High regularity results of solutions to modified p-Navier-Stokes equations

Francesca Crispo

joint work with P. Maremonti

Department of Mathematics and Physics
Second University of Naples

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Introduction

We consider the following problem:

$$\nabla \cdot T(u,\pi) = F + (u \cdot \nabla)u \,, \quad \nabla \cdot u = 0 \text{ in } \mathbb{R}^3 \,, \tag{1}$$

where $u = (u_1, u_2, u_3)$ is a vector field, π is a scalar field,

$T(u,\pi)$ is a special tensor of the kind

$$T(u,\pi) = -\pi I + |\nabla u|^{p-2} \nabla u \,, \tag{2}$$

with I identity operator, $p \in (1,2)$.

The system is nonlinear and singular.
It was studied for the first time by J.L.Lions (1969), through the monotone operators theory.

Tensor T is "close" to the well known stress tensor $\widetilde{T}$ of non-Newtonian fluids, in the singular case,

$$\widetilde{T}(u,\pi) = -\pi I + |\mathcal{D}u|^{p-2}\mathcal{D}u\,,$$

$\mathcal{D}u = \frac{\nabla u + (\nabla u)^T}{2}$, which gives rise to the so called p-Navier-Stokes problem

$$\nabla \cdot \left(|\mathcal{D}u|^{p-2}\,\mathcal{D}u \right) - \nabla \pi = F + (u \cdot \nabla)u\,, \quad \nabla \cdot u = 0 \;\text{ in } \mathbb{R}^3\,. \tag{3}$$

Hence, we call, on the contrary, system (1) *modified* p-Navier-Stokes system.

Introduction

The difference between T and $\widetilde{T}$ makes our results not interesting from a fluid dynamic point of view.

Nevertheless the results gain a special interest from the analytical point of view.

Indeed, the literature concerning the "high regularity" of solutions to the singular p-(Navier-)Stokes problem is not satisfactory. We are just aware of the following results:

Naumann & Wolf (JMFM 2005)

$$\Omega \text{ bounded}, \ p \in (9/5, 2), \ (u, \pi) \in W^{2, \frac{3p}{p+1}}_{\ell oc}(\Omega) \times L^{p'}_{\ell oc}(\Omega),$$

Berselli, Diening & Růžička (JMFM 2010)

$$\Omega \text{ space-periodic}, \ p \in (9/5, 2), \ u \in W^{2, \frac{3p}{p+1}}(]0, 1[^3),$$

Ebmeyer (MMAS 2006)

$$\Omega \text{ bounded, slip boundary conditions}, \ p \in (9/5, 2), \ u \in W^{2, \frac{3p}{p+1}}(\Omega)\,.$$

No global regularity result for the pressure field π.

The corresponding elliptic system is the well known p-Laplacian, whose study has a wide literature.

We quote in particular the results due to:

Acerbi & Fusco ('89), DiBenedetto ('93), Ebmeyer ('06), Iwaniec & Manfredi ('89), Liu & Barrett ('93), Málek & Rajagopal & Růžička ('95), Acerbi & Mingione & Seregin ('04), Tolksdorf ('05), Mingione (06);

high regularity with $D^2u \in L^q(\Omega)$:
Beirão da Veiga & Crispo ('12) and ('13),
Crispo & Maremonti (forthcoming).

These results lead us to wonder if it is possible to deduce analogous results related to the p-(Navier-)Stokes problem.

Our task is to understand if the couple of the vector field u and the scalar field π can give a smooth solution of (1) at least with ∇u in place of $\mathcal{D}u$.
In other words, we investigate if, in spite of the presence of the scalar field π, the regularity of the field u can be compared with the one of solutions to the corresponding elliptic system.

The results I'm going to introduce are part of the papers

F.C. and P. Maremonti, *A high regularity result of solutions to modified p-Stokes equations*, submitted;

F.C. and P.Maremonti, *A high regularity result of solutions to modified p-Navier -Stokes equations*, forthcoming.

Definition 1 (High regular solution)

A pair (u,π) is a high regular solution of (1) if:

- *for some* $q \in (3,+\infty)$, $D^2 u, \nabla\pi \in L^q(\mathbb{R}^3)$, $\nabla u \in L^p(\mathbb{R}^3)$, $\pi \in L^{p'}(\mathbb{R}^3)$,
- $\nabla \cdot u = 0, \; a.\ e.\ in\, \mathbb{R}^3$,
- $(|\nabla u|^{p-2}\nabla u, \nabla\varphi) = (\nabla\pi,\varphi) + ((u\cdot\nabla)u,\varphi) + (F,\varphi), \;$ *for all* $\varphi \in C_0^\infty(\mathbb{R}^3)$.

We set

$$M(r) := 1 - (2-p)H(r')(5 + H(r)),$$

and

$$\overline{M}(2) := 2p - 3 - (2-p)(1 + H(2)),$$

where $H(s)$ is the L^s-singular norm of Calderón-Zygmund type, r' coniugate exponent of r.

Our first result is the following existence result of high regular solution of the modified p-Stokes problem.

Theorem 1:"high regularity" for the modified p-Stokes

Let

- $p \in (\frac{3}{2}, 2]$;
- $\overline{M}(2) > 0,\ M(q_1) > 0$ *with* $q_1 = \frac{3p}{3+p}$;
- $q \in (3, +\infty)$, $F \in L^q(\mathbb{R}^3) \cap L^{q_1}(\mathbb{R}^3)$.

If $M(q) > 0$, then there exists a solution (u, π) in the sense of Definition 1 of the modified p-Stokes system, and

$$\|D^2u\|_q + \|D^2u\|_{q_1} \leq c\,(\|F\|_q + \|F\|_{q_1})^{\frac{1}{p-1}}\,. \tag{4}$$

$$\|\nabla\pi\|_q \leq c\,\|F\|_q, \quad \|\nabla\pi\|_{q_1} \leq c\,\|F\|_{q_1}\,. \tag{5}$$

Moreover, the solution (u, π) is unique in the class of solutions with $\nabla u \in L^p(\mathbb{R}^3)$.

The regularity results contained in the above theorem are the first high regularity results for solutions to a modified p-Stokes system,

in the sense of $D^2u \in L^q(\mathbb{R}^3)$ and also of $C^{1,\alpha}$-regularity, obtained by Sobolev embedding $(\alpha = 1 - \frac{3}{q})$

Approximating solutions

Before giving some comments on the statement, to better explain, we point out that we develop our arguments starting from a nonsingular problem:

$$\nabla \cdot u = 0, \text{ in } \mathbb{R}^n, \quad \nabla \cdot \left[(\mu + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \right] - \nabla \pi = F, \text{ in } \mathbb{R}^3, \qquad (6)$$

where $\mu > 0$.

The idea is to work on this problem and to to obtain all the estimates uniformly with respect to the parameter μ.

Some comments on the statement of the theorem

The choice of the exponent $q_1 = \frac{3p}{3+p}$ has been made in order to get $\nabla u \in L^p(\mathbb{R}^3)$ by Sobolev embedding. $\nabla u \in L^p(\mathbb{R}^3)$ enables us to discuss uniqueness. Hence in particular to compare our solution with weak solutions.

(Note that $q_1 > 1$ if and only if $p > \frac{n}{n-1}$, which excludes the value $n = 2$.)

The requests on the constants $M(q_1) > 0$ and $M(q) > 0$ translate a condition of proximity of p to 2, which is a sufficient condition in order to get the following kind of estimate

$$\left\| \frac{D^2 u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} \right\|_{L^r(\mathbb{R}^3)} \leq c \, \|F\|_{L^r(\mathbb{R}^3)} , \text{ for all } \mu > 0, \tag{7}$$

for exponents $r = q_1, q$.

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for exponents $r = q_1, q$.

Why does the theorem not work under the assumption $\widetilde{T}(u,\pi) = -\pi I + |\mathcal{D}u|^{p-2}\mathcal{D}u$? Hydrodynamic case!

The reason is connected with the fact that we employ the pointwise estimate

$$\frac{|D_{x_i} u_j(x)|^2}{\mu + |\nabla u(x)|^2} \leq 1\,,$$

that clearly does not hold, in general, with

$$\mu + |\mathcal{D}u(x)|^2 \text{ in place of } \mu + |\nabla u(x)|^2.$$

However, we could obtain the results for $\widetilde{T}(u,\pi)$ if we were able to show the following crucial kind of estimates:

$$\int_\Omega \frac{|D_{x_h} u_i D_{x_i} |\mathcal{D}u|^2|^r}{(\mu + |\mathcal{D}u|^2)^{\frac{4-p}{2}r}}\, dx \leq c \int_\Omega \frac{|\mathcal{D}u|^r |D^2 u|^r}{(\mu + |\mathcal{D}u|^2)^{\frac{3-p}{2}r}}\, dx\,,$$

for $r = 2$ and bounded Ω, $u \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, and for $r \neq 2$ and $\Omega = \mathbb{R}^3$, $\nabla u \in L^p(\mathbb{R}^3)$ and $D^2 u \in L^r(\mathbb{R}^3)$.

Why does our proof work? What is new?

- It is new in the approach
- there is the advantage of a domain like $\mathbb{R}^3$

Our approach

The new idea is to investigate the existence and the regularity looking at the problem as a **perturbed elliptic problem**.

Since we work in $\mathbb{R}^3$, we are able to arrive at a representation formula of the pressure field of this kind

$$\pi := (2-p)\int_{\mathbb{R}^3} D_{y_i}\mathcal{E}(x-y)\,\frac{D_{y_i}u_s(y)D_{y_s}|\nabla u|^2}{(\mu+|\nabla u(y)|^2)^{\frac{4-p}{2}}}\,dy\,,$$

$\mathcal{E}(x-y)$ being the fundamental solution of the Laplace equation.
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$\mathcal{E}(x-y)$ being the fundamental solution of the Laplace equation.
This representation is crucial for our aims.

Setting $K[\nabla u] := \pi(x)$, where $\pi(x)$ is the pressure field already defined, then we study the problem

$$\nabla \cdot \left[(\mu + |\nabla u|)^{p-2} \nabla u\right] - \nabla K[\nabla u] = F, \text{ for all } \mu > 0. \tag{8}$$

Here the perturbation is the term $\nabla K[\nabla u]$, for p near 2 it can be small in suitable norms.

The advantage to handle system (8) is that now we can employ the methods of the elliptic problems, the eigenfunctions of the Laplace operator as special Galerkin basis (on a bounded domain), and all the estimates up to the second derivatives are made in L^q spaces and not in the spaces of the hydrodynamic (which intrinsically contain the Helmohltz-Weyl orthogonality).

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In particular, our estimates on second derivatives have, as a starting point, the estimate of the kind:

$$\Big\| \frac{D^2 u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} \Big\|_{L^r(\mathbb{R}^3)} \leq c \, \|F\|_{L^r(\mathbb{R}^3)} \, , \text{ for all } \mu > 0, \tag{9}$$

for exponents $r = q_1 , q$.
A crucial tool for the above estimate is the following one

$$\Big\| \frac{D^2 u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} \Big\|_{L^r(\mathbb{R}^3)} \leq c(r, p) \Big\| \frac{\Delta u}{(\mu + |\nabla u|^2)^{\frac{2-p}{2}}} \Big\|_{L^r(\mathbb{R}^3)}$$

which is a generalization of the well known inequality

$$\|D^2 u\|_{L^r(\mathbb{R}^3)} \leq c(r) \|\Delta u\|_{L^r(\mathbb{R}^3)} \, .$$

Estimate (9) plays the same role as the estimate of $\|P\Delta u\|_r$ for the classical Stokes system.

We gain, roughly speaking, a solution of the problem

$$\nabla \cdot T(u, \pi, \mu) = F\,, \quad \text{in } \mathbb{R}^3\,, \text{ for all } \mu > 0\,, \tag{10}$$

with

$$\pi = (2-p)\int_{\mathbb{R}^3} D_{y_i}\mathcal{E}(x-y)\,\frac{D_{y_i}u_s D_{y_s}|\nabla u|^2}{(\mu + |\nabla u|^2)^{\frac{4-p}{2}}}\,dy\,. \tag{11}$$

So the divergence free equation

$$\nabla \cdot u = 0 \tag{12}$$

seems missing.

But our pressure field has the representation formula of a pressure field arisen from the equations of the pressure:

$$\Delta\pi = \nabla \cdot \left[\nabla \cdot \left((\mu + |\nabla u|)^{p-2}\nabla u\right)\right]\,, \text{ **joint with the condition** } \nabla \cdot u = 0.$$

Therefore equation (12) is satisfied by our solution as a compatibility condition between the equations (10) and the representation formula (11) and finally by the condition $\nabla \cdot u \to 0$ for $|x| \to \infty$.

Indeed, the compatibility between the equation (10) and (11) formally gives, for $\nabla \cdot u$, the equation

$$\Delta(\nabla \cdot u) + \frac{(2-p)}{2}\frac{\nabla(\nabla \cdot u) \cdot \nabla|\nabla u|^2}{(\mu + |\nabla u|^2)} = 0\,, \text{ in } \mathbb{R}^3,$$

we also can append the condition

$$\nabla \cdot u \ \to 0 \text{ as } |x| \to \infty.$$

Therefore the maximum principle ensures that $\nabla \cdot u = 0$.
So, the solution of (10) is divergence free.
Hence u is a solution to the modified p-Stokes problem (1).

In order to obtain a high regularity result for the modified p-Navier-Stokes system, it is crucial to generalize the result obtained for the modified p-Stokes system, by making different assumptions on the integrability of the force term F.
This is the aim of the following theorem, whose proof relies on Theorem 1.

Theorem 2: "high regularity" for the modified p-Stokes

Let

- $p \in (\frac{9}{5}, 2]$;
- $\overline{M}(2) > 0,\ M(q_1) > 0$ *with* $q_1 = \frac{3p}{3+p}$;
- $q \in (3, +\infty)$, $F \in L^q(\mathbb{R}^3) \cap D_0^{-1,p'}(\mathbb{R}^3)$.

If $M(q) > 0$, *then there exists a solution* (u, π) *in the sense of Definition 1 of the modified p-Stokes system, and*

$$\|D^2 u\|_q \leq c\,(\|F\|_q \|F\|_{-1,p'}^{\frac{(1-a)(2-p)}{p-1}})^{\frac{1}{1-a(2-p)}}\,, \tag{13}$$

$a = \frac{3q}{3(q-p)+pq}$. *Moreover, the solution* (u, π) *is unique in the class of solutions with* $\nabla u \in L^p(\mathbb{R}^3)$.

Estimate (13) comes from the fundamental estimate

$$\|D^2 u\|_q \leq c\,(\|F\|_q + \|\nabla \pi\|_q)\|\nabla u\|_\infty^{2-p}\,. \tag{14}$$

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Theorem 3: “high regularity” for the modified p-Navier-Stokes

Let

- $p \in (\frac{9}{5}, 2]$;
- $\overline{M}(2) > 0,\ M(q_1) > 0$ *with* $q_1 = \frac{3p}{3+p}$;
- $q \in (3, +\infty)$, $F \in L^q(\mathbb{R}^3) \cap D_0^{-1,p'}(\mathbb{R}^3)$.

If $M(q) > 0$, *then there exists a solution* (u, π) *in the sense of Definition 1 of the modified p-Navier- Stokes system.*

The high regularity obtained in the above theorem is completely similar to the one given for the p-Stokes system, as it happens for the classical Navier-Stokes equations.
Likewise, no uniqueness result is at disposal.

The leading ideas behind this result are the following.

- We consider the ”regularized” nonlinear system

$$\nabla\cdot\big(|\nabla v|^{p-2}\nabla v\big)-\nabla\pi_v=\chi^\rho J_\varepsilon\big((v\cdot\nabla)J_\varepsilon(v\chi^\rho)\big)+f\,,\quad \nabla\cdot v=0 \text{ in } \mathbb{R}^3\,. \tag{15}$$

Using the Galerkin method and the monotone operator theory, for $f\in D_0^{-1,p'}(\mathbb{R}^3)$ there exists a weak solution of (15) satisfying

$$\|\nabla v\|_p\le c\|f\|_{-1,p'}^{\frac{1}{p-1}}\,,\ \forall\varepsilon>0,\ \forall\rho>0\,. \tag{16}$$

Note that $(\chi^\rho J_\varepsilon\big((v\cdot\nabla)J_\varepsilon(v\chi^\rho)\big),v)=((v\cdot\nabla)J_\varepsilon(v\chi^\rho),J_\varepsilon(v\chi^\rho))=0\,.$

- We consider the modified p-Stokes system

$$\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right) - \nabla \pi_u = F\,, \quad \nabla \cdot u = 0 \;\text{ in } \mathbb{R}^3\,. \tag{17}$$

  with

$$F := f + F_{\varepsilon,\rho} := f + \chi^\rho J_\varepsilon\big((v\cdot\nabla)J_\varepsilon(v\chi^\rho)\big) \in L^q(\mathbb{R}^3)\cap D_0^{-1,p'}(\mathbb{R}^3)\,, \tag{18}$$

  for which our Theorem 2 holds. Hence there exists a high regular solution u.

- By uniqueness in the class of weak solutions, $u \equiv v$.

Now we need to get estimates on $\|D^2u\|_q$, uniformly in ε and ρ, in order to use weak sequential compactness results, and pass to the limit on $\varepsilon \to 0$ and $\rho \to \infty$.

This is possible thanks to estimate (14), that we repreoduce here

$$\|D^2 u\|_q \le c\,(\|F\|_q + \|\nabla\pi\|_q)\|\nabla u\|_\infty^{2-p}\,. \tag{19}$$

Indeed, from Gagliardo-Nirenberg's inequality, Sobolev's inequality and Theorem 1,

$$\|\nabla\pi\|_q \le c\|F\|_q \le c(\|f\|_q + \|F_{\varepsilon,\rho}\|_q),$$

$$\|\nabla u\|_\infty^{2-p} \le c\|D^2 u\|_q^{a(2-p)}\|\nabla u\|_p^{(1-a)(2-p)} \le c\|D^2 u\|_q^{a(2-p)}\|f\|_{-1,p'}^{\frac{(1-a)(2-p)}{p-1}},$$

and

$$\|F_{\varepsilon,\rho}\|_q \le \|u\|_\infty\|\nabla(u\chi_\rho)\|_q \le c\|D^2 u\|_q^b\|\nabla u\|_p^{1-b}(\|\nabla u\|_q + \frac{1}{\rho}\|u\|_{L^q(\rho\le|x|\le2\rho)})$$

$$\le c\|D^2 u\|_q^b\|\nabla u\|_p^{1-b}(\|D^2 u\|_q^d\|\nabla u\|_p^{1-d} + \frac{1}{\rho^{1-\frac{3}{q}}}\|u\|_\infty)$$

$$\le c\|D^2 u\|_q^{b+d}\|\nabla u\|_p^{2-b-d} + \frac{c}{\rho^{1-\frac{3}{q}}}\|D^2 u\|_q^{2b}\|\nabla u\|_p^{2(1-b)}\,.$$

for suitable constants a, b, d.

- All the previous estimates on the convective term, more precisely on $F_{\varepsilon,\rho}$, are not formal, as we already know that the norms on the right -hand side are finite.

  Further the regularity result doesn't require any iterative procedure.

- The ideas of the proof are not dependent on the *modified* problem we are dealing with, neither on the domain $\mathbb{R}^3$. Actually we could consider the p-Stokes problem (also for $p = 2$) in a bounded domain. In this case:

  If u is a high regular solution of the p-Stokes problem (in $\mathbb{R}^3$, in Ω bounded domain with homogeneous Dirichlet boundary conditions) which is unique in the class of weak solution, then if $p > \frac{9}{5}$ there exists a high regular solution of the p-Navier-Stokes equations enjoying the same regularity properties.

Some remarks

- All the previous estimates on the convective term, more precisely on $F_{\varepsilon,\rho}$, are not formal, as we already know that the norms on the right -hand side are finite.

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THANK YOU FOR YOUR ATTENTION