JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Asymptotic Preserving Schemes for Weakly Compressible Flows

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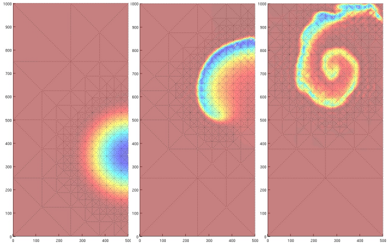
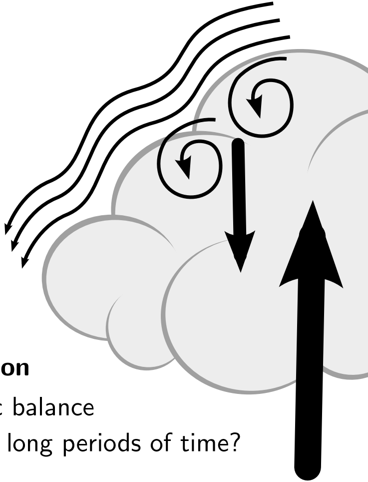
Application





Meteorology: Cloud Simulation

- Gravity induces hydrostatic balance
- How do clouds evolve over long periods of time?

Multiscale phenomena of oceanographical, atmospherical flows

-wave speeds differ by several orders: $\|\mathbf{u}\| << c \Rightarrow \mathbf{M}, \mathbf{Fr} := \frac{\|\mathbf{u}\|}{\mathbf{c}} << \mathbf{1}$

-typically $\mathbf{Fr} \approx \mathbf{10^{-2}}$

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$$\frac{\max(|u| + c, |v| + c)\Delta t}{\Delta x} \leq 1$$

$$\max\left(\left(1 + \frac{1}{\mathbf{Fr}}\right)\sqrt{u^2 + v^2}\right)\frac{\Delta t}{\Delta x} \leq 1$$

- number of time steps $\mathcal{O}(1/\mathbf{Fr})$

-low Mach / low Froude number problem

[Bijl & Wesseling ('98), Klein et al.('95, '01), Meister ('99,01), Munz &Park ('05), Degond et al. ('11) ...]

Cancelation problem

- Sesterhenn et al. ('99)
- h ... water depth in the shallow flow
- "pressure term" $\frac{1}{2\mathbf{Fr}^2}\nabla h^2 \Longrightarrow$
- $h_L, h_R = h_L + \delta h, \quad \delta h \approx \mathcal{O}(\mathbf{Fr}^2)$
- round off errors can yield the cancelation effects

$$
\begin{aligned}
h_R^2 - h_L^2 &= ((h_L^2 + 2h_L\delta h + \delta h^2)(1+\epsilon_1) - h_L^2)(1+\epsilon_2) \\
h_R^2 - h_L^2 &= \delta h \left[(2h_L + \delta h_L) + \epsilon_1 \frac{(h_L + \delta h)^2}{\delta h} + h.o.t.\right]
\end{aligned}
$$

- leading order error in the pressure term $\approx \frac{1}{\mathbf{Fr}^2}\epsilon_1 \mathcal{O}(\frac{1}{\mathbf{Fr}^2}) \approx \mathcal{O}(1) \quad !$

Accuracy problem

- numerical viscosity of upwind methods depends on $\mathbf{Fr}$
- truncation error grows as $\mathbf{Fr} \to \mathbf{0}$ [Guillard, Viozat ('99), Rieper ('10)]

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- **AIM:**
  - reduce adverse effect of $1 + 1/\mathbf{Fr}$
  - large time step scheme: Δt does not depends on $\mathbf{Fr}$
  - efficient scheme for advection effects
  - stability and accuracy of the scheme is independent on $\mathbf{Fr}$

Asymptotic preserving schemes

Goal: *Derive a scheme, which gives a consistent approximation of the limiting equations for* $\varepsilon = \mathbf{Fr} \to \mathbf{0}$

[S.Jin&Pareschi('01), Gosse&Toscani('02), Degond et al.('11), ...]

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- to illustrate the idea: shallow water eqs.

- $z = h + b$, h - water depth, z - mean sea level to the top surface, b - mean sea level to the bottom ($b \leq 0$)

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$$\partial_t z + \partial_x m + \partial_y n = 0$$

$$\partial_t m + \partial_x (m^2/(z-b)) + \partial_y (mn/(z-b)) + \frac{1}{2\mathbf{Fr}^2}\partial_x (z^2) = \frac{1}{\mathbf{Fr}^2} b \partial_x z$$

$$\partial_t n + \partial_x (mn/(z-b)) + \partial_y (n^2/(z-b)) + \frac{1}{2\mathbf{Fr}^2}\partial_y (z^2) = \frac{1}{\mathbf{Fr}^2} b \partial_y z$$

Asymptotic expansion

-rigorous analysis [Klainerman & Majda ('81), Feireisl & Novotný (2009, 2013)]
-formally: $(\varepsilon = \mathbf{Fr})$

$$z^{\varepsilon}(x,t;\varepsilon) = z^{(0)}(x,t) + \varepsilon z^{(1)}(x,t) + \varepsilon^2 z^{(2)}(x,t)$$
$$u^{\varepsilon}(x,t;\varepsilon) = u^{(0)}(x,t) + \varepsilon u^{(1)}(x,t) + \varepsilon^2 u^{(2)}(x,t)$$

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plug into the SWE $\Longrightarrow$

$$z^{(0)} = z^{(0)}(t); \quad \partial_x(h^{(0)} + b) = 0$$
$$\partial_x h^{(1)} = 0$$
$$\partial_t z^{(0)} = \partial_x(h^{(0)} u^{(0)}) \equiv \partial_x m^{(0)}$$
$$\partial_t m^{(0)} + \partial_x(h^{(0)}(u^{(0)})^2) + h^{(0)}\partial_x z^{(2)} = 0$$

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limiting system as $\varepsilon \to 0$ $(\partial_t b = 0)$

$$h^{(0)}(x) = b(x) + const. \qquad (1)$$
$$\partial_t h^{(0)} = \partial_x m^{(0)}$$
$$\partial_t u^{(0)} + u^{(0)} \partial_x u^{(0)} + \partial_x z^{(2)} = 0$$

Does a numerical scheme give a consistent approximation of (1) ?

Time discretization

Key idea:
- semi-implicit time discretization: splitting into the linear and nonlinear part
- linear operator modells gravitational (acoustic) waves are treated implicitly
- rest nonlinear terms are treated explicitly

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$$\frac{\partial \mathbf{w}}{\partial t} = -\nabla \cdot \mathbf{F}(\mathbf{w}) + \mathbf{B}(\mathbf{w}) \equiv \mathcal{L}(\mathbf{w}) + \mathcal{N}(\mathbf{w})$$

$$\mathbf{w} = (z, m, n)^T, \quad z = h + b;\ b < 0$$

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$$\mathcal{L}(\mathbf{w}) := \begin{pmatrix} -\partial_x(m) - \partial_y(n) \\ \frac{b}{\mathbf{Fr}^2}\partial_x z \\ \frac{b}{\mathbf{Fr}^2}\partial_y z \end{pmatrix}$$

- $\mathcal{L}$: spatially varying linear system

$$\mathbf{w}_t + \mathbf{A}_1(b)\mathbf{w}_x + \mathbf{A}_2(b)\mathbf{w}_y = 0$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\mathbf{Fr}^2} b(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{\mathbf{Fr}^2} b(x,y) & 0 & 0 \end{pmatrix} \quad \Longrightarrow E_\Delta^L$$

Multi-d evolution operator in [Arun, M.L., Kraft, Prasad (2009)]

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Multi-d evolution operator in [Arun, M.L., Kraft, Prasad (2009)]

- REST: nonlinear system $\mathcal{N}$

$$z_t = 0$$
$$m_t + (m^2/(z-b))_x + \frac{1}{2\mathbf{Fr}^2}(z^2)_x + (mn/(z-b)))_y = 0$$
$$n_t + (mn/(z-b))_x + (n^2/(z-b))_y + \frac{1}{2\mathbf{Fr}^2}(z^2)_y = 0 \quad \Longrightarrow E_\Delta^N$$

E_Δ^N is the evolution along the characteristics as by Prof. Tabata, Prof. Notsu

Characteristic scheme for $\mathcal{N}$

- characteristic curves: $\mathbf{x}(t) = (x(t), y(t))$ determined by $\dfrac{dx}{dt} = u,\ \dfrac{dy}{dt} = v.$
- time evolution:

$$\frac{Dm}{Dt} = (u^2 - \frac{z}{\mathbf{Fr^2}})z_x + uvz_y - u(m_x + n_y) - u(ub_x + vb_y)$$
$$\frac{Dn}{Dt} = (v^2 - \frac{z}{\mathbf{Fr^2}})z_y + uvz_x - v(m_x + n_y) - v(ub_x + vb_y)$$

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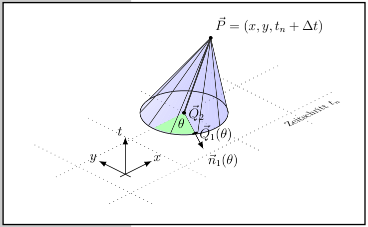
- approximation:

$$m^{n+1}(P) = m + \Delta t \left\{ \left[u^2 - \frac{z}{\mathbf{Fr^2}} \right] z_x + uvz_y - u(m_x + n_y) - u(ub_x + vb_y) \right\} (\mathbf{x}(t^n), t^n)$$
$$n^{n+1}(P) = n + \Delta t \left\{ \left[v^2 - \frac{z}{\mathbf{Fr^2}} \right] z_y + uvz_x - v(m_x + n_y) - v(ub_x + vb_y) \right\} (\mathbf{x}(t^n), t^n),$$

where the characteristic directions are frozen at t^n

$$\frac{dx}{dt} = u(\mathbf{x}(P), t^n), \quad \frac{dy}{dt} = v(\mathbf{x}(P), t^n)$$



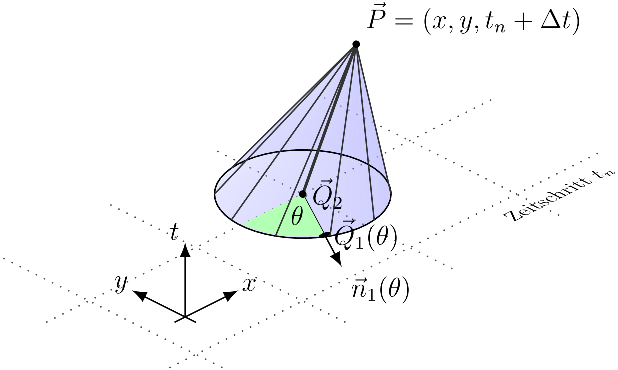
2

Bicharacteristic scheme for linear operator $\mathcal{L}$

Wave propagation for the hyperbolic balance laws

Information travels along **bicharacteristic curves**

Integration along each curve + averaging over the cone mantle yields integral representation for the solution at the pick of the cone



M. Lukáčová-Medvid'ová, K.W. Morton, and Gerald Warnecke.
Finite volume evolution Galerkin methods for hyperbolic systems.
J. Sci. Comp. 2004.

Short derivation of integral representation

Step 1: Formulation as a quasilinear system

$$\partial_t \mathbf{w} + \underline{A}_1(\mathbf{w})\,\partial_x \mathbf{w} + \underline{A}_2(\mathbf{w})\,\partial_y \mathbf{w} = \mathbf{s}(\mathbf{w})$$

with matrices $\underline{A}_1(\mathbf{w})$, $\underline{A}_2(\mathbf{w})$ and source term $\mathbf{s}(\mathbf{w})$.
Freeze Jacobians $\underline{A}_1$, $\underline{A}_2$ if they depend on $\mathbf{w}$

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Freeze Jacobians $\underline{A}_1$, $\underline{A}_2$ if they depend on $\mathbf{w}$
Step 2: Quasi-diagonalization
Let $\underline{R}$ denote the right eigenvectors of $\underline{P} = \underline{A}_1 n_x + \underline{A}_2 n_y$.
Change of variables $\mathbf{v} = \underline{R}^{-1}\mathbf{w}$ yields a quasi-diagonal system

$$\boxed{\partial_t \mathbf{v} + \mathrm{diag}(\underline{B}_1)\partial_x \mathbf{v} + \mathrm{diag}(\underline{B}_2)\partial_y \mathbf{v} = \mathbf{S} + \mathbf{r}}$$

where $\underline{B}_{1/2} := \underline{R}^{-1}\underline{A}_{1/2}\underline{R}$

$\mathbf{r} := \underline{R}^{-1}\mathbf{s}(\mathbf{w})$, $\quad \mathbf{S} := -(\underline{B}_1 - \mathrm{diag}(\underline{B}_1))\partial_x \mathbf{v} - (\underline{B}_2 - \mathrm{diag}(\underline{B}_2))\partial_y \mathbf{v}$.

Short derivation of integral representation (cont'd)

Step 3: Averaging over the cone mantle
For every direction $[n_x, n_y] = [\cos(\theta), \sin(\theta)]$ with $\theta \in [0, 2\pi]$:

The system

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can be solved exactly.

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Temporal integration over $[t_n, t_n + \tau]$ and averaging over the *wave-front*, i. e. $\theta \in [0, 2\pi]$, yields an integral representation for $[x, y, t_n + \tau]$.

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Step 4: Back transform to primitive variables
Change of variables $\mathbf{w} = \underline{R}\mathbf{v}$.

Back to our linear subsystem $\mathcal{L}$

$$\boxed{\partial_t \mathbf{w} - \mathcal{L}(\mathbf{w}) = 0}$$

$$\mathbf{w} := \begin{pmatrix} z \\ m \\ n \end{pmatrix} \qquad \mathcal{L}(\mathbf{w}) := \begin{pmatrix} -\mathrm{div}(m,n)^T \\ \frac{b}{\mathbf{Fr^2}} \partial z / \partial x \\ \frac{b}{\mathbf{Fr^2}} \partial z / \partial y \end{pmatrix}$$

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Quasilinear form

$$\boxed{\mathbf{w}_t + \mathbf{A}_1\mathbf{w}_x + \mathbf{A}_2\mathbf{w}_y = 0}$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-1}{\mathbf{Fr}^2}b(x,y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{-1}{\mathbf{Fr}^2}b(x,y) & 0 & 0 \end{pmatrix}$$

eigenstructure: $\lambda_1 = -a,\ \lambda_2 = 0, \lambda_3 = a, \quad a := \frac{\sqrt{-b}}{\mathbf{Fr}}$

Exact integral representation

$$
\begin{aligned}
(az)(P) \quad = \quad & \frac{1}{2\pi}\int\limits_0^{2\pi} \{az - m\cos\theta - n\sin\theta\}\left(\mathbf{x}^1(t^n;\omega), t^n\right)\, d\omega \\
& - \frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_{t^n}^{t^{n+1}} \left\{azD_\theta^+[s] + D_\theta^-[ma]\sin\theta - D_\theta^-[na]\cos\theta\right\}\left(\mathbf{x}^1(t;\omega), t\right)
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where

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and the **bicharacteristics** are:

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\mathbf{x}^1(t;\omega), t) = (x^1(t,\theta), y^1(t,\theta)), \quad \theta(t^{n+1}) = \omega
$$

$$
\frac{dx^1(t)}{dt} = -a(\mathbf{x}^1)\cos(\theta^1), \quad \frac{dy^1(t)}{dt} = -a(\mathbf{x}^1)\sin(\theta^1), \frac{d\theta^1(t)}{dt} = -D_{\theta^1}^-[a](\mathbf{x}^1)
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$$

Expression intractable for a numerical scheme – Simplify!

- temporal integrals by the rectangle rule
- to avoid large sonic circles use local evolution for $\tau \to 0$ [Sun & Ren ('09)]
- in practice τ fixed, $\dfrac{a\tau}{\Delta x} \approx 0.01$

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$$\boxed{(az)(P) = \frac{1}{2\pi} \int_0^{2\pi} \left\{ az^n - m^n \cos\omega - n^n \sin\omega - zD_\omega^+[a] \right\} (Q_\tau)\, d\omega,}$$

where

$$Q_\tau(\omega) = \begin{pmatrix} x_p + \tau\bar{a}\cos(\omega) \\ y_p + \tau\bar{a}\sin(\omega) \end{pmatrix}, \quad \bar{a} \equiv a(P)$$

$$D_\theta^+[a] := \cos(\theta)a_x + \sin(\theta)a_y$$

- The resulting operator is a predictor for the cell-interface values of fluxes in the Finite Volume update
- **The operator is asymptotic preserving !**
- It can be shown to be of order $\mathcal{O}(\tau^2)$.

Semi-implicit time discretization

$$\mathbf{w}^{n+1} = \mathbf{w}^n + \frac{\Delta t}{2}\left[\mathcal{L}(\mathbf{w}^n) + \mathcal{L}(\mathbf{w}^{n+1})\right] + \Delta t \mathcal{N}(\mathbf{w}^{n+1/2})$$

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- spatial discretization: Finite Volume update using flux differences
+ EG-evolution operator to evaluate fluxes at interfaces (multi-d Riemann solver)

$$\mathcal{L}(\mathbf{w}^\ell) = \frac{1}{\Delta x_k}\sum_{k=1}^{2} \delta_{x_k}(\mathbf{F}_L(\mathbf{E}_0(\mathbf{w}^\ell))), \quad \ell = n, n+1$$

$$\mathcal{N}(\mathbf{w}^{n+1/2}) = \frac{1}{\Delta x_k}\sum_{k=1}^{2} \delta_{x_k}(\mathbf{F}_N(\mathbf{E}_{\Delta t/2}(\mathbf{w}^n)))$$

$$\delta_x f_i \equiv f_{i+1/2} - f_{i-1/2}$$

AP property for the semi-implicit time discretization scheme

semi-discrete scheme:

$$z^{n+1} = z^n - \frac{\Delta t}{2}\left[m_x^{n+1} + m_x^n\right] \tag{2}$$

$$m^{n+1} = m^n + \frac{\Delta t}{2}\left[\frac{b}{\varepsilon^2} z_x^{n+1} + \frac{b}{\varepsilon^2} z_x^n\right] - \Delta t\left[\frac{1}{2\varepsilon^2}(z_x^{n+1/2})^2 + (mu)_x^{n+1/2}\right] \tag{3}$$

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- we assume that $z^n, z^{n+1/2}, m^n, m^{n+1/2}$ approximate the limiting eqs. (1)
• Eq.(3) yields for ε^{-2}

$$\frac{b}{2}\left(z_x^{(0),n+1} + z_x^{(0),n}\right) - \frac{1}{2} z^{(0),n+1/2} z_x^{(0),n+1/2} = 0$$

$\Longrightarrow z^{(0),n+1}(x) = const.$

$$z^{n+1} = z^n - \frac{\Delta t}{2}\left[m_x^{n+1} + m_x^n\right] \qquad (2)$$

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- Eq.(2) yields for ε^0 consistent approx. of

$$\partial_t z^{(0)} = -\partial_x m^{(0)}$$

- periodic, slip BC $\Longrightarrow z^{(0),n+1}(x) = z^{(0),n}(x)$
- $m^{(0),n+1}(x) = const.$

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- Eq.(3) yields for ε^0 terms :

$$\begin{aligned} m^{(0),n+1} &= m^{(0),n} - \frac{\Delta t}{2}\Big[-b\Big(z_x^{(2),n+1} + z_x^{(2),n}\Big) \\ &\qquad + z^{(0),n+1/2} z_x^{(2),n+1/2} - (mu)_x^{(0),n+1/2}\Big] \\ &\approx m^{(0),n} - \Delta t\Big[h^{(0),n+1/2} z_x^{(2),n+1/2} - (hu^2)^{(0),n+1/2}\Big] \end{aligned}$$

- which is a consistent approx. of the momentum eq.

$$\partial_t u^{(0)} = u^{(0)} \partial_x u^{(0)} + \partial_x z^{(2)}$$

Example1: Travelling vortex

- Ricchiuto & Bollermann ('09)

$$h(x,y,0) = 110 + \begin{cases} \left(\frac{\varepsilon\Gamma}{\omega}\right)^2 (k(\omega r_c) - k(\pi)) & \text{if } \omega r_c \leq \pi \\ 0 & \text{else} \end{cases} \quad (4)$$

$$u(x,y,0) = 0.6 + \begin{cases} \Gamma(1+\cos(\omega r_c))(0.5-y) & \text{if } \omega r_c \leq \pi \\ 0 & \text{else} \end{cases}$$

$$v(x,y,0) = \begin{cases} \Gamma(1+\cos(\omega r_c))(x-0.5) & \text{if } \omega r_c \leq \pi \\ 0 & \text{else} \end{cases}$$

$$r_c = \|\mathbf{x} - (0.5, 0.5)\|, \quad \Gamma = 1.5, \quad \omega = 4\pi$$

$$k(r) = 2\cos(r) + 2r\sin(r) + \frac{1}{8}\cos(2r) + \frac{r}{4}\sin(2r) + \frac{3}{4}r^2 \quad (5)$$

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- rotating vortex initially at $(0.5, 0.5)$
- transported with $\mathbf{u_{ref}} = (0.6, 0)$
- period $T = 5/3$
- exact solution: $w(x,y,t) = w(x - t/T, y, 0)$; $w = (z, m, n)^T$

Experimental error analysis

- First order method; $\varepsilon = 0.8$ and 0.05

$\varepsilon = 0.8$, $CFL_u = 0.45$, $CFL \approx 0.9$, $T = 0.1$

N	L^1-error in z	EOC	L^1-error in m	EOC	L^1-error in n	EOC
20	0.21019		0.50860		0.44681	
40	0.14303	0.55539	0.29634	0.77926	0.25680	0.7990
80	0.08408	0.76648	0.16136	0.87697	0.13759	0.9002
160	0.04578	0.87704	0.08455	0.93239	0.07160	0.9422

$\varepsilon = 0.05$, $CFL_u = 0.45$, $\mathbf{CFL \approx 7.25}$, $T = 0.1$

N	L^1-error in z	EOC	L^1-error in m	EOC	L^1-error in n	EOC
20	0.00408		1.18800		1.16980	
40	0.00320	0.34894	0.87983	0.43328	0.87707	0.41547
80	0.00210	0.60779	0.57048	0.62504	0.57483	0.60955
160	0.00123	0.77580	0.33396	0.77250	0.33783	0.76682

Experimental error analysis

- Second order method; $\varepsilon = 0.8$ and 0.01

$\varepsilon = 0.8$, $CFL_u = 0.9$, $CFL \approx 1.75$, $T = 0.1$

N	L^1-error in z	EOC	L^1-error in m	EOC	L^1-error in n	EOC
20	0.06944		0.17415		0.18840	
40	0.01584	2.1323	0.03977	2.1306	0.05377	1.8089
80	0.00327	2.2766	0.00906	2.1349	0.01609	1.7407
160	0.00085	1.9419	0.00230	1.9780	0.00445	1.8534

$\varepsilon = 0.01$, $CFL_u = 0.9$, $\mathbf{CFL \approx 69}$, $T = 0.1$

N	L^1-error in z	EOC	L^1-error in m	EOC	L^1-error in n	EOC
20	5.07e-4		1.14180		1.17160	
40	1.23e-4	2.0472	0.35999	1.6653	0.36423	1.6855
80	3.20e-5	1.9363	0.07283	2.3054	0.07454	2.2888
160	8.25e-6	**1.9569**	0.01347	**2.4348**	0.01434	**2.3781**

Well-balancing

• preserve EXACTLY equilibrium states of the dynamical system for given discrete data

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Theorem

The IMEX type large time step schemes are well-balanced for the lake at rest uniformly with respect to the Froude number $\varepsilon > 0$.

$z =$ const., $m = 0 = n \Longrightarrow \nabla \cdot \mathcal{F}_{NL}(w^n) = 0$

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$$\begin{aligned} w^{n+1} + \Delta t \nabla \cdot (\mathcal{F}_L - K)(w^{n+1}) &= w^n - \Delta t \nabla \cdot \mathcal{F}_{NL}(w^n) \\ w^{n+1} + \Delta t \Phi(w^{n+1}) &= w^n \end{aligned} \tag{6}$$

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$$w^{n+1} + \Phi(w^{n+1}) \equiv w^{n+1} + \Delta t \begin{bmatrix} \nabla \cdot \mathbf{m}^{n+1} \\ -gb\nabla z^{n+1} \end{bmatrix}$$

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$\rightarrow$ (6), (7) lake at rest is a solution of the IMEX-type semi-discrete equation

Lemma

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz-continuous domain and the bottom topography $b \in W^{1,\infty}(\Omega)$, $b \leq 0$. Then the following problem

$$w + \Delta t \Phi(w) = 0 \tag{7}$$

has a unique solution $w \in H^1(\Omega)$, provided

$$\int_{\partial\Omega} bz\partial_\nu z \, ds \geq 0. \tag{8}$$

Proof of lemma

- for the linear part we have: $z = -\Delta t \nabla \cdot \mathbf{m}$ and $\mathbf{m} = \Delta t g b \nabla z$

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-plugging m into z-equation: elliptic eigenvalue problem

$$-\nabla \cdot (b \nabla z) = \lambda z, \quad \lambda := \frac{1}{g \Delta t^2} > 0. \tag{9}$$

$$0 \le \lambda \|z\|^2_{L^2(\Omega)} = \langle z, -\nabla (b \nabla z) \rangle_{L^2(\Omega)} = \int_\Omega b \nabla z \cdot \nabla z \, d\mathbf{x} - \int_{\partial\Omega} b z \partial_\nu z \, ds \le 0.$$

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$\rightarrow z = 0$ and $m = 0 = n$ ✓

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- Do linear fluxes balance the source term discretization ?

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- discretization of $\mathcal{F}_L(w^{n+1})$

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-discretization of $K(w^{n+1})$

$$\int_{C_{ij}} K(w^{n+1}) dx \approx -\frac{1}{\varepsilon^2} \sum_{k=-1}^{1} \gamma_k (\mu_x z^{*,n+1}_{i,j+\frac{k}{2}}) \, (\delta_x b_{i,j+\frac{k}{2}}) \tag{11}$$

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But for $z = \text{const.}$ it holds

$$(\mu_x z^{*,n+1}_{i,j+\frac{k}{2}}) \, (\delta_x b_{i,j+\frac{k}{2}}) = \delta_x (z^{*,n+1} b)_{i,j+\frac{k}{2}} \qquad \checkmark$$

Application to atmospheric flow

Compressible Euler equations

$$\begin{aligned}
\partial_t \rho' &+ \nabla \cdot (\rho \mathbf{u}) &&= 0 \\
\partial_t (\rho \mathbf{u}) &+ \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p' \, \mathrm{Id}) &&= -\rho' g \mathbf{k} \\
\partial_t (\rho\theta)' &+ \nabla \cdot (\rho\theta \mathbf{u}) &&= 0
\end{aligned}$$

with background state $\bar{p}$, $\bar{\rho}$, $\bar{\theta}$ in **hydrostatic balance**

$$\boxed{\partial_y \bar{p} = -\bar{\rho} g}$$

State variables: $\mathbf{w} = [\rho', \rho u, \rho v, (\rho\theta)']^{\mathsf{T}}$

- Potential temperature $\theta := T/\pi$
- *Exner-pressure* $\pi(y) := 1 - \frac{gy}{c_p \theta}$

Exact evolution operator for the linear subsystem

linear part for the Euler system

$$\partial_t \mathbf{w} + \mathcal{L}(\mathbf{w}) = 0$$

$$\mathbf{w} := \begin{pmatrix} \rho' \\ \rho u \\ \rho v \\ (\rho\theta)' \end{pmatrix} \qquad \mathcal{L}(\mathbf{w}) := \begin{pmatrix} \mathrm{div}(\rho\mathbf{u}) \\ \partial p'/\partial x \\ \partial p'/\partial y + g\rho' \\ \mathrm{div}(\overline{\theta}\rho\mathbf{u}) \end{pmatrix}$$

- linearized version of p': $p' = \dfrac{c_p\overline{p}}{c_v\overline{\rho}\overline{\theta}}(\rho\theta)'$

Exact evolution operator for the linear subsystem

$$\partial \mathbf{w} + \mathbf{A}_1 \mathbf{w}_x + \mathbf{A}_2 \mathbf{w}_y = S(\mathbf{w})$$

$$\mathbf{A}_1 = \begin{pmatrix} 0 & \bar{\theta} & 0 & 0 \\ \tilde{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \mathbf{A}_2 = \begin{pmatrix} 0 & 0 & \bar{\theta} & 0 \\ 0 & 0 & 0 & 0 \\ \tilde{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $\bar{\theta} = \bar{\theta}(y)$, $\tilde{\gamma} = \dfrac{c_p \bar{p}}{c_v \bar{\rho} \bar{\theta}}$

eigenstructure: $\lambda_1 = -a,\ \lambda_{2,3} = 0, \lambda_4 = a, \quad a := \sqrt{\tilde{\gamma}\bar{\theta}}$

Note: in the non-dimensional form $\tilde{\gamma} = \dfrac{\gamma R}{\mathbf{M}^2}$

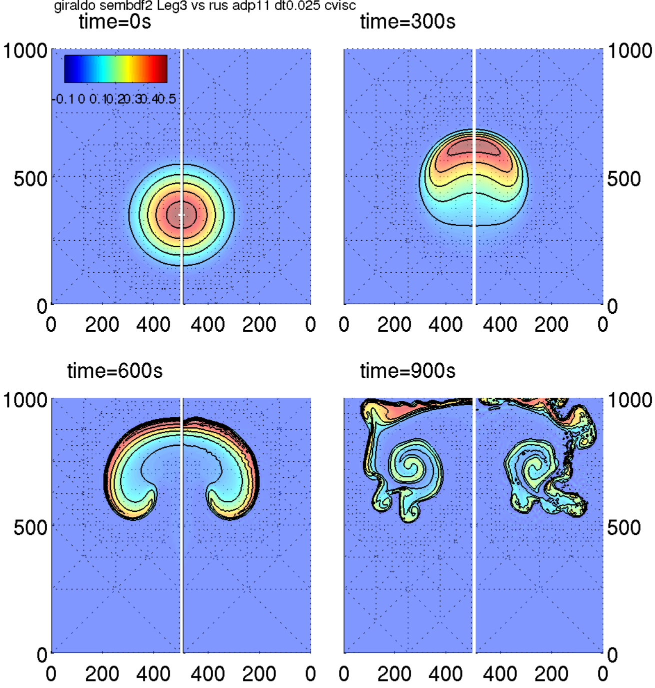
Test 1: rising warm air bubble

- bubble with a cosine profile in $\theta = \overline{\theta} + \theta'$:

$$\theta' = \begin{cases} 0 & r > r_C,\ r = \|\mathbf{x} - \mathbf{x}_C\| \\ 0.25[1 + \cos(\pi_c r / r_C)] & r \leq r_C \end{cases}$$

$$\mathbf{x}_C = (500, 350),\ r_C = 250m,\ \overline{\theta} = 300K,$$
$$\mathbf{x} \in [0, 1000]^2,\ t \in [0, 700]$$

• in the momentum and energy eqs. regularized viscous terms with a small viscosity μ are added
$\mu = 0.1 m^2/s$


giraldo sembdf2 Leg3 vs rus adp11 dt0.025 cvisc
time=0s
time=300s
time=600s
time=900s
-0.1 0 0.1 0.2 0.3 0.4 0.5
1000
500
0
0 200 400 400 200 0

Test 2: small cold bubble on the top of large warm bubble

- **Robert test** (1993)

- both bubbles: a Gaussian profile
- warm air bubble: amplitude of 0.5 K
- cold air bubble: amplitude 0.17 K
- $\mu = 0.1 m^2/s$