On Liouville problems for the planer Navier-Stokes equations with the no-slip boundary condition

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Equations

We study a backward solution to the Navier-Stokes equations in the half plane

$$\partial_t u + \nabla \cdot (u \otimes u) - \Delta u + \nabla p = 0, \quad \mathrm{div}\ u = 0 \quad \text{in} \ \ (-\infty, 0) \times \mathbb{R}^2_+ \tag{1}$$

subject to the no-slip boundary condition

$$u = 0 \qquad\qquad \text{on} \ \ (-\infty, 0) \times \partial\mathbb{R}^2_+. \tag{2}$$

Here $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$,
$u(t, x) = (u_1(t, x), u_2(t, x)) \in \mathbb{R}^2$ and $p(t, x) \in \mathbb{R}$.

As is well known, in the study of evolution equations the Liouville problem for bounded *backward* solutions plays an important role in obtaining an a priori bound of *forward* solutions through a suitable scaling argument called a blow-up argument.

Our goal: To solve the Liouville problem for (1) - (2), that is, the nonexistence of nontrivial bounded global solutions to (1) - (2).

Liouville type results

Whole space

Liouville type theorem
If u is a bounded mild solution of NS in $\mathbb{R}^2 \times (-\infty, 0)$, it must be a constant solution.

- Koch-Nadrashvilli-Seregin-Sverak, 2007
  (based on integral estimates)
- Giga-miura, 2011
  (based on strong Maximum principle of vorticity equation)

Under boundary conditions

(1)Slip boundary condition

The rescaled two-dimensional vorticity equations still enjoy the maximum principle since there is no vorticity production from the boundary.

(2)Dirichlet boundary condition

Vorticity is expected to be created on the boundary. There is even a counterexample of Poiseuille type flow (Giga, 2011)

$$u = (u_1(t, x_3), 0, 0), \quad p(t, x_1) = -x_1 f(t), \tag{3}$$

solves the initial-boundary value problem provided that u_1 solves the heat equation

$$\begin{aligned} \partial_t u_1 - \partial_3^2 u_1 = f(t) \quad &\text{in } (0, T) \times \{x_3 > 0\}, \\ u_1 = 0 \quad &\text{on } (0, T) \times \{x_3 = 0\}. \end{aligned}$$

with some f depending only on time.

Liouville type result under the Dirichlet BC

Theorem 1.1 (Giga-H-Maekawa 2013)

Let (u, p) *be the solution to* (1)-(2) *satisfying the following conditions.*

(C1) $\sup_{-\infty<t<0} \left(\|u(t)\|_{C^{2+\mu}} + \|\partial_t u(t)\|_{C^{\mu}} \right) < \infty$ *for some* $\mu \in (0,1)$.

(C2) $p = p_F + p_H$

(C3) $\sup_{-\infty<t<0} (-t)^{1/2} \|u(t)\|_\infty < \infty$.

(C4) $\omega \geq 0$ *in* $(-\infty, 0) \times \mathbb{R}^2_+$.

Then u is identically zero.

Here p_F is the solution of

$$\begin{cases} -\Delta p_F = \sum_{i,j=1}^{3} \partial_i\partial_j(u_i u_j), F = -u\otimes u & \text{in } \partial\mathbb{R}^2 \\ \frac{\partial p_F}{\partial n} = 0 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{4}$$

such that

$$\|p_F\|_{BMO} \le C\|F\|_\infty, \qquad \|\nabla p_F\|_{C^\mu} \le C\|F\|_{C^{1+\mu}}, \qquad 0<\mu<1. \tag{5}$$

p_H: harmonic pressure; the solution of

$$\begin{cases} \Delta p_H = 0 & , \quad \text{in } \partial\mathbb{R}^2 \\ \partial_2 p_H = \partial_1\omega & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{6}$$

such that

$$\sup_{x\in\mathbb{R}^2_+} x_2|\nabla p_H(x)| \le C\|\omega\|_\infty, \tag{7}$$

Geometric regularity criterion

Theorem 1.2 (Giga-Miura 2011 (simplest form))

Let u be a spatially bounded mild solution for NS in $(-1,0)\times\mathbb{R}^3$. Assume that blow-up at zero is type I, i.e.

$$\sup_{x,t} |u(t,x)|(-t)^{1/2} < \infty.$$

If the vorticity direction $\xi=\omega/|\omega|$ is uniformly continuous in space in the sense that
$(\mathrm{CA})\ |\xi(t,x)-\xi(t,y)|\leq\eta(|x-y|)$,
for $(t,x),(t,y)\in\Omega_d=\{(t,x)\mid|\omega(t,x)|>d\}$ for some $d>0$ and η a modulus of continuity. Then u does not blow-up at $t=0$.

Problem: Can we prove similar result in half space?

Geometric regularity criterion in the half space

Theorem 1.3 (Giga-H-Maekawa 2013)

Let u be a spatially bounded mild solution for NSD in $(-1, 0) \times \mathbb{R}^3_+$ *.*
If u is type I near $t = 0$ *and u satisfies* (CA)*, then u is bounded up to* $t = 0$*.*

Geometric regularity criterion in the half space

Theorem 1.3 (Giga-H-Maekawa 2013)

Let u be a spatially bounded mild solution for NSD in $(-1,0)\times\mathbb{R}^3_+$. If u is type I near $t=0$ and u satisfies (CA), then u is bounded up to $t=0$.

Note: (1) - (2) is naturally derived from a blow-up argument for the three-dimensional Navier-Stokes equations in the half space.

Lemma 1.4 (Decay in normal)

Under the conditions (C1), (C2), *and* (C3) *of Theorem 1.1 the vorticity ω satisfies*

$$\sup_{(t,x)\in(-\infty,0)\times\mathbb{R}^2_+} x_2^{1+\theta}|\omega(t,x)| < \infty \qquad \text{for all} \quad \theta \in (0,1). \tag{8}$$

Lemma 1.5 (Biot-Savart law)

Under the conditions (C1), (C2), *and* (C3) *of Theorem 1.1 the velocity u is represented as*

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2_+} \Big(\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x-y^*)^\perp}{|x-y^*|^2}\Big)\omega(t,y)\,\mathrm{d}y. \tag{9}$$

Here $x^\perp = (-x_2, x_1)^\top$ and $y^* = (y_1, -y_2)^\top$.

Proof admitting Lemma 1.4 and Lemma 1.5

Lemma 1.4, Lemma 1.5 and the Lebesgue convergence theorem implies

$$u_1(t, x_1, 0) = \frac{1}{\pi} \int_{\mathbb{R}^2+} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega(t, y) \, \mathrm{d}y$$

The Dirichlet condition implies that $u_1(t, x_1, 0) \equiv 0$. Then (C4) ($\omega \geq 0$) implies $\omega \equiv 0$. By the classical Liouville theorem for harmonic functions u must be a constant. By (C3) $u \equiv 0$.

vorticity representation formula
(C.R. Anderson 1989; Y. Maekawa 2013)

Lemma. Under the Dirichlet condition
ω satisfies

$$\partial_t \omega - \Delta\omega = -\nabla^\perp \cdot \mathrm{div}\ F \qquad \text{in} \ \ (-L,0) \times \mathbb{R}^2_+$$

$$\partial_2 \omega + (-\partial_1^2)^{\frac{1}{2}} \omega = -\partial_1 p_F \qquad \text{on} \ \ (-L,0) \times \partial\mathbb{R}^2_+.$$

Lemma. (Representation)

$$\begin{aligned}\omega(t) = {} & T(t-s)u(s) \\ & + \int_s^t T(t-\tau)\operatorname{div} F(\tau)\,\mathrm{d}\tau \\ & + \int_s^t e^{(t-\tau)B}(\partial_1 p_{F(\tau)}\delta_{\partial\mathbb{R}^2_+})\,\mathrm{d}\tau\end{aligned}$$

for $-L < s < t < 0$.

Definition of some operators

Let $G(t,x) = (4\pi t)^{-1} \exp\big(-|x|^2/(4t)\big)$ be the two-dimensional Gaussian. Then for each $t > 0$ we introduce the operator e^{tB} defined by

$$e^{tB} f = G(t) * f + G(t) \star f + \Gamma(t) \star f, \tag{10}$$

where

$$\Gamma(t) = 2 \int_0^\infty \big(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}} \partial_2\big)\, G(t+\tau)\, \mathrm{d}\tau \tag{11}$$

with the notations

$$f * h(x) = \int_{\mathbb{R}^2_+} f(x-y) h(y)\, \mathrm{d}y, \qquad f \star h(x) = \int_{\mathbb{R}^2_+} f(x-y^*) h(y)\, \mathrm{d}y,$$

For each $t > 0$ we also set the operator
$T(t) : (L^\infty(\mathbb{R}^2_+))^2 \to L^\infty(\mathbb{R}^2_+)$ as follows:

$$\langle T(t) v, f \rangle_{L^2} = \langle v_1, \partial_2 e^{tB} f \rangle_{L^2} - \langle v_2, \partial_1 e^{tB} f \rangle_{L^2} \qquad \text{for all} \quad f \in L^1(\mathbb{R}^2_+). \tag{12}$$

Thank you for your attention!!

our preprint is available on
http://arxiv.org/abs/1310.6471