

$$= c(w)\left\{\int_{t_{n-1}}^{t_n}\left(\left\|\nabla\eta_h\right\|_0^2 + \left\|\partial_t\eta_h\right\|_0^2\right)dt\right\}^{1/2} \qquad \big(\because y = x-(1-s)w^n(x)\Delta t\big)$$

$$\big(\because t = t_n-(1-s)\Delta t\big)$$

$$\le c(\nu, w, u, p)h^k$$

$$\because \quad R_{h3}^n = -\frac{e_h^{n-1} - e_h^{n-1}\circ X^n}{\Delta t}, \qquad \left\|R_{h3}^n\right\|_0 \le c\left\|w^n\right\|_{L^\infty}\left\|\nabla e_h^{n-1}\right\|_0 \qquad \blacksquare$$

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Discrete energy inequality in $\left\|\sqrt{\nu}D\left(e_h^n\right)\right\|_0$

<u>Lemma</u>

$$\bar{D}_{\Delta t}\left\|\sqrt{\nu}D\left(e_h^n\right)\right\|_0^2 + \frac{1}{2}\left\|\bar{D}_{\Delta t}e_h^n\right\|_0^2$$

$$\le c_1(\nu, w)\left\|\sqrt{\nu}D\left(e_h^{n-1}\right)\right\|_0^2 + c_2(\nu, w, u, p)\left(h^{2k} + \Delta t^2\right)$$

$$(n = 1, \cdots, N_T)$$

$$\left\|\sqrt{\nu}D\left(e_h^0\right)\right\|_0 \le c\left\|p^0\right\|_k h^k$$

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Proof. Substitute $\bar{D}_{\Delta t} e_h^n$ into v_h in the error equation 2.

$$\left(\bar{D}_{\Delta t} e_h^n, \bar{D}_{\Delta t} e_h^n\right) + a\left(e_h^n, \bar{D}_{\Delta t} e_h^n\right) = \left(R_h^n + R_{h3}^n, \bar{D}_{\Delta t} e_h^n\right) \ , n \ge 1$$

$$\left(\because b\left(e_h^n, \varepsilon_h^n\right) = b\left(u_h^n, \varepsilon_h^n\right) - b\left(\hat{u}_h^n, \varepsilon_h^n\right) = 0 - 0 = 0, n \ge 0\right)$$

$$\left\|\bar{D}_{\Delta t} e_h^n\right\|_0^2 + \nu \bar{D}_{\Delta t} \left\|D\left(e_h^n\right)\right\|_0^2 \le \left(\left\|R_{h1}^n\right\|_0 + \left\|R_{h2}^n\right\|_0 + \left\|R_{h3}^n\right\|_0\right)\left\|\bar{D}_{\Delta t} e_h^n\right\|_0$$

$$\frac{1}{2}\left\|\bar{D}_{\Delta t} e_h^n\right\|_0^2 + \bar{D}_{\Delta t}\left\|\sqrt{\nu} D\left(e_h^n\right)\right\|_0^2 \le c_1(\nu, w)\left\|\sqrt{\nu} D\left(e_h^{n-1}\right)\right\|_0^2 + c_2(\nu, w, u, p)\left(h^{2k} + \Delta t^2\right)$$

$$e_h^0 = \left(\Pi_h^S\left(0, -p^0\right)\right)_1 = \left(\left(\Pi_h^S - I\right)\left(0, -p^0\right)\right)_1$$

$$\left\|\sqrt{\nu} e_h^0\right\|_1 \le c\left\|p^0\right\|_k h^k$$

$$\left\|\sqrt{\nu} D\left(e_h^0\right)\right\|_0 \le c\left\|p^0\right\|_k h^k$$

■

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Proof of Theorem

Discrte energy inequality :

$$\bar{D}_{\Delta t}\left\|\sqrt{\nu} D\left(e_h^n\right)\right\|_0^2 + \frac{1}{2}\left\|\bar{D}_{\Delta t} e_h^n\right\|_0^2$$

$$\le c_1(\nu, w)\left\|\sqrt{\nu} D\left(e_h^{n-1}\right)\right\|_0^2 + c_2(\nu, w, u, p)\left(h^{2k} + \Delta t^2\right) \quad \left(n = 1, \cdots, N_T\right)$$

$$\left\|\sqrt{\nu} D\left(e_h^0\right)\right\|_0 \le c(p) h^k$$

Apply the discrte Gronwall's inequality.

$$\left\|\sqrt{\nu} D\left(e_h\right)\right\|_{l^\infty(L^2)}^2 + \frac{1}{2}\left\|\bar{D}_{\Delta t} e_h^n\right\|_{l^2(L^2)}^2 \le c(\nu, w, u, p, T)\left(h^{2k} + \Delta t^2\right)$$

$$\left\|\sqrt{\nu} D\left(e_h\right)\right\|_{l^\infty(L^2)} + \frac{1}{2}\left\|\bar{D}_{\Delta t} e_h^n\right\|_{l^2(L^2)} \le c(\nu, w, u, p, T)\left(h^k + \Delta t\right)$$

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Estimate of the pressure.

$$\left\|\varepsilon_h^n\right\|_0 \le c\ \sup_{v_h\in V_h} \frac{b\left(v_h,\varepsilon_h^n\right)}{\left\|v_h\right\|_V}$$

$$= c\ \sup_{v_h\in V_h} \frac{1}{\left\|v_h\right\|_V}\left\{\left(R_h^n + R_{h3}^n, v_h\right) - \left(\bar{D}_{\Delta t}e_h^n, v_h\right) - a\left(e_h^n, v_h\right)\right\}$$

$$\le c\ \left\{\left\|R_h^n\right\|_0 + \left\|R_{h3}^n\right\|_0 + \left\|\bar{D}_{\Delta t}e_h^n\right\|_0 + 2\nu\left\|D\left(e_h^n\right)\right\|_0\right\}$$

$$\le c\left(\nu, w, u, p\right)\left\{h^k + \Delta t + \left\|\nabla e_h^{n-1}\right\|_0 + \left\|\bar{D}_{\Delta t}e_h^n\right\|_0 + 2\nu\left\|D\left(e_h^n\right)\right\|_0\right\}$$

$$\le c\left(\nu, w, u, p\right)\left\{h^k + \Delta t + \left\|\bar{D}_{\Delta t}e_h^n\right\|_0\right\}$$

$$\left\|\varepsilon_h\right\|_{\ell^2\left(L^2\right)} \le c\left(\nu, w, u, p, T\right)\left\{h^k + \Delta t + \left\|\bar{D}_{\Delta t}e_h^n\right\|_{\ell^2\left(L^2\right)}\right\}$$

$$\le c\left(\nu, w, u, p, T\right)\left(h^k + \Delta t\right) \qquad \blacksquare$$

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Galerkin-Characteristics Finite Element Methods for Flow Problems, III

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III. Navier-Stokes Equations

Navier-Stokes Equations

$\Omega \subset \mathbf{R}^d \ (d = 2,3)$, bounded

$\Gamma = \partial\Omega \qquad T > 0$

Find $(u, p): \Omega \times (0,T) \to \mathbf{R}^d \times \mathbf{R}$ such that

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f \quad \text{in } \Omega\times(0,T)$$
$$\nabla\cdot u = 0 \quad \text{in } \Omega\times(0,T)$$
$$u = 0 \quad \text{on } \Gamma\times(0,T)$$
$$u = u^0 \quad \text{at } t = 0 \text{ in } \Omega$$

where $f \in L^2(\Omega)^d,\ u^0 \in W_0^{1,\infty}(\Omega)^d, \nabla\cdot u^0 = 0$

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Weak formulation

$V \equiv H_0^1(\Omega)^d,\ Q = L_0^2(\Omega)$

Find $(u, p): (0,T) \to V\times Q$ such that

$$\left(\frac{Du}{Dt}, v\right) + a(u,v) + b(v,p) = (f,v), \quad \forall v \in V$$
$$b(u,q) = 0, \quad \forall q \in Q$$
$$u(0) = u^0$$

where $\dfrac{Du}{Dt} \equiv \dfrac{D^{(u)}u}{Dt},\quad \dfrac{D^{(w)}u}{Dt} \equiv \dfrac{\partial u}{\partial t} + (w\cdot\nabla)u$

$a(u,v) \equiv 2\nu(D(u), D(v)),\ D_{ij}(v) \equiv \dfrac{1}{2}\left(\dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i}\right)$

$b(u,q) \equiv -(\nabla\cdot u, q)$

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1^{st} order scheme in time

Δt : time increment, $N_T \equiv \lfloor T / \Delta t \rfloor$

$V_h \subset V,\ Q_h \subset Q$: FE-space

Find $\left(u_h^n, p_h^n\right) \in V_h \times Q_h,\ n = 1, \ldots, N_T$, such that

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_h^{n-1}}{\Delta t}, v_h\right) + a\left(u_h^n, v_h\right) + b\left(v_h, p_h^n\right) = \left(f_h^n, v_h\right), \quad \forall v_h \in V_h$$

$$b\left(u_h^n, q_h\right) = 0, \qquad \forall q_h \in Q_h$$

$$u_h^0 = \left(\Pi_h^S\left(u^0, 0\right)\right)_1$$

where

$$X_h^{n-1}(x) \equiv x - u_h^{n-1}(x)\Delta t$$

Note. Pironneau[1982]

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1^{st} order scheme in time (cont.)

<u>Theorem</u> V_h / Q_h : Stokes projection of order k

$$\Rightarrow \quad \exists h_0, c_0 > 0, \quad h \le h_0,\ \Delta t \le c_0 h^{d/4}$$

$$\exists c\left(\nu, T, u, p\right) > 0,$$

$$\left\|u_h - u\right\|_{\ell^\infty\left(H^1\right)}, \left\|p_h - p\right\|_{\ell^2\left(L^2\right)} \le c\left(\Delta t + h^k\right)$$

where

$$\left\|v_h\right\|_{\ell^\infty(X)} = \max\left\{ \left\|v_h^n\right\|_X ;\ n = 0, \ldots, N_T \right\},$$

$$\left\|v_h\right\|_{\ell^2(X)} = \left\{\Delta t \sum_{n=1}^{N_T} \left\|v_h^n\right\|_X^2\right\}^{1/2}$$

Note. Süli[1988]

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Framework for the Navier-Stokes equations

Hilbert spaces.

$$V = H_0^1(\Omega)^d, \ d = 2,3, \quad \Omega: \text{bdd.} \subset \Re^d$$

$$Q = L_0(\Omega) \equiv \left\{ q \in L^2(\Omega); \int_\Omega q(x)\,dx = 0 \right\}$$

FEM spaces. $V_h \subset V, \ Q_h \subset Q$

$\{\mathcal{T}_h\}_{h\downarrow 0}$: regular, inverse ineq.

$$\exists \sigma > 0, \ \forall h, \forall K \in \mathcal{T}_h, \ \text{diam}(K) \le \sigma \rho(K)$$

$\rho(K)$: radius of the inscribe ball

$$\exists c_2, c_2 > 0, \ \forall h, \forall K \in \mathcal{T}_h, \ c_1 h \le \text{diam}(K) \le c_2 h$$

Bilinear forms. $a: V \times V \to \Re, \ b: V \times Q \to \Re$

$$a(u,v) = 2\nu \int_\Omega D(u) : D(v)\,dx \ , \ D_{ij}(v) = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

$$b(v,q) = -\int_\Omega q \nabla\cdot v \ dx \qquad (\text{strain rate tensor})$$

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Stokes projection

$$\Pi_h^S : V \times Q \to V_h \times Q_h, \quad \Pi_h^S(u,p) \equiv (\hat{u}_h, \hat{p}_h)$$

$$a(\hat{u}_h, v_h) + b(v_h, \hat{p}_h) = a(u, v_h) + b(v_h, p), \ \forall v_h \in V_h$$

$$b(\hat{u}_h, q_h) = b(u, q_h), \ \forall q_h \in Q_h$$

$$\exists c, k > 0, \quad \left\|(\hat{u}_h - u, \hat{p}_h - p)\right\|_{V\times Q} \le c \left\|(u,p)\right\|_{H^{k+1}\times H^k} h^k$$

$$i.e., \quad \left\|I - \Pi_h^S\right\|_{\mathcal{L}(H^{k+1}\times H^k, V\times Q)} \le c h^k$$

Ex. 1 $V_h / Q_h : P2/P1$, Taylor-Hood element, $k = 2$

Ex. 2 $V_h / Q_h : P1 + \text{bubble} / P1$, MINI element, $k = 1$

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Plan of the proof

- Stokes projection
- Error equation in $(e_h, \varepsilon_h) \equiv (u_h, p_h) - \Pi_h^S(u, p)$

$$(u_h - u, p_h - p) = (e_h, \varepsilon_h) + \left(\Pi_h^S - I\right)(u, p)$$

- Discrete Gronwall's inequality
- Induction is employed to evaluate $\Delta t \left\| u_h^{n-1} \right\|_{W^{1,\infty}}$ and $\left\| u_h^{n-1} \right\|_{L^\infty}$ at each step.

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New terms to be evaluated

- The transformation depends on the function u_h^{n-1}.

$$X_h^{n-1}(x) = x - \Delta t \ u_h^{n-1}(x)$$

The estimate $\Delta t \left\| u_h^{n-1} \right\|_{W^{1,\infty}}$ is requied.

- u_h^{n-1} appears as coefficients in the error equation.

The estimate $\left\| u_h^{n-1} \right\|_{L^\infty}$ is requied.

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Tools for the proof

- Korn's inequality

$$\exists c>0,\ \left\|D(v)\right\|_0+\|v\|_0\ge c\|v\|_1,\quad \forall v\in H^1(\Omega)^d$$

- Poincare's inequality

$$\exists c>0,\ \|v\|_0\le c\|\nabla v\|_0,\quad \forall v\in H_0^1(\Omega)$$

$$\exists c_1,c_2>0,\quad c_1\|v\|_1\le\left\|D(v)\right\|_0\le c_2\|v\|_1,\quad \forall v\in H_0^1(\Omega)^d$$

- Sobolev's imbedding

$$\exists c>0,\ \|v\|_{L^q}\le c\|v\|_{W^{m,p}},\quad \forall v\in W^{m,p}(\Omega),\ q=\frac{dp}{d-mp}$$

- Inverse inequality $1\le p<q\le\infty$

$$\exists c(p,q)>0,\ \|v_h\|_{L^q}\le c\,h^{d/q-d/p}\|v_h\|_{L^p},\quad \forall v_h\in V_h$$

- Lagrang interpolation operator $\Pi_h:V\to V_h$

$$\exists c>0,\ \|v-\Pi_h v\|_V\le ch^k\|v\|_{H^{k+1}},\quad \forall v\in H^{k+1}(\Omega)$$

$$\|\Pi_h v\|_{L^\infty}\le\|v\|_{L^\infty},\quad \forall v\in L^\infty(\Omega)$$

$$\exists c>0,\ \|\Pi_h v\|_{W^{1,\infty}}\le c\|v\|_{W^{1,\infty}},\quad \forall v\in W^{1,\infty}(\Omega),$$

Error equation1 in (e_h,ε_h)

$(e_h,\varepsilon_h)\equiv(u_h-\hat{u}_h,p_h-\hat{p}_h),\ (\hat{u}_h,\hat{p}_h)\equiv\Pi_h^S(u,p)$, satisfies

$$\left(\frac{e_h^n-e_h^{n-1}\circ X_h^{n-1}}{\Delta t},v_h\right)+a(e_h^n,v_h)+b(v_h,\varepsilon_h^n)=(R_h^n,v_h),\quad \forall v_h\in V_h$$

$$b(e_h^n,q_h)=0,\qquad \forall q_h\in Q_h,\quad n=1,\cdots,N_T$$

$$e_h^0=\left(\Pi_h^S(0,-p^0)\right)_1$$

where

$$R_h^n=R_{h1}^n+R_{h2}^n+R_{h3}^n,\qquad R_{h1}^n\equiv\frac{Du^n}{Dt}-\frac{u^n-u^{n-1}\circ X^{n-1}}{\Delta t}$$

$$R_{h2}^n\equiv\frac{\eta_h^n-\eta_h^{n-1}\circ X_h^{n-1}}{\Delta t},\quad R_{h3}^n\equiv\frac{u^{n-1}\circ X_h^{n-1}-u^{n-1}\circ X^{n-1}}{\Delta t}$$

$$\eta_h\equiv u-\hat{u}_h:\Omega\times[0,T]\to\Re^d,\quad X^{n-1}(x)\equiv x-\Delta t\,u^{n-1}(x)$$

$$\because \quad \left(\frac{u_h^n - u_h^{n-1} \circ X_h^{n-1}}{\Delta t}, v_h \right) + a\left(u_h^n, v_h\right) + b\left(v_h, p_h^n\right) = \left(f^n, v_h\right)$$

$$\left(\frac{Du^n}{Dt}, v_h \right) + a\left(\hat{u}^n, v_h\right) + b\left(v_h, \hat{p}^n\right) = \left(f^n, v_h\right)$$

$\left((u, p) : \text{solution of NS, Stokes projection}\right)$

$$R_h^n = \frac{Du^n}{Dt} - \frac{\hat{u}^n - \hat{u}^{n-1} \circ X_h^{n-1}}{\Delta t}$$

$$= \frac{Du^n}{Dt} - \frac{u^n - u^{n-1} \circ X^{n-1}}{\Delta t} \quad \left(R_{h1}^n : \text{truncation error}\right)$$

$$+ \frac{\eta_h^n - \eta_h^{n-1} \circ X_h^{n-1}}{\Delta t} \quad \left(R_{h2}^n : \text{projection error}\right)$$

$$+ \frac{u^{n-1} \circ X_h^{n-1} - u^{n-1} \circ X^{n-1}}{\Delta t} \quad \left(R_{h3}^n : \text{perturbation error}\right)$$

$b\left(e_h^n, q_h\right) = b\left(u_h^n, q_h\right) - b\left(\hat{u}_h^n, q_h\right) = 0 - 0 = 0.$

$$e_h^0 = u_h^0 - \hat{u}_h^0 = \left(\Pi_h^S\left(u^0, 0\right)\right)_1 - \left(\Pi_h^S\left(u^0, p^0\right)\right)_1 = \left(\Pi_h^S\left(0, -p^0\right)\right)_1 \quad \blacksquare$$

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Error equation2 in (e_h, ε_h)

$(e_h, \varepsilon_h) \equiv (u_h - \hat{u}_h, p_h - \hat{p}_h)$, $(\hat{u}_h, \hat{p}_h) \equiv \Pi_h^S(u, p)$, satisfies

$$\left(\frac{e_h^n - e_h^{n-1}}{\Delta t}, v_h \right) + a\left(e_h^n, v_h\right) + b\left(v_h, \varepsilon_h^n\right) = \left(R_h^n + R_{h4}^n, v_h\right), \quad \forall v_h \in V_h$$

$$b\left(e_h^n, q_h\right) = 0, \qquad \forall q_h \in Q_h, \quad n = 1, \cdots, N_T$$

$$e_h^0 = \left(\Pi_h^S\left(0, -p^0\right)\right)_1$$

where

$$R_h^n = R_{h1}^n + R_{h2}^n + R_{h3}^n, \qquad R_{h1}^n \equiv \frac{Du^n}{Dt} - \frac{u^n - u^{n-1} \circ X^{n-1}}{\Delta t}$$

$$R_{h2}^n \equiv \frac{\eta_h^n - \eta_h^{n-1} \circ X_h^{n-1}}{\Delta t}, \qquad R_{h3}^n \equiv \frac{u^{n-1} \circ X_h^{n-1} - u^{n-1} \circ X^{n-1}}{\Delta t}$$

$$R_{h4}^n \equiv -\frac{e_h^{n-1} - e_h^{n-1} \circ X_h^{n-1}}{\Delta t}$$

$$\eta_h \equiv u - \hat{u}_h : \Omega \times [0, T] \to \Re^d, \quad X^{n-1}(x) \equiv x - \Delta t \, u^{n-1}(x)$$

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Estimates of the remainders

Lemma

$$\left\|u_h^{n-1}\right\|_{W^{1,\infty}} \Delta t \le \delta < 1$$

$\Rightarrow$

$$\left\|R_{h1}^n\right\|_0 \le c(u)\Delta t$$

$$\left\|R_{h2}^n\right\|_0 \le c(\nu,u,p)\left(\left\|u_h^{n-1}\right\|_{L^\infty}+1\right)h^k$$

$$\left\|R_{h3}^n\right\|_0 \le c(u,p)\left\{\left\|e_h^{n-1}\right\|_0 + h^k\right\}$$

$$\left\|R_{h4}^n\right\|_0 \le c\left\|u_h^{n-1}\right\|_{L^\infty}\left\|\nabla e_h^{n-1}\right\|_0$$

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$\because$

$$R_{h1}^n(x) = \frac{Du^n}{Dt}(x) - \frac{u^n(x)-u^{n-1}\left(X^{n-1}(x)\right)}{\Delta t}$$

$$\frac{1}{\Delta t}\left\{u^n(x)-u^{n-1}\left(X^{n-1}(x)\right)\right\} = \frac{1}{\Delta t}\Big[u\left(x-(1-s)u^{n-1}(x)\Delta t, t_n-(1-s)\Delta t\right)\Big]_{s=0}^1$$

$$\left(\because X^{n-1}(x) \equiv x - u^{n-1}(x)\Delta t\right)$$

$$= \int_{s=0}^1 \left\{u^{n-1}(x)\cdot\nabla u + \partial_t u\right\}\left(x-(1-s)u^{n-1}(x)\Delta t, t_n-(1-s)\Delta t\right)ds$$

$$R_{h1}^n(x) \equiv R_{h11}^n(x) + R_{h12}^n(x),\quad R_{h1}^n(x) \equiv \left(u^n(x)-u^{n-1}(x)\right)\cdot\nabla u(x,t_n)$$

$$R_{h12}^n(x) = \left(u^{n-1}(x)\cdot\nabla u + \partial_t u\right)(x,t_n)$$

$$-\int_{s=0}^1 \left\{u^{n-1}(x)\cdot\nabla u + \partial_t u\right\}\left(x-(1-s)u^{n-1}(x)\Delta t, t_n-(1-s)\Delta t\right)ds$$

$$= \int_{s=0}^1 \Big[\left\{u^{n-1}(x)\cdot\nabla u + \partial_t u\right\}\left(x-(1-s_1)(1-s)u^{n-1}(x)\Delta t, t_n-(1-s_1)(1-s)\Delta t\right)\Big]_{s_1=0}^1 ds$$

$$= \Delta t\int_{s=0}^1 (1-s)\,ds\int_{s_1=0}^1 \left\{u_j^{n-1}(x)u_k^{n-1}(x)u_{,jk} + 2u_j^{n-1}(x)u_{,jt} + u_{,tt}\right\}$$

$$\left(x-(1-s_1)(1-s)u^{n-1}(x)\Delta t, t_n-(1-s_1)(1-s)\Delta t\right)ds_1$$

$$\left\|R_{h12}^n\right\|_0 \le c(u)\Delta t$$

$$\left\|R_{h11}^n\right\|_0 \le c(u)\Delta t$$

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$$\because \quad R_{h2}^n = \frac{\eta_h^n - \eta_h^{n-1} \circ X_h^{n-1}}{\Delta t} \quad \left(R_{h2}^n : \text{projection error} \right)$$

$$R_{h2}^n(x) = \frac{1}{\Delta t} \left[\eta_h \left(x - (1-s) u_h^{n-1}(x) \Delta t, t_n - (1-s) \Delta t \right) \right]_{s=0}^1$$

$$= \int_{s=0}^1 \left\{ u_h^{n-1}(x) \cdot \nabla \eta_h + \partial_t \eta_h \right\} \left(x - (1-s) u_h^{n-1}(x) \Delta t, t_n - (1-s) \Delta t \right) ds$$

$$\left| R_{h2}^n(x) \right|^2 \le \int_{s=0}^1 \left| u_h^{n-1}(x) \cdot \nabla \eta_h + \partial_t \eta_h \right|^2 \left(x - (1-s) u_h^{n-1}(x) \Delta t, t_n - (1-s) \Delta t \right) ds$$

$$\left\| R_{h2}^n \right\|_0 \le \left\{ \int_\Omega dx \int_{s=0}^1 \left| u_h^{n-1}(x) \cdot \nabla \eta_h + \partial_t \eta_h \right|^2 \left(x - (1-s) u_h^{n-1}(x) \Delta t, t_n - (1-s) \Delta t \right) ds \right\}^{1/2}$$

$$= \left\{ \int_{s=0}^1 ds \int_\Omega \left| u_h^{n-1}(x) \cdot \nabla \eta_h + \partial_t \eta_h \right|^2 \left(x - (1-s) u_h^{n-1}(x) \Delta t, t_n - (1-s) \Delta t \right) dx \right\}^{1/2}$$

$$= \left\{ \int_{s=0}^1 ds \int_\Omega \left| u_h^{n-1}(x) \cdot \nabla \eta_h + \partial_t \eta_h \right|^2 \left(y, t_n - (1-s) \Delta t \right) \left| \frac{\partial x}{\partial y} \right| dy \right\}^{1/2}$$

$$\le c \left\| u_h^{n-1} \right\|_{L^\infty} \left\{ \int_{t_{n-1}}^{t_n} \left(\left\| \nabla \eta_h \right\|_0^2 + \left\| \partial_t \eta_h \right\|_0^2 \right) dt \right\}^{1/2} \quad \left(\because y = x - (1-s) u_h^{n-1}(x) \Delta t \right) \quad \left(\because t = t_n - (1-s) \Delta t \right)$$

$$\le c(\nu, u, p) \left(\left\| u_h^{n-1} \right\|_{L^\infty} + 1 \right) h^k \quad \left(\because \partial_t \eta_h(t) = \left(I - \Pi_h^S \right) \left(\partial_t u, \partial_t p \right)_1 (t) \right)$$

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$$\because \quad R_{h3}^n = \frac{u^{n-1} \circ X_h^{n-1} - u^{n-1} \circ X^{n-1}}{\Delta t} \quad \left(R_{h3}^n : \text{perturbation error} \right)$$

$$\left\| R_{h3}^n \right\|_0 \le c \left\| \nabla u^{n-1} \right\|_{L^\infty} \left\| u^{n-1} - u_h^{n-1} \right\|_0$$

$$\le c \left\| \nabla u^{n-1} \right\|_{L^\infty} \left(\left\| u^{n-1} - \hat{u}_h^{n-1} \right\|_0 + \left\| \hat{u}_h^{n-1} - u_h^{n-1} \right\|_0 \right)$$

$$\le c(u, p) \left(h^k + \left\| e_h^{n-1} \right\|_0 \right)$$

$$R_{h4}^n = -\frac{e_h^{n-1} - e_h^{n-1} \circ X_h^n}{\Delta t} \quad \left(R_{h4}^n : \text{convection term} \right)$$

$$\left\| R_{h4}^n \right\|_0 \le c \left\| u_h^{n-1} \right\|_{L^\infty} \left\| \nabla e_h^{n-1} \right\|_0$$

■

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Discrete energy inequality in $\left\|\sqrt{\nu}D\left(e_h^n\right)\right\|_0$

Lemma

(1) $\Delta t\left\|u_h^{n-1}\right\|_{W^{1,\infty}} \le \delta < 1,\ ,\ 1 \le n \le N_T$

$\Rightarrow$

$$\bar{D}_{\Delta t}\left\|\sqrt{\nu}D\left(e_h^n\right)\right\|_0^2 + \frac{1}{2}\left\|\bar{D}_{\Delta t}e_h^n\right\|_0^2$$

$$\le c_1\left(\nu,u,p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left\|\sqrt{\nu}D\left(e_h^{n-1}\right)\right\|_0^2 + c_2\left(\nu,u,p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left(h^{2k}+\Delta t^2\right)$$

(2) $\left\|\sqrt{\nu}D\left(e_h^0\right)\right\|_0 \le c_3(p)h^k$

Proof of (1). Error equation2:

$$\left(\frac{e_h^n - e_h^{n-1}}{\Delta t}, v_h\right) + a\left(e_h^n, v_h\right) + b\left(v_h, \varepsilon_h^n\right) = \left(R_h^n + R_{h4}^n, v_h\right),\quad \forall v_h \in V_h$$

$$b\left(e_h^n, q_h\right) = 0, \qquad \forall q_h \in Q_h,\quad n = 1,\cdots,N_T$$

Substitute $\bar{D}_{\Delta t}e_h^n$ into v_h.

$$\left(\bar{D}_{\Delta t}e_h^n, \bar{D}_{\Delta t}e_h^n\right) + a\left(e_h^n, \bar{D}_{\Delta t}e_h^n\right) = \left(R_h^n + R_{h4}^n, \bar{D}_{\Delta t}e_h^n\right)\ ,\ n \ge 1$$

$$\left(\because b\left(e_h^n, \varepsilon_h^n\right) = b\left(u_h^n, \varepsilon_h^n\right) - b\left(\hat{u}_h^n, \varepsilon_h^n\right) = 0 - 0 = 0, n \ge 0\right)$$

$$\left\|\bar{D}_{\Delta t}e_h^n\right\|_0^2 + \nu\bar{D}_{\Delta t}\left\|D\left(e_h^n\right)\right\|_0^2 \le \left(\left\|R_{h1}^n\right\|_0 + \left\|R_{h2}^n\right\|_0 + \left\|R_{h3}^n\right\|_0 + \left\|R_{h4}^n\right\|_0\right)\left\|\bar{D}_{\Delta t}e_h^n\right\|_0$$

$$\frac{1}{2}\left\|\bar{D}_{\Delta t}e_h^n\right\|_0^2 + \bar{D}_{\Delta t}\left\|\sqrt{\nu}D\left(e_h^n\right)\right\|_0^2 \le \left\|R_{h1}^n\right\|_0^2 + \left\|R_{h2}^n\right\|_0^2 + \left\|R_{h3}^n\right\|_0^2 + \left\|R_{h4}^n\right\|_0^2$$

$$\frac{1}{2}\left\|\bar{D}_{\Delta t}e_h^n\right\|_0^2 + \bar{D}_{\Delta t}\left\|\sqrt{\nu}D\left(e_h^n\right)\right\|_0^2 \le \left\|R_{h1}^n\right\|_0^2 + \left\|R_{h2}^n\right\|_0^2 + \left\|R_{h3}^n\right\|_0^2 + \left\|R_{h4}^n\right\|_0^2$$

$$\le c(u)\Delta t^2 + c(\nu,u,p)\left(\left\|u_h^{n-1}\right\|_{L^\infty}^2 + 1\right)h^{2k} + c(u,p)\left\{\left\|e_h^{n-1}\right\|_0^2 + h^{2k}\right\}$$

$$+c\left\|u_h^{n-1}\right\|_{L^\infty}^2 \left\|\nabla e_h^{n-1}\right\|_0^2$$

$$\le c_1\left(u,p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left\|D\left(e_h^{n-1}\right)\right\|_0^2 + c_2\left(\nu,u,p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left(h^{2k}+\Delta t^2\right)$$

$$\left(\because \left\|e_h^{n-1}\right\|_0 \le c\left\|\nabla e_h^{n-1}\right\|_0 \le c'\left\|D\left(e_h^{n-1}\right)\right\|_0\right)$$

$$\le c_1\left(\nu,u,p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left\|\sqrt{\nu}D\left(e_h^{n-1}\right)\right\|_0^2 + c_2\left(\nu,u,p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left(h^{2k}+\Delta t^2\right)$$

$$(2)\ e_h^0 = \left(\Pi_h^S\left(0,-p^0\right)\right)_1 = \left(\left(\Pi_h^S - I\right)\left(0,-p^0\right)\right)_1$$

$$\left\|\sqrt{\nu}e_h^0\right\|_1 \le c\left\|p^0\right\|_{H^k} h^k\ ,\ \left\|\sqrt{\nu}D\left(e_h^0\right)\right\|_0 \le c\left\|p^0\right\|_k h^k$$

■

Proof of Theorem

From the discrete energy inequality,

$\Delta t\left\|u_h^{n-1}\right\|_{W^{1,\infty}} \le \delta < 1,\ ,\ 1 \le n \le N_T$ implies

$$\bar{D}_{\Delta t}\left\|e_h^n\right\|_1^2 \le c_1\left(\left\|u_h^{n-1}\right\|_{L^\infty};\nu,u,p\right)\left\|e_h^{n-1}\right\|_1^2 + c_2\left(\left\|u_h^{n-1}\right\|_{L^\infty};\nu,u,p\right)\left(h^{2k}+\Delta t^2\right)$$

where c_1, c_2 :monotone increasing w.r.t. the first variable.

Choose small $h_0, c_0 > 0.\quad h \le h_0, \Delta t \le c_0 h^{d/4}$.

We show the following by induction, $n = 0,\cdots,N_T$:

$(1)_n\ \left\|e_h^n\right\|_1^2 \le \exp\left(c_1\left(\|u\|_{L^\infty(L^\infty)}+1\right)n\Delta t\right)\left\{\left\|e_h^0\right\|_1^2 + c_2\left(\|u\|_{L^\infty(L^\infty)}+1\right)\left(h^{2k}+\Delta t^2\right)n\Delta t\right\}$

$(2)_n\ \Delta t\left\|u_h^n\right\|_{W^{1,\infty}} \le \delta$

$(3)_n\ \left\|u_h^n\right\|_{L^\infty} \le \|u\|_{L^\infty(L^\infty)} + 1$

Initial step, $n=0$.

$(1)_0$ LHS=RHS

$(2)_0$ $\Delta t \left\| u_h^0 \right\|_{W^{1,\infty}} \le \delta$. Proved similarly to that of $(2)_n$.

$(3)_0$ $\left\| u_h^0 \right\|_{L^\infty} \le \left\| u \right\|_{L^\infty(L^\infty)} + 1$. Proved similarly to that of $(3)_n$.

General step, $n \ge 1$. Assume $(1)_{n-1}, (2)_{n-1}$ and $(3)_{n-1}$ are valid.

$\Delta t \left\| u_h^{n-1} \right\|_{W^{1,\infty}} \le \delta$ implies the existence of $\left(u_h^n, p_h^n\right)$.

$(1)_n$ From the discrete energy inequality and $(3)_{n-1}$, we obtain

$$\bar{D}_{\Delta t} \left\| e_h^n \right\|_1^2 \le c_1 \left(\left\| u_h^{n-1} \right\|_{L^\infty}; \nu, u, p \right) \left\| e_h^{n-1} \right\|_1^2 + c_2 \left(\left\| u_h^{n-1} \right\|_{L^\infty}; \nu, u, p \right) \left(h^{2k} + \Delta t^2 \right)$$
$$\le c_1 \left(\left\| u \right\|_{L^\infty} + 1 \right) \left\| e_h^{n-1} \right\|_1^2 + c_2 \left(\left\| u \right\|_{L^\infty} + 1 \right) \left(h^{2k} + \Delta t^2 \right).$$

Applying the discrete Gronwall's inequality, we obtain

$$\left\| e_h^n \right\|_1^2 \le \exp\left(c_1 \left(\left\| u \right\|_{L^\infty(L^\infty)} + 1 \right) n\Delta t \right) \left\{ \left\| e_h^0 \right\|_1^2 + c_2 \left(\left\| u \right\|_{L^\infty(L^\infty)} + 1 \right) \left(h^{2k} + \Delta t^2 \right) n\Delta t \right\}.$$

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$(2)_n$

$$\left\| u_h^n \right\|_{W^{1,\infty}} \le \left\| u_h^n - \Pi_h u^n \right\|_{W^{1,\infty}} + \left\| \Pi_h u^n \right\|_{W^{1,\infty}}$$
$$\le c_{20} h^{-d/2} \left\| u_h^n - \Pi_h u^n \right\|_{H^1} + c_{21} \left\| u^n \right\|_{W^{1,\infty}}$$
$$\le c_{20} h^{-d/2} \left(\left\| u_h^n - \hat{u}_h^n \right\|_1 + \left\| \hat{u}_h^n - u^n \right\|_1 + \left\| u^n - \Pi_h u^n \right\|_1 \right) + c_{21} \left\| u^n \right\|_{W^{1,\infty}}$$
$$\le c_{20} h^{-d/2} \left(c_3 \left(\Delta t + h^k \right) + c_{22} h^k + c_{23} h^k \right) + c_{21} \left\| u^n \right\|_{W^{1,\infty}}$$
$$\le c_{24} h^{-d/2} \left(\Delta t + h^k \right) + c_{21} \left\| u^n \right\|_{W^{1,\infty}}$$

$$\Delta t \left\| u_h^n \right\|_{W^{1,\infty}} \le \Delta t \left\{ c_{24} h^{-d/2} \left(\Delta t + h^k \right) + c_{21} \left\| u^n \right\|_{W^{1,\infty}} \right\}$$
$$\le c_{24} \left(h^{-d/2} \Delta t^2 + h^{k-d/2} \Delta t \right) + c_{21} \Delta t \left\| u^n \right\|_{W^{1,\infty}}$$
$$\le c_0 \left(c_{24} c_0 + c_{24} h^{k-d/4} + c_{21} h^{d/4} \left\| u^n \right\|_{W^{1,\infty}} \right)$$
$$\le c_0 \left(c_{24} c_0 + c_{24} h_0^{k-d/4} + c_{21} h_0^{d/4} \left\| u^n \right\|_{W^{1,\infty}} \right)$$
$$\le \delta$$

$\Delta t \le c_0 h^{d/4}$

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$$(3)_n \quad \left\|u_h^n\right\|_{L^\infty} \le \left\|u_h^n - \Pi_h u^n\right\|_{L^\infty} + \left\|\Pi_h u^n\right\|_{L^\infty}$$
$$\le c_{30} h^{-d/6} \left\|u_h^n - \Pi_h u^n\right\|_{L^6} + \left\|u^n\right\|_{L^\infty}$$
$$\le c_{30} h^{-d/6} \left(\left\|u_h^n - \hat{u}_h^n\right\|_1 + \left\|\hat{u}_h^n - u^n\right\|_1 + \left\|u^n - \Pi_h u^n\right\|_1 \right) + \left\|u^n\right\|_{L^\infty}$$
$$\le c_{30} h^{-d/6} \left(c_3 \left(\Delta t + h^k\right) + c_{31} h^k + c_{32} h^k \right) + \left\|u^n\right\|_{L^\infty}$$
$$\le c_{34} \left(h^{-d/6} \Delta t + h^{k-d/6} \right) + \left\|u^n\right\|_{L^\infty}$$
$$\le c_{34} \left(c_0 h^{d/12} + h^{k-d/6} \right) + \left\|u^n\right\|_{L^\infty} \qquad \Delta t \le c_0 h^{d/4}$$
$$\le c_{34} \left(c_0 h_0^{d/12} + h_0^{k-d/6} \right) + \left\|u^n\right\|_{L^\infty}$$
$$\le 1 + \left\|u^n\right\|_{L^\infty}$$

The induction, $n = 0, \cdots, N_T$,

$$(1)_n \quad \left\|e_h^n\right\|_1^2 \le \exp\left(c_1 \left(\|u\|_{L^\infty(L^\infty)} + 1 \right) n\Delta t \right) \left\{ \left\|e_h^0\right\|_1^2 + c_2 \left(\|u\|_{L^\infty(L^\infty)} + 1 \right) \left(h^{2k} + \Delta t^2 \right) n\Delta t \right\}$$
$$(2)_n \quad \Delta t \left\|u_h^n\right\|_{W^{1,\infty}} \le \delta$$
$$(3)_n \quad \left\|u_h^n\right\|_{L^\infty} \le \|u\|_{L^\infty(L^\infty)} + 1$$

has been completed.

$$\left\|e_h^n\right\|_1 \le c_3 \left(\nu, T, u, p\right) \left(h^k + \Delta t\right), \ \forall n$$
$$\left\|e_h\right\|_{\ell^\infty(H^1)} \le c_3 \left(\nu, T, u, p\right) \left(h^k + \Delta t\right).$$

- Estimate of $\left\|\bar{D}_{\Delta t} e_h^n\right\|_0$

Now the discrete energy inequaliy

$$\bar{D}_{\Delta t}\left\|\sqrt{\nu} D\left(e_h^n\right)\right\|_0^2+\frac{1}{2}\left\|\bar{D}_{\Delta t} e_h^n\right\|_0^2$$
$$\leq c_1\left(\nu, u, p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left\|\sqrt{\nu} D\left(e_h^{n-1}\right)\right\|_0^2+c_2\left(\nu, u, p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left(h^{2k}+\Delta t^2\right)$$

can be written as

$$\bar{D}_{\Delta t}\left\|\sqrt{\nu} D\left(e_h^n\right)\right\|_0^2+\frac{1}{2}\left\|\bar{D}_{\Delta t} e_h^n\right\|_0^2$$
$$\leq c_4(\nu, u, p, T)\left\|\sqrt{\nu} D\left(e_h^{n-1}\right)\right\|_0^2+c_5(\nu, u, p, T)\left(h^{2k}+\Delta t^2\right).$$

Applying the discrete Gronwall's inequaliy, we obtain

$$\left\|\bar{D}_{\Delta t} e_h\right\|_{\ell^2\left(L^2\right)} \leq c_6(\nu, u, p, T)\left(h^k+\Delta t\right).$$

- Estimate of the pressure.

$$\left\|\varepsilon_h^n\right\|_0 \leq c \sup_{v_h \in V_h} \frac{b\left(v_h, \varepsilon_h^n\right)}{\left\|v_h\right\|_V}$$
$$=c \sup_{v_h \in V_h} \frac{1}{\left\|v_h\right\|_V}\left\{\left(R_h^n+R_{h4}^n, v_h\right)-\left(\bar{D}_{\Delta t} e_h^n, v_h\right)-a\left(e_h^n, v_h\right)\right\}$$
$$\leq c\left\{\left\|R_h^n\right\|_0+\left\|R_{h4}^n\right\|_0+\left\|\bar{D}_{\Delta t} e_h^n\right\|_0+2\nu\left\|D\left(e_h^n\right)\right\|_0\right\}$$
$$\leq c\left(\nu, u, p,\left\|u_h^{n-1}\right\|_{L^\infty}\right)\left\{h^k+\Delta t+\left\|\nabla e_h^{n-1}\right\|_0+\left\|\bar{D}_{\Delta t} e_h^n\right\|_0+2\nu\left\|D\left(e_h^n\right)\right\|_0\right\}$$
$$\leq c(\nu, u, p)\left\{h^k+\Delta t+\left\|\bar{D}_{\Delta t} e_h^n\right\|_0\right\}$$
$$\left\|\varepsilon_h\right\|_{\ell^2\left(L^2\right)} \leq c(\nu, u, p, T)\left\{h^k+\Delta t+\left\|\bar{D}_{\Delta t} e_h^n\right\|_{\ell^2\left(L^2\right)}\right\}$$
$$\leq c(\nu, u, p, T)\left(h^k+\Delta t\right) \quad \blacksquare$$

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Galerkin-Characteristics Finite Element Methods for Flow Problems, IV

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IV. 2nd Order Schemes in Time and a Stabilized Scheme

2^{nd} Order Schemes in Time

2^{nd} order Galerkin-characteristics schemes

- 2 step scheme

$$\frac{3f(t)-4f(t-\Delta t)+f(t-2\Delta t)}{\Delta t}-\frac{df}{dt}(t)=O(\Delta t^2)$$

$$u_h^n \leftarrow u_h^{n-1},\ u_h^{n-2} \qquad (n=2,\cdots,N_T)$$

$u_h^1,\ u_h^0$: initial functions

- Single step scheme

$$\frac{f(t)-f(t-\Delta t)}{\Delta t}-\frac{df}{dt}\left(t-\frac{\Delta t}{2}\right)=O(\Delta t^2)$$

$$u_h^n \leftarrow u_h^{n-1} \qquad (n=2,\cdots,N_T)$$

u_h^0 : initial function

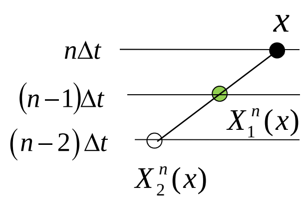
Note. $\frac{f(t)-f(t-\Delta t)}{\Delta t}-\frac{df}{dt}(t)=O(\Delta t)$

Idea of 2nd order two-step scheme

$$\frac{3\phi^n - 4\phi^{n-1}\circ X_1^n + \phi^{n-2}\circ X_2^n}{2\Delta t}(x) = \left(\frac{\partial \phi}{\partial t} + (u\cdot\nabla)\phi\right)^n (x) + O\left(\Delta t^2\right)$$

where

$$X_1^n(x) \equiv x - u^n(x)\Delta t$$
$$X_2^n(x) \equiv x - 2u^n(x)\Delta t$$



or

$$X_1^n(x) \equiv x - u^*(x)\Delta t$$
$$X_2^n(x) \equiv x - 2u^*(x)\Delta t$$
$$u^*(x) \equiv 2u^{n-1}(x) - u^{n-2}(x)$$

Note. $u^*(x) = u^n(x) + O(\Delta t^2)$

Ewing-Russell[1981]

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Navier-Stokes Equations

$\Omega \subset \mathbf{R}^d \ (d = 2,3)$, bounded

$\Gamma = \partial\Omega \qquad T > 0$

Find $(u, p): \Omega\times(0,T) \to \mathbf{R}^d \times \mathbf{R}$ such that

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = f \quad \text{in } \Omega\times(0,T)$$
$$\nabla\cdot u = 0 \quad \text{in } \Omega\times(0,T)$$
$$u = 0 \quad \text{on } \Gamma\times(0,T)$$
$$u = u^0 \quad \text{at } t = 0 \text{ in } \Omega$$

where $f \in L^2(\Omega)^d,\ u^0 \in W_0^{1,\infty}(\Omega)^d, \nabla\cdot u^0 = 0$

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Weak formulation

$$V \equiv H_0^1(\Omega)^d,\ Q = L_0^2(\Omega)$$

Find $(u,p):(0,T) \to V \times Q$ such that

$$\left(\frac{Du}{Dt}, v\right) + a(u,v) + b(v,p) = (f,v), \quad \forall v \in V$$
$$b(u,q) = 0, \quad \forall q \in Q$$
$$u(0) = u^0$$

where $\dfrac{Du}{Dt} \equiv \dfrac{D^{(u)}u}{Dt}, \quad \dfrac{D^{(w)}u}{Dt} \equiv \dfrac{\partial u}{\partial t} + (w \cdot \nabla)u$

$$a(u,v) \equiv 2\nu(D(u), D(v)), \quad D_{ij}(v) \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

$$b(u,q) \equiv -(\nabla \cdot u, q)$$

2^{nd} order two-step scheme

Δt : time increment, $N_T \equiv \lfloor T/\Delta t \rfloor$

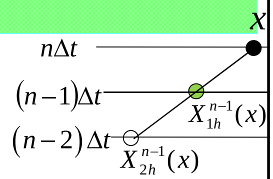
$V_h \subset V,\ Q_h \subset Q$

Find $(u_h^n, p_h^n) \in V_h \times Q_h$, $n = 2, \ldots, N_T$, such that

$$\left(\frac{3u_h^n - 4u_h^{n-1} \circ X_{1h}^{n-1} + u_h^{n-2} \circ X_{2h}^{n-1}}{2\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n) = (f_h^n, v_h), \quad \forall v_h \in V_h$$
$$b(u_h^n, q_h) = 0, \quad \forall q_h \in Q_h$$
$$u_h^1 : *$$
$$u_h^0 = \left(\Pi_h^S(u^0, 0)\right)_1$$

where

$$X_{1h}^{n-1}(x) \equiv x - u_h^*(x)\Delta t, \quad u_h^* \equiv 2u_h^{n-1} - u_h^{n-2}$$
$$X_{2h}^{n-1}(x) \equiv x - 2u_h^*(x)\Delta t$$



Note. $u_h^1 = u^1 + O(\Delta t^2 + h^k)$ should be found by another method.

2^{nd} order two-step scheme (cont.)

<u>Theorem</u> V_h / Q_h : Stokes projection of order k

$$\Rightarrow \begin{array}{l} \exists h_0, c_0 > 0, \quad h \le h_0, \ \Delta t \le c_0 h^{d/6} \\ \exists c(\nu, T, u, p) > 0, \end{array}$$

$$\|u_h - u\|_{\ell^\infty(H^1)}, \|p_h - p\|_{\ell^2(L^2)} \le c\left(\Delta t^2 + h^k\right)$$

where

$$\|v_h\|_{\ell^\infty(X)} = \max\left\{ \|v_h^n\|_X ; n = 0, \dots, N_T \right\},$$

$$\|v_h\|_{\ell^2(X)} = \left\{\Delta t \sum_{n=1}^{N_T} \|v_h^n\|_X^2 \right\}^{1/2}$$

Note. Boukir et al.[1997]

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Error equation1 in (e_h, ε_h)

$(e_h, \varepsilon_h) \equiv (u_h - \hat{u}_h, p_h - \hat{p}_h)$, $(\hat{u}_h, \hat{p}_h) \equiv \Pi_h^S(u, p)$, satisfies

$$\left(\frac{3e_h^n - 4e_h^{n-1} \circ X_{1h}^{n-1} + e_h^{n-2} \circ X_{2h}^{n-1}}{\Delta t}, v_h\right) + a\left(e_h^n, v_h\right) + b\left(v_h, \varepsilon_h^n\right) = \left(R_h^n, v_h\right), \quad \forall v_h \in V_h$$

$$b\left(e_h^n, q_h\right) = 0, \qquad \forall q_h \in Q_h, \quad n = 1, \cdots, N_T$$

$$e_h^1 = O\left(\Delta t^2 + h^k\right)$$

$$e_h^0 = \left(\Pi_h^S\left(0, -p^0\right)\right)_1$$

where $R_h^n = R_{h1}^n + R_{h2}^n + R_{h3}^n$,

$$R_{h1}^n \equiv= \frac{Du^n}{Dt} - \frac{3u^n - 4u^{n-1} \circ X_1^{n-1} + u^{n-2} \circ X_2^{n-1}}{\Delta t} \qquad R_{h2}^n \equiv \frac{3\eta_h^n - 4\eta_h^{n-1} \circ X_{1h}^{n-1} + \eta_h^{n-2} \circ X_{2h}^{n-1}}{\Delta t}$$

$$R_{h3}^n \equiv 4\frac{u^{n-1} \circ X_{1h}^{n-1} - u^{n-1} \circ X_1^{n-1}}{\Delta t} - \frac{u^{n-2} \circ X_{2h}^{n-1} - u^{n-2} \circ X_2^{n-1}}{\Delta t}$$

$\eta_h \equiv u - \hat{u}_h : \Omega \times [0, T] \to \Re^d$,

$$X_1^{n-1}(x) \equiv x - u^*(x)\Delta t, \ u^* \equiv 2u^{n-1} - u^{n-2}$$
$$X_2^{n-1}(x) \equiv x - 2u^*(x)\Delta t$$

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$$\left(\frac{3u_h^n - 4u_h^{n-1}\circ X_{1h}^{n-1} + u_h^{n-2}\circ X_{2h}^{n-1}}{\Delta t}, v_h\right) + a\left(u_h^n, v_h\right) + b\left(v_h, p_h^n\right) = \left(f^n, v_h\right)$$

$$\left(\frac{Du^n}{Dt}, v_h\right) + a\left(\hat{u}^n, v_h\right) + b\left(v_h, \hat{p}^n\right) = \left(f^n, v_h\right)$$

$\left((u,p)\text{: solution of NS, Stokes projection}\right)$

$$R_h^n = \frac{Du^n}{Dt} - \frac{3\hat{u}^n - 4\hat{u}^{n-1}\circ X_{1h}^{n-1} + \hat{u}^{n-2}\circ X_{2h}^{n-1}}{\Delta t}$$

$$= \frac{Du^n}{Dt} - \frac{3u^n - 4u^{n-1}\circ X_1^{n-1} + u^{n-2}\circ X_2^{n-1}}{\Delta t} \quad \left(R_{h1}^n\text{: truncation error}\right)$$

$$+ \frac{3\eta_h^n - 4\eta_h^{n-1}\circ X_{1h}^{n-1} + \eta_h^{n-2}\circ X_{2h}^{n-1}}{\Delta t} \quad \left(R_{h2}^n\text{: projection error}\right)$$

$$+4\frac{u^{n-1}\circ X_{1h}^{n-1} - u^{n-1}\circ X_1^{n-1}}{\Delta t} - \frac{u^{n-2}\circ X_{2h}^{n-1} - u^{n-2}\circ X_2^{n-1}}{\Delta t} \quad \left(R_{h3}^n\text{: perturbation error}\right)$$

$$b\left(e_h^n, q_h\right) = b\left(u_h^n, q_h\right) - b\left(\hat{u}_h^n, q_h\right) = 0 - 0 = 0.$$

$$e_h^0 = u_h^0 - \hat{u}_h^0 = \left(\Pi_h^S\left(u^0, 0\right)\right)_1 - \left(\Pi_h^S\left(u^0, p^0\right)\right)_1 = \left(\Pi_h^S\left(0, -p^0\right)\right)_1 \quad \blacksquare$$

Error equation2 in (e_h, ε_h)

$(e_h, \varepsilon_h) \equiv (u_h - \hat{u}_h, p_h - \hat{p}_h)$, $(\hat{u}_h, \hat{p}_h) \equiv \Pi_h^S(u, p)$, satisfies

$$\left(\frac{3e_h^n - 4e_h^{n-1} + e_h^{n-2}}{\Delta t}, v_h\right) + a\left(e_h^n, v_h\right) + b\left(v_h, \varepsilon_h^n\right) = \left(R_h^n + R_{h4}^n, v_h\right), \quad \forall v_h \in V_h$$

$$b\left(e_h^n, q_h\right) = 0, \qquad \forall q_h \in Q_h, \quad n = 1, \cdots, N_T$$

$$e_h^1 = O\left(\Delta t^2 + h^k\right)$$

$$e_h^0 = \left(\Pi_h^S\left(0, -p^0\right)\right)_1$$

where $R_h^n = R_{h1}^n + R_{h2}^n + R_{h3}^n$,

$$R_{h1}^n \equiv= \frac{Du^n}{Dt} - \frac{3u^n - 4u^{n-1}\circ X_1^{n-1} + u^{n-2}\circ X_2^{n-1}}{\Delta t} \qquad R_{h2}^n \equiv \frac{3\eta_h^n - 4\eta_h^{n-1}\circ X_{1h}^{n-1} + \eta_h^{n-2}\circ X_{2h}^{n-1}}{\Delta t}$$

$$R_{h3}^n \equiv 4\frac{u^{n-1}\circ X_{1h}^{n-1} - u^{n-1}\circ X_1^{n-1}}{\Delta t} - \frac{u^{n-2}\circ X_{2h}^{n-1} - u^{n-2}\circ X_2^{n-1}}{\Delta t}$$

$$R_{h4}^n \equiv -4\frac{e_h^{n-1} - e_h^{n-1}\circ X_{1h}^{n-1}}{\Delta t} + \frac{e_h^{n-2} - e_h^{n-2}\circ X_{2h}^{n-1}}{\Delta t}$$

Estimates of the remainders

Lemma

$$\left\| u_h^{n-1} \right\|_{W^{1,\infty}} \Delta t \le \delta < 1$$

$\Rightarrow$

$$\left\| R_{h1}^n \right\|_0 \le c(u)\Delta t^2$$

$$\left\| R_{h2}^n \right\|_0 \le c(\nu, u, p)\left(\left\| u_h^{n-1} \right\|_{L^\infty} + 1\right) h^k$$

$$\left\| R_{h3}^n \right\|_0 \le c(u, p)\left\{\left\| e_h^{n-1} \right\|_0 + h^k\right\}$$

$$\left\| R_{h4}^n \right\|_0 \le c\left\| u_h^{n-1} \right\|_{L^\infty} \left\| \nabla e_h^{n-1} \right\|_0$$

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Estimate of $\left\| u_h^n \right\|_{W^{1,\infty}}$ for the 2nd order scheme

$$(2)_n \quad \left\| u_h^n \right\|_{W^{1,\infty}} \le \left\| u_h^n - \Pi_h u^n \right\|_{W^{1,\infty}} + \left\| \Pi_h u^n \right\|_{W^{1,\infty}}$$

$$\le c_{20} h^{-d/2} \left\| u_h^n - \Pi_h u^n \right\|_{H^1} + c_{21} \left\| u^n \right\|_{W^{1,\infty}}$$

$$\le c_{20} h^{-d/2} \left(\left\| u_h^n - \hat{u}_h^n \right\|_1 + \left\| \hat{u}_h^n - u^n \right\|_1 + \left\| u^n - \Pi_h u^n \right\|_1 \right) + c_{21} \left\| u^n \right\|_{W^{1,\infty}}$$

$$\le c_{20} h^{-d/2} \left(c_3 \left(\Delta t^2 + h^k \right) + c_{22} h^k + c_{23} h^k \right) + c_{21} \left\| u^n \right\|_{W^{1,\infty}}$$

$$\le c_{24} h^{-d/2} \left(\Delta t^2 + h^k \right) + c_{21} \left\| u^n \right\|_{W^{1,\infty}}$$

$$\Delta t \left\| u_h^n \right\|_{W^{1,\infty}} \le \Delta t \left\{ c_{24} h^{-d/2} \left(\Delta t^2 + h^k \right) + c_{21} \left\| u^n \right\|_{W^{1,\infty}} \right\}$$

$$\le c_{24} \left(h^{-d/2} \Delta t^3 + h^{k-d/2} \Delta t \right) + c_{21} \Delta t \left\| u^n \right\|_{W^{1,\infty}}$$

$$\Delta t \le c_0 h^{d/6}$$

$$\le c_0 \left(c_{24} c_0^2 + c_{24} h^{k-d/3} + c_{21} h^{d/6} \left\| u^n \right\|_{W^{1,\infty}} \right)$$

$$\le c_0 \left(c_{24} c_0^2 + c_{24} h_0^{k-d/3} + c_{21} h_0^{d/6} \left\| u^n \right\|_{W^{1,\infty}} \right)$$

$$\le \delta$$

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Estimate of $\left\|u_h^n\right\|_{L^\infty}$ for the 2nd order scheme

$$
\begin{aligned}
(3)_n \quad \left\|u_h^n\right\|_{L^\infty} &\le \left\|u_h^n - \Pi_h u^n\right\|_{L^\infty} + \left\|\Pi_h u^n\right\|_{L^\infty} \\
&\le c_{30} h^{-d/6} \left\|u_h^n - \Pi_h u^n\right\|_{L^6} + \left\|u^n\right\|_{L^\infty} \\
&\le c_{30} h^{-d/6} \left(\left\|u_h^n - \hat{u}_h^n\right\|_1 + \left\|\hat{u}_h^n - u^n\right\|_1 + \left\|u^n - \Pi_h u^n\right\|_1 \right) + \left\|u^n\right\|_{L^\infty} \\
&\le c_{30} h^{-d/6} \left(c_3 \left(\Delta t^2 + h^k\right) + c_{31} h^k + c_{32} h^k \right) + \left\|u^n\right\|_{L^\infty} \\
&\le c_{34} \left(h^{-d/6} \Delta t^2 + h^{k-d/6} \right) + \left\|u^n\right\|_{L^\infty} \\
&\le c_{34} \left(c_0^2 h^{d/6} + h^{k-d/6} \right) + \left\|u^n\right\|_{L^\infty} \qquad \Delta t \le c_0 h^{d/6} \\
&\le c_{34} \left(c_0^2 h_0^{d/6} + h_0^{k-d/6} \right) + \left\|u^n\right\|_{L^\infty} \\
&\le 1 + \left\|u^n\right\|_{L^\infty}
\end{aligned}
$$

2^{nd} order one-step scheme

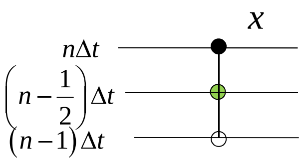
$$\frac{\partial \phi}{\partial t} - \nu \Delta \phi = f \quad \left(x \in \Omega, t \in (0,T)\right), \ \phi = 0 \ \left(x \in \partial\Omega\right)$$

$$\left(\frac{\partial \phi}{\partial t}, \psi\right) + \nu\left(\nabla\phi, \nabla\psi\right) = \left(f, \psi\right) \quad \left(\forall \psi \in H_0^1(\Omega)\right)$$

$$\left(\frac{\phi^n - \phi^{n-1}}{\Delta t}, \psi\right) + \nu\left(\nabla\phi^{n-1/2}, \nabla\psi\right) = \left(f^{n-1/2}, \psi\right) \quad \left(\forall \psi \in H_0^1(\Omega)\right)$$

$$\left(\frac{\phi_h^n - \phi_h^{n-1}}{\Delta t}, \psi_h\right) + \frac{\nu}{2}\left(\nabla\left(\phi_h^n + \phi_h^{n-1}\right), \nabla\psi_h\right) = \frac{1}{2}\left(f^n + f^{n-1}, \psi_h\right) \quad \left(\forall \psi_h \in V_h\right)$$

Crank-Nicolson scheme: Δt^2



Convection-Diffusion Equation

$\Omega \subset \mathbf{R}^d \ (d = 2,3)$, bounded

$T > 0$

Find $\phi : \Omega \times (0,T) \to \mathbf{R}$ such that

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi = f \qquad \text{in } \Omega \times (0,T)$$

$$\phi = 0 \qquad \text{on } \Gamma \times (0,T)$$

$$\phi = \phi^0 \qquad \text{at } t = 0 \ \text{ in } \ \Omega$$

where

$$u \in W_0^{1,\infty}(\Omega)^d, \ f \in L^2(\Omega), \ \phi^0 \in L^2(\Omega)$$

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Pure Crank-Nicholson approximation

Find $\phi_h^n \in V_h, n = 1,\dots,N_T,$ such that

$$\left(\frac{\phi_h^n - \phi_h^{n-1} \circ X_2^n}{\Delta t}, \psi_h \right) + \frac{\nu}{2}\left(\nabla \phi_h^n + \nabla \phi_h^{n-1}, \nabla \psi_h \right) = \frac{1}{2}\left(f_h^n + f_h^{n-1}, \psi_h \right)$$

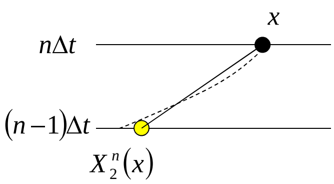
$$\forall \psi_h \in V_h$$

$$\phi_h^0 = \Pi_h \phi^0$$

where

$$X_2^n(x) = x - u^{n-1/2}\left(x - u^n(x)\frac{\Delta t}{2} \right)\Delta t$$

2nd order Runge-Kutta method



or

$$X_2^n(x) = x - \frac{\Delta t}{2}\left(u^n(x) + u^{n-1}\left(x - \Delta t u^n(x) \right) \right)$$

Heun method

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Pure Crank-Nicholson approximation (cont.)

- This scheme is not of $O(\Delta t^2)$.

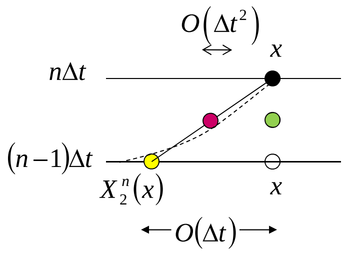
$$\left(\frac{\phi^n - \phi^{n-1}\circ X_2^n}{\Delta t}\right)(x) = \left(\frac{\partial \phi}{\partial t} + (u\cdot\nabla)\phi\right)^{n-1/2}\left(\frac{x + X_2^n(x)}{2}\right) + O\left(\Delta t^2\right)$$

$$\frac{\nu}{2}\left(\nabla\phi^n + \nabla\phi^{n-1}\right)(x) = \nu\,\nabla\phi^{n-1/2}(x) + O\left(\Delta t^2\right)$$

$$\frac{1}{2}\left(f^n + f^{n-1}\right)(x) = f^{n-1/2}(x) + O\left(\Delta t^2\right)$$

However, $\left|\frac{x + X_2^n(x)}{2} - x\right| = O(\Delta t)$

The total accuracy is $O(\Delta t)$!



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A Second Order Characteristic FEM (Rui-T[2002])

Find $\phi_h^n \in V_h,\, n = 1,\dots,N_T$, such that

$$\left(\frac{\phi_h^n - \phi_h^{n-1}\circ X_2^n}{\Delta t}, \psi_h\right) + \frac{\nu}{2}\left(\nabla\phi_h^n + \nabla\phi_h^{n-1}\circ X_1^n, \nabla\psi_h\right)$$
$$+ \frac{\nu\Delta t}{2}\left\{\left(J^n\nabla\phi_h^{n-1}\circ X_1^n, \nabla\psi_h\right) + \left(\nabla \mathrm{div}u^n\cdot\nabla\phi_h^{n-1}, \psi_h\right)\right\}$$
$$= \frac{1}{2}\left(f_h^n + f_h^{n-1}\circ X_1^n, \psi_h\right) \qquad \forall\psi_h \in V_h$$

where $\phi_h^0 = \Pi_h\phi^0$

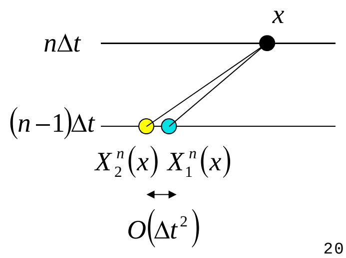
$$X_2^n(x) = x - u^{n-1/2}\left(x - u^n(x)\frac{\Delta t}{2}\right)\Delta t$$

or

$$X_2^n(x) = x - \frac{\Delta t}{2}\left(u^n(x) + u^{n-1}\left(x - \Delta t u^n(x)\right)\right)$$

$$X_1^n(x) = x - u^n(x)\Delta t$$

$$[J^n]_{ij} = \frac{\partial u_i^n}{\partial x_j}$$



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A Second Order Characteristic FEM (Cont.)

<u>Theorem</u> (Rui-T[2002]) P_k - element

$\Rightarrow \quad \exists c\left(\phi, u, T\right) > 0$

$$\left\|\phi_h - \phi\right\|_{\ell^\infty\left(L^2\right)}, \sqrt{\nu}\left|\phi_h - \phi\right|'_{\ell^2\left(H^1\right)} \le c\left(\Delta t^2 + h^k\right)$$

where

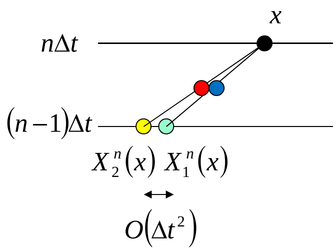
$$\left\|\varphi_h\right\|_{\ell^\infty\left(L^2\right)} = \max\left\{ \left\|\varphi_h^n\right\|_{L^2} ; n = 0, \ldots, N_T \right\},$$

$$\left|\varphi\right|'_{\ell^2\left(H^1\right)} = \left\{ \Delta t \sum_{n=1}^{N_T} \left\| \frac{\nabla\varphi^n + \nabla\varphi^{n-1} \circ X_1^n}{2} \right\|_{L^2}^2 \right\}^{1/2}$$

Key Expression

$$\frac{1}{2}\left(\Delta\phi^n + (\Delta\phi^{n-1}) \circ X_1^n\right) = \Delta\phi^{n-1/2}\left(\frac{1}{2}\left(x + X_1^n\right)\right) + O\left(\Delta t^2\right)$$

$$\frac{1}{2}(\Delta\phi^{n-1}) \circ X_1^n$$
$$= \frac{1}{2}\nabla\cdot\left(\nabla\phi^{n-1} \circ X_1^n\right) + \frac{\Delta t}{2}\sum_{i,j=1}^{d} \frac{\partial u_j^n}{\partial x_i} \frac{\partial}{\partial x_j}\left(\frac{\partial\phi^{n-1}}{\partial x_i}\right) + O\left(\Delta t^2\right)$$



Example 1 (rotating Gaussian hill)

$$\Omega = (-0.5, 0.5) \times (-0.5, 0.5), \quad T = \pi$$

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi = 0, \quad u = \left(-x_2, x_1\right)$$

$$\nu = 1.25 \times 10^{-4}, 2.5 \times 10^{-4}, 5.0 \times 10^{-4}, 1.0 \times 10^{-3}$$

Exact solution:

$$\phi(x_1, x_2, t) = \frac{\sigma}{\sigma + 4\nu t} \exp\left(-\frac{|\bar{x}(t) - x_c|^2}{\sigma + 4\nu t} \right)$$

$$\bar{x}(t) = (x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t)$$

$$x_c = (0.25, 0), \sigma = 0.01$$

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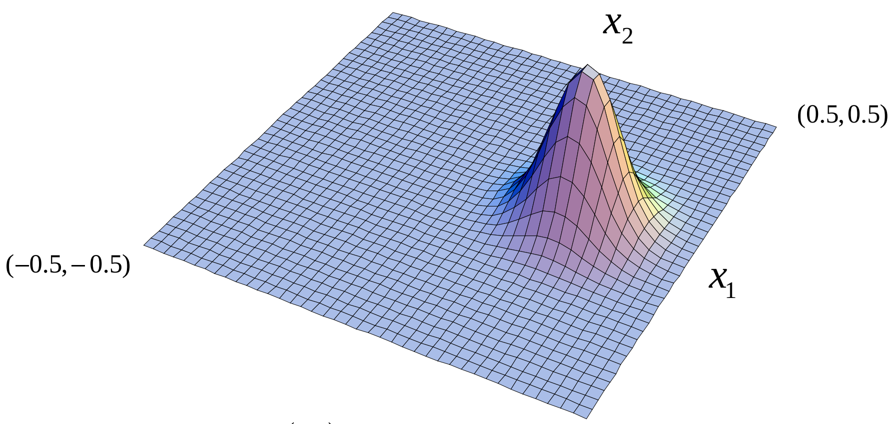
Rotating Gaussian Hill



$\phi(\cdot, t) : t = 0$

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Example 1 (cont.)

Ω is divided into a union of $2N^2$ triangles, $N = 32, 48, 64, 80$

$h = \sqrt{2}/N, \quad \Delta t = 2T/(5\sqrt{N}) \qquad h \propto \Delta t^2$

P1 element

Pure Crank-Nicolson scheme: $O(h+\Delta t) = O(\Delta t)$

$$\left(\frac{\phi_h^n - \phi_h^{n-1}\circ X_2^n}{\Delta t}, \psi_h\right) + \frac{\nu}{2}\left(\nabla\phi_h^n + \nabla\phi_h^{n-1}, \nabla\psi_h\right) = \frac{1}{2}\left(f_h^n + f_h^{n-1}, \psi_h\right)$$

2nd order scheme: $O(h+\Delta t^2) = O(\Delta t^2)$

$$\left(\frac{\phi_h^n - \phi_h^{n-1}\circ X_2^n}{\Delta t}, \psi_h\right) + \frac{\nu}{2}\left(\nabla\phi_h^n + \nabla\phi_h^{n-1}\circ X_1^n, \nabla\psi_h\right)$$
$$+\frac{\nu\Delta t}{2}\left(J^n\nabla\phi_h^{n-1}\circ X_1^n, \nabla\psi_h\right) = \frac{1}{2}\left(f_h^n + f_h^{n-1}\circ X_1^n, \psi_h\right)$$

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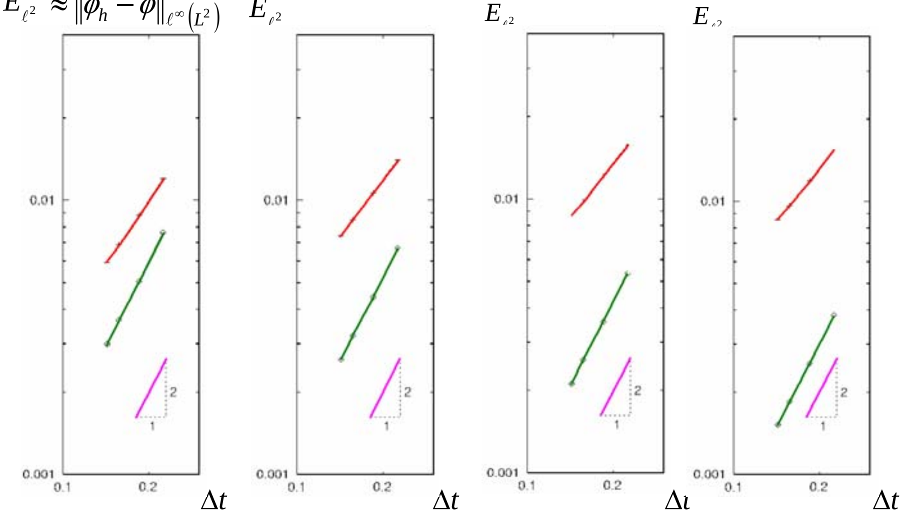
L^2-error vs. Δt

— : Pure Crank-Nicholson scheme
— : 2nd order characteristics scheme

$E_{\ell^2} \approx \|\phi_h - \phi\|_{\ell^\infty(L^2)}$



$\nu = 1.25\times10^{-4}$ $\quad\nu = 2.5\times10^{-4}$ $\quad\nu = 5.0\times10^{-4}$ $\quad\nu = 1.0\times10^{-3}$

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Navier-Stokes Equations

$\Omega \subset \mathbf{R}^d \ (d = 2,3)$, bounded

$\Gamma = \partial\Omega \qquad T > 0$

Find $(u, p): \Omega \times (0,T) \to \mathbf{R}^d \times \mathbf{R}$ such that

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f && \text{in } \Omega \times (0,T) \\
&\nabla \cdot u = 0 && \text{in } \Omega \times (0,T) \\
&u = 0 && \text{on } \Gamma \times (0,T) \\
&u = u^0 && \text{at } t = 0 \text{ in } \Omega
\end{aligned}
$$

where $f \in L^2(\Omega)^d$, $u^0 \in W_0^{1,\infty}(\Omega)^d$, $\nabla \cdot u^0 = 0$

2^{nd} order single-step scheme, Notsu-T[2009]

Δt : time increment, $N_T \equiv \lfloor T / \Delta t \rfloor$

$V_h \subset V,\ Q_h \subset Q$

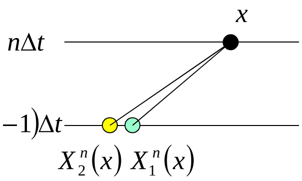
Find $\left(u_h^n, p_h^n\right) \in V_h \times Q_h,\ n = 1, \dots, N_T$, such that

$$
\begin{aligned}
&\left(\frac{u_h^n - u_h^{n-1} \circ X_2^n(u_h^n, u_h^{n-1})}{\Delta t}, v_h\right) + \nu\left(D\left(u_h^n\right) + D\left(u_h^{n-1}\right) \circ X_1^n, D\left(v_h\right)\right) - \frac{1}{2}\left(\nabla \cdot v_h, p_h^n + p_h^{n-1} \circ X_1^n\right) \\
&+\Delta t\left(\nu J\left(u_h^{n-1}\right) J\left(u_h^{n-1}\right)^T - \frac{1}{2} p_h^{n-1} J\left(u_h^{n-1}\right)^T, J\left(v_h\right)\right) = \frac{1}{2}\left(f_h^n + f_h^n \circ X_1^n, v_h\right), \quad \forall v_h \in V_h \\
&b\left(u_h^n, q_h\right) = 0, \qquad \forall q_h \in Q_h \\
&u_h^0 = \Pi_h u^0
\end{aligned}
$$

where

$X_1^n(x) \equiv x - u_h^{n-1}(x)\Delta t,$

$X_2^n(w,u)(x) \equiv x - \frac{\Delta t}{2}\left\{w(x) + u\left(x - u(x)\Delta\right)\right\}$



2nd order single-step scheme (cont.)

Internal iteration: $w_h^0 = u_h^{n-1}$

$$w_h^{\ell-1}, \rightarrow \left(w_h^{\ell}, r_h^{\ell}\right) \rightarrow \cdots \rightarrow \left(u_h^n, p_h^n\right)$$

Find $\left(w_h^l, r_h^l\right) \in V_h \times Q_h,\ \ell = 1, \ldots, N_T,$ such that

$$\left(\frac{w_h^{\ell} - u_h^{n-1} \circ X_2^n(w_h^{\ell-1}, u_h^{n-1})}{\Delta t}, v_h\right) + \nu\left(D\left(w_h^{\ell}\right) + D\left(u_h^{n-1}\right) \circ X_1^n, D\left(v_h\right)\right) - \frac{1}{2}\left(\nabla \cdot v_h, r_h^{\ell} + p_h^{n-1} \circ X_1^n\right)$$

$$+\Delta t\left(\nu J\left(u_h^{n-1}\right) J\left(u_h^{n-1}\right)^T - \frac{1}{2} p_h^{n-1} J\left(u_h^{n-1}\right)^T, J\left(v_h\right)\right) = \frac{1}{2}\left(f_h^n + f_h^n \circ X_1^n, v_h\right), \quad \forall v_h \in V_h$$

$$b\left(w_h^{\ell}, q_h\right) = 0, \qquad \forall q_h \in Q_h$$

$$u_h^0 = \Pi_h u^0$$

Note. This scheme can be used to get u_h^1 in two-step method.

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A stabilized Galerkin-characteristics scheme for the Navier-Stokes Equations

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Stabilized P1/P1 scheme

- P1/P1 element for NS equations.
  Especially useful for 3D computation.
- Stabilizing term is necessary.
  P1/P1 element does not satisfy the inf-sup condition.
- Stabilized P1/P1 Galerkin-characteristics FE scheme.
  symmetric matrix, small DOF
  → less computation time
  → finer subdivision for 3D problems

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Navier-Stokes Equations

$\Omega \subset \mathbf{R}^d \ (d = 2, 3)$, bounded

$\Gamma = \partial\Omega \qquad T > 0$

Find $(u, p): \Omega \times (0,T) \to \mathbf{R}^d \times \mathbf{R}$ such that

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f && \text{in } \Omega \times (0,T) \\
&\nabla \cdot u = 0 && \text{in } \Omega \times (0,T) \\
&u = 0 && \text{on } \Gamma \times (0,T) \\
&u = u^0 && \text{at } t = 0 \text{ in } \Omega
\end{aligned}
$$

where $f \in L^2(\Omega)^d$, $u^0 \in W_0^{1,\infty}(\Omega)^d$, $\nabla \cdot u^0 = 0$

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Weak formulation

$V \equiv H_0^1(\Omega)^d,\ Q = L_0^2(\Omega)$

Find $(u, p) : (0,T) \to V \times Q$ such that

$$\left(\frac{Du}{Dt}, v\right) + a(u,v) + b(v,p) = (f, v), \quad \forall v \in V$$
$$b(u,q) = 0, \quad \forall q \in Q$$
$$u(0) = u^0$$

where $\displaystyle \frac{Du}{Dt} \equiv \frac{D^{(u)}u}{Dt}, \quad \frac{D^{(w)}u}{Dt} \equiv \frac{\partial u}{\partial t} + (w \cdot \nabla) u$

$$a(u,v) \equiv 2\nu (D(u), D(v)), \quad D_{ij}(v) \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$
$$b(u,q) \equiv -(\nabla \cdot u, q)$$

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P1/P1 stabilized scheme

Δt : time increment, $N_T \equiv \lfloor T / \Delta t \rfloor$

$V_h \subset V,\ Q_h \subset Q$: P1– FE space

Find $(u_h^n, p_h^n) \in V_h \times Q_h,\ n = 1, \ldots, N_T,$ such that

$$\left(\frac{u_h^n - u_h^{n-1} \circ X_h^{n-1}}{\Delta t}, v_h\right) + a(u_h^n, v_h) + b(v_h, p_h^n) = (f_h^n, v_h), \quad \forall v_h \in V_h$$
$$b(u_h^n, q_h) - \mathcal{C}_h(p_h^n, q_h) = 0, \quad \forall q_h \in Q_h$$
$$u_h^0 = \left(\Pi_h^S(u^0, 0)\right)_1$$

where

$$X_h^{n-1}(x) \equiv x - u_h^{n-1}(x)\Delta t, \quad \mathcal{C}_h(p,q) = \frac{\delta_0}{\nu} \sum_{k \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K$$

Note. Notsu-T[2008]

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P1/P1 stabilized scheme (cont.)

Theorem V_h / Q_h : P1/P1 element

$$\exists h_0, c_0 > 0, \quad h \le h_0, \ \Delta t \le c_0 h^{d/4}$$

$$\Rightarrow \exists c(\nu, T, u, p, \delta_0) > 0,$$

$$\|u_h - u\|_{\ell^\infty(H^1)}, \|p_h - p\|_{\ell^2(L^2)} \le c(\Delta t + h)$$

where

$$\|v_h\|_{\ell^\infty(X)} = \max\left\{ \|v_h^n\|_X ; n = 0, \dots, N_T \right\},$$

$$\|v_h\|_{\ell^2(X)} = \left\{ \Delta t \sum_{n=1}^{N_T} \|v_h^n\|_X^2 \right\}^{1/2}$$

Ref. Notsu-T[2013]

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Framework for the Navier-Stokes equations

Hilbert spaces.

$$V = H_0^1(\Omega)^d, \ d = 2,3, \ \ \Omega : \text{bdd.} \subset \Re^d$$

$$Q = L_0(\Omega) \equiv \left\{ q \in L^2(\Omega); \int_\Omega q(x)\, dx = 0 \right\}$$

FEM spaces. $V_h \subset V, \ Q_h \subset Q$

$\{\mathcal{T}_h\}_{h \downarrow 0}$: regular, inverse ineq.

$$\exists \sigma > 0, \ \forall h, \forall K \in \mathcal{T}_h, \ \text{diam}(K) \le \sigma \rho(K)$$

$\rho(K)$: radius of the inscribe ball

$$\exists c_2, c_2 > 0, \ \forall h, \forall K \in \mathcal{T}_h, \ c_1 h \le \text{diam}(K) \le c_2 h$$

Bilinear forms. $a : V \times V \to \Re, \ b : V \times Q \to \Re, \ \mathcal{C}_h : Q \times Q \to \Re$

$$a(u,v) = 2\nu \int_\Omega D(u) : D(v)\, dx$$

$$b(v,q) = -\int_\Omega q \nabla \cdot v \ dx, \ \mathcal{C}_h(p,q) = \frac{\delta_0}{\nu} \sum_{k \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K$$

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Stokes projection

$$\Pi_h^S : V \times Q \to V_h \times Q_h, \quad \Pi_h^S(u, p) \equiv (\hat{u}_h, \hat{p}_h)$$

$$a(\hat{u}_h, v_h) + b(v_h, \hat{p}_h) = a(u, v_h) + b(v_h, p), \ \forall v_h \in V_h$$
$$b(\hat{u}_h, q_h) - \mathcal{C}_h(\hat{p}_h, q_h) = b(u, q_h) - \mathcal{C}_h(p, q_h), \ \forall q_h \in Q_h$$

$$\exists c > 0, \quad \|(\hat{u}_h - u, \hat{p}_h - p)\|_{V\times Q} \le c \|(u, p)\|_{H^2 \times H^1} h$$
$$i.e., \quad \|I - \Pi_h^S\|_{\mathcal{L}(H^2\times H^1, V\times Q)} \le ch$$

Ref. Brezzi-Douglas[1988]

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Stokes projection (cont.)

Key condition for the proof: Stability inequality.

$\exists \gamma_0 > 0,$

$$\inf_{(u_h, p_h)\in V_h\times Q_h} \sup_{(v_h, q_h)\in V_h\times Q_h} \frac{a(u_h, v_h) + b(v_h, p_h) + b(u_h, q_h) - \mathcal{C}_h(p_h, q_h)}{\|(u_h, p_h)\|_{V\times Q} \|(v_h, q_h)\|_{V\times Q}} \ge \gamma_0$$

Ref. Brezzi-Douglas[1988], Franca-Stenberg[1991]

Note. Inf-sup condition,

$$\exists \beta_0 > 0, \quad \inf_{q_h\in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|q_h\|_Q \|v_h\|_V} \ge \beta_0$$

does not hold for the P1/P1 element.

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Outline of the proof

- Stokes projection of the stabilized type
- Error equation in $(e_h, \varepsilon_h) \equiv (u_h, p_h) - \Pi_h^S(u, p)$

$$(u_h - u, p_h - p) = (e_h, \varepsilon_h) + \left(\Pi_h^S - I\right)(u, p)$$

- Estimates of the term $\mathscr{e}_h$
- Discrete Gronwall's inequality
- Induction is employed to evaluate $\Delta t \left\| u_h^{n-1} \right\|_{W^{1,\infty}}$ and $\left\| u_h^{n-1} \right\|_{L^\infty}$ at each step.

Refer to Notsu-T[2013] for the details.

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Schemes free from numerical integration error

- Mass lumping technique:
  L^∞ – esitmate
  Pironneau-T[2010]
- Finite difference method:
  discrete L^2 – esitmate
  Notsu-Rui-T[2013]
- Exact integration for approximated velocity:
  T-Uchiumi[2013]

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Galerkin-characteristics FEM of lumped mass type

Δt : time increment, $N_T \equiv \lfloor T/\Delta t \rfloor$

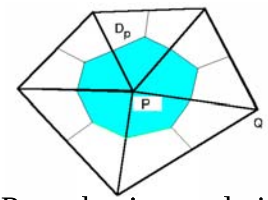
$V_h \subset V$: $P1-$FEM space

Find $\phi_h^n \in V_h, n = 1,\ldots,N_T$, such that

$$\left(\frac{\bar{\phi}_h^n - \bar{I}_h\left(\phi_h^{n-1} \circ X_1^n\right)}{\Delta t}, \bar{\psi}_h\right) + \nu(\nabla \phi_h^n, \nabla \psi_h) = (\bar{I}_h f^n, \bar{\psi}_h), \quad \forall \psi_h \in V_h$$

$$\phi_h^0 = I_h \phi^0$$

where $X_1^n(x) \equiv x - u^n(x)\Delta t$



$I_h : C(\bar{\Omega}) \to V_h$, $(I_h v)(P) = v(P)$, $\forall P$: node, interpolation

$\bar{\ }: V_h \to L^2(\Omega)$, $\bar{v}_h(x) = v_h(P)$, $(x \in D_P) \forall P$: node, lumping

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A First Order Characteristic FEM (cont.)

<u>Theorem</u>

P_1-element, weakly acute type triangulation

$$\Rightarrow \quad \left\|\phi_h - I_h \phi\right\|_{\ell^\infty(L^\infty)} \le c_\varepsilon \left(h + \Delta t + \frac{h^{2-\varepsilon}}{\Delta t}\right)$$

$$\varepsilon \in (0,1), d = 2; \ \varepsilon = 0, d = 3$$

$$\Rightarrow \quad \left\|\phi_h - I_h \phi\right\|_{\ell^\infty(L^\infty)} \le c_\varepsilon h^{1-\varepsilon}, \quad \Delta t = h$$

Note. $\left\|\phi_h\right\|_{\ell^\infty(X)} = \max\left\{ \left\|\phi_h^n\right\|_X ; n = 0,\ldots,N_T \right\}$

Pironneau-T[2010]

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Application to a Two-Fluid Flow Problem

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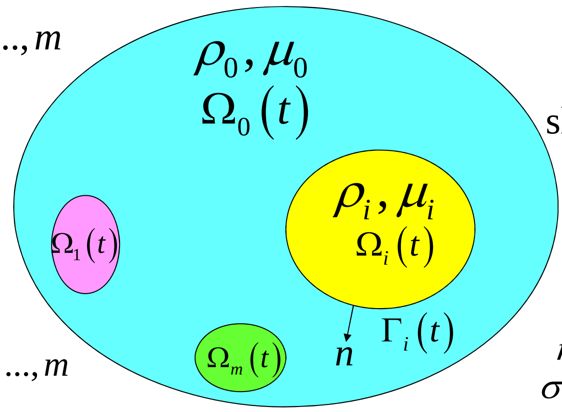
Multiphase flow problems with interface tension(1)

Fluids $0, 1, .., m$

NS eqns



slip BC
or $u = 0$

$i = 1, \ldots, m$

κ : curvature
σ_i : coefficient

Interface condition: $[u]_{\Gamma_i} = 0, \left[-pn + 2\mu D(u)n\right]_{\Gamma_i} = \sigma_i \kappa n$

Note. $\bar{\Omega}_i(t) \subset \Omega \quad (i = 1, \ldots, m)$ $\quad D(u) \equiv \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$

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Multiphase flow problems with interface tension(2)

(u, p) satisfies NS eqns. in each domain $\Omega_k(t),\ \forall k = 0,\dots,m$

$$\rho_k\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) - \nabla\left(2\mu_k D(u) - pI\right) = \rho_k f$$
$$\nabla\cdot u = 0$$

Interface conditions on $\Gamma_i(t) \equiv \bar{\Omega}_0(t)\cap\bar{\Omega}_i(t)$

$$[u] = 0,\quad \left[-pn + 2\mu D(u)n\right]_{\Gamma_i} = \sigma_i \kappa n$$

Boundary conditions on $\Gamma\times(0,T),\ \Gamma\equiv\partial\Omega$

$$u\cdot n = 0,\quad D(u)n\times n = 0\quad (\text{or}\quad u = 0)$$

Evolution of $\Gamma_i(t) \equiv \{\chi_i(s,t); s\in[0,1)\}$

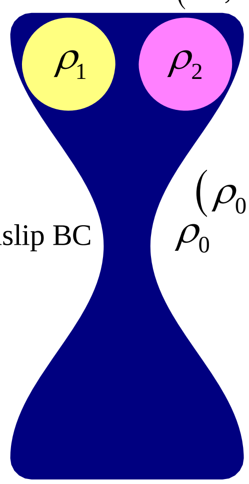
$$\frac{\partial \chi_i}{\partial t} = u(\chi_i, t),\quad i = 1,\cdots,m$$

Initial conditions at $t = 0$

$$u = u^0,\quad \Omega_k(0) = \Omega_k^0,\ \forall k = 0,\dots,m$$

Different density fluids in an "Hourglass"

$(0.5, 2)$



$(\rho_1, \mu_1, \sigma_1) = (100, 0.2, 0.1)$

$(\rho_2, \mu_2, \sigma_2) = (120, 0.2, 0.1)$

$$f = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$(\rho_0, \mu_0) = (1,1)$

nonslip BC $\quad \rho_0$

$T = 60$

$N = 32,\qquad \Delta t = \frac{1}{8}$

$(-0.5, 0)$ $\quad$ t=0.00000

Ref. T[2010]

References

- M. Tabata. A finite element approximation corresponding to the upwind finite differencing. Memoirs of Numerical Mathematics, 4(1977), 47-63.
- R. E. Ewing and T. F. Russell. Multistep Galerkin methods along characteristics for convection-diffusion problems. In R. Vichnevetsky and R. S. Stepleman, editors, Advances in Computer Methods for Partial Differential Equations, 4, 28-36, IMACS, 1981.
- O. Pironneau. On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numerische Mathematik, 38(1982), 309-332.
- F. Brezzi and J. Jr. Douglas. Stabilized mixed methods for the Stokes problem. Numerische Mathematik, 53(1988), 225-235.
- E. Suli. Convergence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations. Numerische Mathematik, 53(1988), 459-483.
- L. P. Franca and R. Stenberg. Error analysis of some Galerkin least squares methods for the elasticity equations. SIAM Journal on Numerical Analysis, 28(1991), 1680-1697.
- K. Boukir, Y. Maday, B. Metivet, and E. Razafindrakoto. A high-order characteristics/finite element method for the incompressible Navier-Stokes equations. International Journal for Numerical Methods in Fluids, 25(1997), 1421-1454.
- H. Rui and M. Tabata. A second order characteristic finite element scheme for convection-diffusion problems. Numerische Mathematik, 92(2002), 161-177.
- H. Notsu and M. Tabata. A single-step characteristic-curve finite element scheme of second order in time for the incompressible Navier-Stokes equations. Journal of Scientific Computing, 38(2009), 1-14.

- O. Pironneau and M. Tabata. Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type. International Journal for Numerical Methods in Fluids, 64(2010), 1240-1253.
- M. Tabata. Numerical simulation of fluid movement in an hourglass by an energy-stable finite element scheme. In M. N. Hafez, K. Oshima, and D. Kwak, editors, Computational Fluid Dynamics Review 2010, 29-50. World Scientific, Singapore, 2010.
- H. Notsu, H. Rui, and M. Tabata. Development and L2-analysis of a single-step characteristics finite difference scheme of second order in time for convection-diffusion problems. Journal of Algorithms & Technology, 7(2013), 343-380.
- H. Notsu and M. Tabata. Error estimates of a pressure-stabilized characteristics finite element scheme for the Oseen equations. WIAS-DP-2013-001, Waseda Univ., 2013.
- H. Notsu and M. Tabata. Error estimates of a pressure-stabilized characteristics finite element scheme for the Navier-Stokes equations. WIAS-DP-2013-002, Waseda Univ., 2013.