An inverse problem with time periodicity for a non-newtonian liquid in an infinite pipe.

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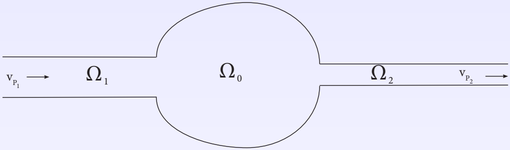
The piping system



$$\Sigma_1,\ \Sigma_2 \subset \mathbb{R}^2 \text{ compact, connected, Lipschitz}$$

$$\Omega_1 = \{x \in \mathbb{R}^3 : x_1 < 0, (x_2, x_3) \in \Sigma_1\}$$

$$\Omega_2 = \{x \in \mathbb{R}^3 : x_1 > 0, (x_2, x_3) \in \Sigma_2\}$$

$$\Omega_0 \subset \mathbb{R}^3 \text{ compact, connected, Lipschitz}$$

$$\Omega = \Omega_1 \cup \Omega_0 \cup \Omega_2.$$

The constitutive law

Non-Newtonian fluid, shear-thinning or shear-thickening

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- $S \longrightarrow$ coercivity, growth, monotonicity conditions.

The Leray's problem

- V' time derivative

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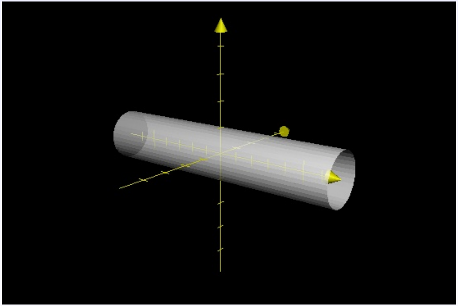
\text{asymptotic conditions for } x_1 \to \pm\infty &
\end{cases}
$$

The asymptotic conditions

Single infinite pipe of constant cross section Σ

$$\Omega = \{x \in \mathbb{R}^3 : (x_2, x_3) \in \Sigma\}$$



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Velocity field directed along the axis of the pipe and not depending on the axial coordinate

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$$p(x,t) = -\Gamma(t)x_1.$$

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$$V \cdot \nabla V = V_i \partial_i V_j = v \partial_1 v = 0.$$

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Stretching tensor

$$(DV)_{1,2} = (DV)_{2,1} = \frac{1}{2}\partial_2 v, \quad (DV)_{1,3} = (DV)_{3,1} = \frac{1}{2}\partial_3 v, \quad |DV|^2 = \frac{1}{2}|\nabla v|^2.$$

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Up to multiplicative factors use the same notation

$$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad S(\nabla v) = h(|\nabla v|^2)\nabla v.$$

Scalar problem

$$\begin{cases} v' = \mu_0 \Delta v + \Gamma(t) + \nabla \cdot S(\nabla v) & \text{in } \Sigma \times \mathbb{R} \\ \displaystyle\int_\Sigma v(x,t)\, dx = \alpha(t) & \text{T - periodic} \\ v = 0 & \text{on } \partial\Sigma \end{cases}$$

$V = v(x_2, x_3, t)e_1$ *fully developed flow*

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- Newtonian fluids

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- Time periodic: *Womersley flow*

- HAGEN G.: On the Motion of Water in Narrow Cylindrical Tubes, *Pogg. Ann.* **46** (1839), 423-442.
- POISEUILLE J.L.M.: Recherches Experimentales sur le Mouvement des Liquides dans les Tubes de Tres Petits Diameters, *C. R. Acad. Sci. Paris* **11** (1840), 961-967.
- WOMERSLEY J.R.: Method for the Calculation of Velocity, Rate of Flow and Viscous Drag in Arteries when the Pressure Gradient is Known, *J. Physiol.* **127** (1955), 553-556.

The Hagen-Poiseuille flow

Newtonian stationary case

$$\begin{cases} \mu_0 \Delta v = -\Gamma & \text{in } \Sigma \\ \displaystyle\int_\Sigma v\,dx = \alpha \\ v|_{\partial\Sigma} = 0 \end{cases}$$

pressure gradient Γ constant.

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- Direct problem: find v knowing Γ without the flow-rate constraint
- Inverse problem: find v and Γ knowing only the flow-rate α.

Direct vs inverse

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Stationary Newtonian problem (Hagen - Poiseuille)

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φ solution of

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Direct vs inverse

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$$v = \frac{\alpha}{\int_\Sigma \varphi}\varphi, \qquad \Gamma = \frac{\alpha}{\int_\Sigma \varphi}.$$

The Womersley flow - Newtonian fluids

Time periodic case: connection between the two problems more involved.

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- GALDI G.P. & ROBERTSON A.M. - *J. Math. Fluid Mech.* (2005):
  inverse problem, strong solutions, one-to-one correspondence between pressure gradient and flow rate

Leray's steady problem - Newtonian fluids

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V \cdot \nabla V = \mu_0 \Delta V - \nabla p & \text{in } \Omega \\
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V_{P_j} fully developed flows in the pipes Ω_j corresponding to the flow rate α.

Leray's steady problem - Newtonian fluids

- LADYZHENSKAYA O.A. - *Dokl. Akad. Nauk. SSSR* (1959).
- AMICK C.J. - *Ann. Sc. Norm. Sup. Pisa* (1977).
- LADYZHENSKAYA O.A., SOLONNIKOV V.A. - *Zap. Nauchn. Sem. LOMI* (1980).
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- SOLONNIKOV V.A. - *Collége de France Seminar, Vol. IV, Pitman Research Notes in Mathematics Series* (1983).
- KAPITANSKIĬ L.V. & PILECKAS K. - *Trudy Mat. Inst. Steklov* **159** (1983), 11-49.
- NAZAROV S.A. & PILECKAS K. - *Trudy Mat. Inst. Steklov* **159** (1983), 137-149.
- GALDI G.P. - *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II. Nonlinear steady problems* 1994.

Leray's steady problem - Non-Newtonian

$$\begin{cases} V \cdot \nabla V = \mu_0 \Delta V + \nabla \cdot S(DV) - \nabla p & \text{in } \Omega \\ \nabla \cdot V = 0 & \text{in } \Omega \\ \displaystyle\int_\Sigma V \cdot n \, d\sigma = \alpha & \\ V|_{\partial\Omega} = 0 & \text{on } \partial\Omega \\ \displaystyle\lim_{|x|\to\infty,\, x\in\Omega_j} V(x) - V_{P_j}(x) = 0 & j = 1, 2. \end{cases}$$

Leray's steady problem - Non-Newtonian

- E. MARUŠIĆ-PALOKA - *Math. Mod. Meth. Appl. Sci.* **10** (2000): existence and uniqueness of a weak solutions, circular section, power-law model, shear-thickening fluids.

The Leray's periodic problem - Non-Newtonian

$$\begin{cases} V' + V \cdot \nabla V = \mu_0 \Delta V + \nabla \cdot S(DV) - \nabla p & \text{in } \Omega \times \mathbb{R} \\ \nabla \cdot V = 0 & \text{in } \Omega \times \mathbb{R} \\ \displaystyle\int_\Sigma V \cdot n \, d\sigma = \alpha(t) & \text{T - periodic} \\ V|_{\partial\Omega} = 0 & \end{cases}$$

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$$V_{P_j}(x,t) = ?$$

The mathematical description: Fully developed flows

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$$\begin{cases} v' = \mu_0 \Delta v + \nabla \cdot S(\nabla v) + \Gamma(t) & \text{in } \mathfrak{D}'(0, T; W^{-1,2}(\Sigma)) \\ \displaystyle\int_\Sigma v(x, t)\, dx = \alpha(t) & \text{in } [0, T] \\ v \in L^2(0, T; W_0^{1,2}(\Sigma)) \cap L^\infty(0, T; L^2(\Sigma)) & \\ v(x, 0) = v(x, T) & \text{a.e. in } \Sigma \end{cases}$$

The weak formulation

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Test functions

$$\mathcal{V}_T = \Big\{ \phi \in C_0^\infty \left(\Sigma \times [0, T]\right) : \int_\Sigma \phi(x,t)\, dx = 0,\ \forall t \in [0, T], \\ \phi(x,0) = \phi(x, T)\ \forall x \in \Sigma \Big\}$$

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Definition (Weak solution)

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- $\displaystyle\int_\Sigma v(x,t)\, dx = \alpha(t) \qquad$ for a.e. $t \in [0, T]$

The extra stress

$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

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S locally Lipschitz

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S locally Lipschitz

$S(D) \cdot D \geq k_1 |D|^q - k_2 \qquad k_1 > 0 \qquad$ (coercivity),

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$(S(D) - S(C)) \cdot (D - C) > 0 \quad$ if $C \neq D \quad$ (monotonicity).

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$q > 1 \qquad$ shear-thinning: $1 < q < 2$, $\qquad$ shear-thickening: $q > 2$

The main result

Theorem (Shear–thinning, $1 < q < 2$)

Let $\Sigma \subset \mathbb{R}^2$ be a Lipschitz bounded domain and $\alpha \in C^1([0,T])$ such that $\alpha(0) = \alpha(T)$. There exist a unique weak solution v and a distribution Γ such that the pair (v, Γ) solves the equation in a distributional sense.

$$v \in L^\infty(0,T; W_0^{1,2}(\Sigma)) \cap L^\infty(\Sigma \times [0,T]),\ v' \in L^2(\Sigma \times (0,T)),\ \Gamma \in L^2(0,T)$$

and there exists a constant K depending on $\Sigma, k_1, k_3, \mu_0, q$ such that

$$\|v(x,t)\|_\infty^2 + \int_0^T \|\nabla v(\cdot,t)\|_2^2\, dt \le K\|\alpha\|_{C^1}$$

Moreover the map $\Phi : C^1([0,T]) \longrightarrow L^2(\Sigma \times (0,T))$ defined by $\Phi(\alpha) = v$ is continuous.

The main result

Theorem (Shear–thickening, $q > 2$)

Let $\Sigma \subset \mathbb{R}^2$ be a Lipschitz bounded domain and $\alpha \in C^1([0,T])$ such that $\alpha(0) = \alpha(T)$. There exist a unique weak solution v and a distribution Γ such that the pair (v, Γ) solves the equation in a distributional sense.

$$v \in L^\infty(0,T; W_0^{1,2}(\Sigma)) \cap L^\infty(\Sigma \times [0,T]),\ v' \in L^2(\Sigma \times (0,T)),\ \Gamma \in L^{q'}(0,T)$$

and there exists a constant K depending on $\Sigma, k_1, k_3, \mu_0, q$ such that

$$\|v(x,t)\|_\infty^2 + \int_0^T \|\nabla v(\cdot,t)\|_q^q \, dt \le K\left(\|\alpha\|_{C^1} + k_2\right)$$

Moreover the map $\Phi : C^1([0,T]) \longrightarrow L^2(\Sigma \times (0,T))$ defined by $\Phi(\alpha) = v$ is continuous.
If $k_2 = 0$ then $v \in L^\infty(0,T; W_0^{1,q}(\Sigma))$ and $\|v(x,t)\|_\infty \le c\|\alpha\|_{C^1}$.

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The L^∞ control

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$$\begin{cases} V' + V\cdot\nabla V = \mu_0 \Delta V + \nabla\cdot S(DV) - \nabla p & \text{in } \Omega\times\mathbb{R} \\ \nabla\cdot V = 0 & \text{in } \Omega\times\mathbb{R} \\ \displaystyle\int_\Sigma V\cdot n\, d\sigma = \alpha(t) & \text{T - periodic} \\ V|_{\partial\Omega} = 0 & \\ \displaystyle\lim_{|x|\to\infty,\, x\in\Omega_j} V(x,t) - V_{P_j}(x,t) = 0 & j = 1,2,\ t\in\mathbb{R} \end{cases}$$

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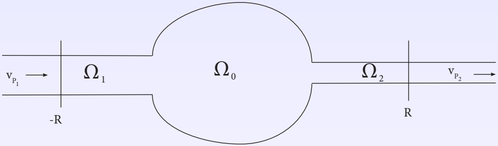

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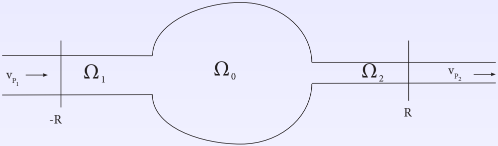

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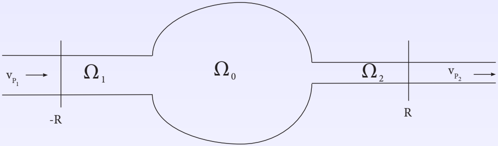

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- continuos dependence by the flow-rate;
- L^∞ estimate: parabolic regularity.