Numerical analysis of a viscoelastic fluid flow

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Existence of a weak solution

The Peterlin viscoelastic model (P)

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded smooth domain and $T > 0$.

We consider the system of equations describing a motion of an incompressible viscoelastic fluid together with the boundary and initial conditions

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \eta \Delta \mathbf{v} + \operatorname{div} \mathbf{T} - \nabla p \qquad (\mathbf{v})$$

$$\operatorname{div} \mathbf{v} = 0$$

$$\mathbf{T} = \psi(\operatorname{tr} \mathbf{C})\mathbf{C}$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C} = (\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T + \chi(\operatorname{tr} \mathbf{C})\mathbf{I} - \phi(\operatorname{tr} \mathbf{C})\mathbf{C} + \varepsilon \Delta \mathbf{C} \qquad (\mathbf{C})$$

$$\text{in } \Omega \times (0, T)$$

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial\Omega \qquad\qquad \mathbf{v}(0) = \mathbf{v}_0 \quad \text{in} \quad \Omega$$

$$\frac{\partial \mathbf{C}}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on} \quad \partial\Omega \qquad\qquad \mathbf{C}(0) = \mathbf{C}_0 \quad \text{in} \quad \Omega.$$

Michael Renardy: Mathematical Analysis of Viscoelastic Flows

Assumptions (A)

(A1) ψ, χ, ϕ are positive functions definied on $[0,\infty)$
(A2) $\psi \in \mathcal{C}^1$ is nondecreasing
(A3) χ and ϕ are Lipschitz - continuous functions
(A4) there exist constants α, β and γ s.t.

$$\lim_{s\to\infty} \frac{\phi(s)}{s^\alpha} = A > 0$$

$$\lim_{s\to\infty} \frac{\psi(s)}{s^\beta} = B > 0$$

$$\lim_{s\to\infty} \frac{\chi(s)}{s^\gamma} = C > 0$$

and

$d = 2$:	$d = 3$:
$\alpha \geq 2$	$2 \leq \alpha < 3$
$\beta = \alpha - 1$	$\beta = \alpha - 1$
$\gamma < \alpha + 1$	$\gamma < \alpha + 1$

$$\mathbf{T} = \psi(\mathrm{tr}\ \mathbf{C})\mathbf{C}$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{C} = (\nabla\mathbf{v})\mathbf{C} + \mathbf{C}(\nabla\mathbf{v})^T + \chi(\mathrm{tr}\ \mathbf{C})\mathbf{I} - \phi(\mathrm{tr}\ \mathbf{C})\mathbf{C} + \varepsilon\Delta\mathbf{C}$$

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Energy estimates

- multiply (**v**) by **v** and integrate

$$\frac{\rho}{2}\int_\Omega |\mathbf{v}|^2\, d\mathbf{x} \;-\; \frac{\rho}{2}\int_\Omega |\mathbf{v}_0|^2\, d\mathbf{x} \;+\eta\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\, d\mathbf{x}\, dt \;=$$
$$= -\int_0^t\int_\Omega \nabla\mathbf{v} : \psi(\mathrm{tr}\,\mathbf{C})\mathbf{C}\, d\mathbf{x}\, dt \qquad (E_v)$$

- multiply (**C**) by $\psi(\mathrm{tr}\,\mathbf{C})$, take half the trace and integrate

$$\frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C})\, d\mathbf{x} \;-\frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C}_0)\, d\mathbf{x} \;+$$
$$+\frac{\varepsilon}{2}\int_0^t\int_\Omega \psi'(\mathrm{tr}\,\mathbf{C})|\nabla \mathrm{tr}\,\mathbf{C}|^2\, d\mathbf{x}\, dt \;+\frac{1}{2}\int_0^t\int_\Omega \psi(\mathrm{tr}\,\mathbf{C})\phi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\,\mathbf{C}\, d\mathbf{x}\, dt \;=$$
$$=\frac{1}{2}\int_0^t\int_\Omega \psi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\left[(\nabla\mathbf{v})\mathbf{C}+\mathbf{C}(\nabla\mathbf{v})^T\right]\, d\mathbf{x}\, dt \;+$$
$$+\frac{1}{2}\int_0^t\int_\Omega 3\psi(\mathrm{tr}\,\mathbf{C})\chi(\mathrm{tr}\,\mathbf{C})\, d\mathbf{x}\, dt \qquad (E_C)$$

Energy estimates

- multiply ($\mathbf{v}$) by $\mathbf{v}$ and integrate

$$\frac{\rho}{2}\int_\Omega |\mathbf{v}|^2\, d\mathbf{x} - \frac{\rho}{2}\int_\Omega |\mathbf{v}_0|^2\, d\mathbf{x} + \eta \int_0^t \int_\Omega |\nabla \mathbf{v}|^2\, d\mathbf{x}\, dt =$$

$$= -\int_0^t \int_\Omega \nabla \mathbf{v} : \psi(\mathrm{tr}\,\mathbf{C})\mathbf{C}\, d\mathbf{x}\, dt \qquad (E_v)$$

- multiply ($\mathbf{C}$) by $\psi(\mathrm{tr}\,\mathbf{C})$, take half the trace and integrate

$$\frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C})\, d\mathbf{x} - \frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C}_0)\, d\mathbf{x} +$$

$$+ \frac{\varepsilon}{2}\int_0^t \int_\Omega \psi'(\mathrm{tr}\,\mathbf{C})|\nabla \mathrm{tr}\,\mathbf{C}|^2\, d\mathbf{x}\, dt + \frac{1}{2}\int_0^t \int_\Omega \psi(\mathrm{tr}\,\mathbf{C})\phi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\,\mathbf{C}\, d\mathbf{x}\, dt =$$

$$= +\frac{1}{2}\int_0^t \int_\Omega \psi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\left[(\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T\right] d\mathbf{x}\, dt +$$

$$+ \frac{1}{2}\int_0^t \int_\Omega 3\psi(\mathrm{tr}\,\mathbf{C})\chi(\mathrm{tr}\,\mathbf{C})\, d\mathbf{x}\, dt \qquad (E_C)$$

$$= 0$$

Energy estimates

Thus, adding (E_v) and (E_C) yields

$$\frac{\rho}{2}\int_\Omega |\mathbf{v}|^2\, d\mathbf{x} + \frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C})\, d\mathbf{x} + \eta\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\, d\mathbf{x}\, dt +$$
$$+ \frac{\varepsilon}{2}\int_0^t\int_\Omega \psi'(\mathrm{tr}\,\mathbf{C})|\nabla \mathrm{tr}\,\mathbf{C}|^2\, d\mathbf{x}\, dt + \frac{1}{2}\int_0^t\int_\Omega \psi(\mathrm{tr}\,\mathbf{C})\phi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\,\mathbf{C}\, d\mathbf{x}\, dt =$$
$$= \frac{\rho}{2}\int_\Omega |\mathbf{v}_0|^2\, d\mathbf{x} + \frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C}_0)\, d\mathbf{x} + \frac{1}{2}\int_0^t\int_\Omega 3\psi(\mathrm{tr}\,\mathbf{C})\chi(\mathrm{tr}\,\mathbf{C})\, d\mathbf{x}\, dt\ ,$$

which, using the assumptions (A), gives us the a priori bounds

$$\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$$
$$\mathrm{tr}\,\mathbf{C} \in L^p(0,T;L^p(\Omega)),$$

where $p = 2\alpha$.

Energy estimates

- $\mathbf{C}$ is positive - definite, tr $\mathbf{C} \in L^p(0, T; L^p(\Omega))$, thus $\mathbf{C} \in \mathbf{L}^p(0, T; \mathbf{L}^p(\Omega))$
- multiply ($\mathbf{C}$) by $\mathbf{C}$ and integrate

$$\frac{1}{2}\int_\Omega |\mathbf{C}|^2 \, d\mathbf{x} \; - \frac{1}{2}\int_\Omega |\mathbf{C}_0|^2 \, d\mathbf{x} \; + \varepsilon \int_0^t \int_\Omega |\nabla \mathbf{C}|^2 \, d\mathbf{x} \, dt \; + $$
$$+ \int_0^t \int_\Omega \phi(\mathrm{tr}\, \mathbf{C})\mathbf{C}^2 \, d\mathbf{x} \, dt \; =$$
$$= \int_0^t \int_\Omega \left[(\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T\right] : \mathbf{C} \, d\mathbf{x} \, dt \; + \int_0^t \int_\Omega 3\chi(\mathrm{tr}\, \mathbf{C})\mathrm{tr}\, \mathbf{C} \, d\mathbf{x} \, dt$$

These estimates together with the assumptions (A) yield the a priori bound for $\mathbf{C}$

$$\mathbf{C} \in \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^p(0, T; \mathbf{L}^p(\Omega)).$$

Existence of a weak solution

Theorem

Let the assumptions (A) be satisfied. Then there exists a weak solution to the Peterlin viscoelastic model (P) s.t.

$$\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,div}(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2_{div}(\Omega))$$
$$\mathbf{C} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)) \cap \mathbf{L}^p(0,T;\mathbf{L}^p(\Omega))$$

satisfying $\mathbf{v}(0) = \mathbf{v}_0$, $\mathbf{C}(0) = \mathbf{C}_0$ *and*

$$\rho \int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{w}\; dx + \rho \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\; dx + \eta \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{w}\; dx =$$
$$= -\int_\Omega \nabla\mathbf{w} : \psi(\mathrm{tr}\;\mathbf{C})\mathbf{C}\; dx$$
$$\forall\; \mathbf{w} \in \mathbf{H}^1_{0,div}(\Omega),\;\text{a.e. } t \in (0,T)$$

$$\int_\Omega \frac{\partial \mathbf{C}}{\partial t} : \mathbf{D}\; dx + \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{C} : \mathbf{D}\; dx + \varepsilon \int_\Omega \nabla\mathbf{C} : \nabla\mathbf{D}\; dx =$$
$$= \int_\Omega [(\nabla\mathbf{v})\mathbf{C} + \mathbf{C}(\nabla\mathbf{v})^T] : \mathbf{D}\; dx + \int_\Omega [\chi(\mathrm{tr}\;\mathbf{C})\mathbf{I} - \phi(\mathrm{tr}\;\mathbf{C})\mathbf{C}] : \mathbf{D}\; dx$$
$$\forall\; \mathbf{D} \in \mathbf{H}^1(\Omega),\;\text{a.e. } t \in (0,T).$$

The Galerkin approximations:

$$\mathbf{v}_m(t) = \sum_{i=1}^{m} g_{im}(t)\mathbf{w}_i \quad \mathbf{C}_m(t) = \sum_{i=1}^{m} G_{im}(t)\mathbf{D}_i$$

$$\rho\big(\mathbf{v}'_m(t), \mathbf{w}_j\big) + \rho\big((\mathbf{v}_m(t)\cdot\nabla)\mathbf{v}_m(t), \mathbf{w}_j\big) + \eta\big(\nabla\mathbf{v}_m(t), \nabla\mathbf{w}_j\big) =$$
$$= -\big(\nabla\mathbf{w}_j, \psi(\mathrm{tr}\ \mathbf{C}_m(t))\mathbf{C}_m(t)\big)$$

$$\big(\mathbf{C}'_m(t), \mathbf{D}_j\big) + \big((\mathbf{v}_m(t)\cdot\nabla)\mathbf{C}_m(t), \mathbf{D}_j\big) + \varepsilon\big(\nabla\mathbf{C}_m(t), \nabla\mathbf{D}_j\big) =$$
$$= \big((\nabla\mathbf{v}_m(t))\mathbf{C}_m(t) + \mathbf{C}_m(t)(\nabla\mathbf{v}_m(t))^T, \mathbf{D}_j\big) +$$
$$+ \big(\chi(\mathrm{tr}\ \mathbf{C}_m(t))\mathbf{I}, \mathbf{D}_j\big) - \big(\phi(\mathrm{tr}\ \mathbf{C}_m(t))\mathbf{C}_m(t), \mathbf{D}_j\big)$$
$$\mathbf{v}_m(0) = \mathbf{v}_{0m}$$
$$\mathbf{C}_m(0) = \mathbf{C}_{0m}$$

for $j = 1, \dots, m, \ \ t \in [0, T]$

The Galerkin approximations

A priori estimates:

$$\mathbf{v}_m \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,div}(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2_{div}(\Omega))$$

$$\mathbf{C}_m \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega)) \cap \mathbf{L}^p(0,T;\mathbf{L}^p(\Omega))$$

$$\mathbf{v}'_m \in \mathbf{L}^{4/3}(0,T;\mathbf{H}^{-1}_{div,0}(\Omega)) \qquad (d=3)$$

$$\mathbf{v}'_m \in \mathbf{L}^2(0,T;\mathbf{H}^{-1}_{div,0}(\Omega)) \qquad (d=2)$$

$$\mathbf{C}'_m \in \mathbf{L}^{4/3}(0,T;\mathbf{H}^{-1}(\Omega))$$

Compact embeddings:

The Lions - Aubin lemma $\Rightarrow$

$$\left\{\mathbf{v}_m \in \mathbf{L}^2(0,T;\mathbf{H}^1_{0,div}(\Omega)),\ \mathbf{v}'_m \in \mathbf{L}^e(0,T;\mathbf{H}^{-1}_{div,0}(\Omega))\right\} \hookrightarrow\hookrightarrow \mathbf{L}^2(0,T;\mathbf{L}^p_{div}(\Omega))$$

$$\left\{\mathbf{C}_m \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega)),\ \mathbf{C}'_m \in \mathbf{L}^{4/3}(0,T;\mathbf{H}^{-1}(\Omega))\right\} \hookrightarrow\hookrightarrow \mathbf{L}^2(0,T;\mathbf{L}^p(\Omega))$$

Passing to the limit:

$$\mathbf{v}_m \rightharpoonup^* \mathbf{v} \ \text{ in } \ \mathbf{L}^\infty(0,T;\mathbf{L}^2_{div}(\Omega)) \qquad \mathbf{C}_m \rightharpoonup^* \mathbf{C} \ \text{ in } \ \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$$
$$\mathbf{v}_m \rightharpoonup \mathbf{v} \ \text{ in } \ \mathbf{L}^2(0,T;\mathbf{H}^1_{0,div}(\Omega)) \qquad \mathbf{C}_m \rightharpoonup \mathbf{C} \ \text{ in } \ \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$$
$$\mathbf{v}_m \to \mathbf{v} \ \text{ in } \ \mathbf{L}^2(0,T;\mathbf{L}^p_{div}(\Omega)) \qquad \mathbf{C}_m \to \mathbf{C} \ \text{ in } \ \mathbf{L}^2(0,T;\mathbf{L}^p(\Omega))$$

Using the assumptions (A) and the convergences above, we can pass to the limit in each term, for instance:

$$\left| \int_0^T \int_\Omega \nabla(\varphi(t)\mathbf{w}_j) : \left(\psi(\text{tr }\mathbf{C}(t))\mathbf{C}(t) - \psi(\text{tr }\mathbf{C}_m(t))\mathbf{C}_m(t)\right) \, d\mathbf{x}\, dt \right| \leq$$

$$\leq c_1 \|\nabla \mathbf{w}_j\|_{\mathbf{L}^2(\Omega)} \left(\int_0^T \left(\int_\Omega |\psi(\text{tr }\mathbf{C}(t)) - \psi(\text{tr }\mathbf{C}_m(t))|^{\frac{2p}{p-2}} \, d\mathbf{x} \right)^{\frac{p-2}{2p}} \left(\int_\Omega |\mathbf{C}_m(t)|^p \, d\mathbf{x} \right)^{1/p} dt + \right.$$

$$\left. + \int_0^T \left(\int_\Omega |\psi(\text{tr }\mathbf{C}(t))|^{\frac{2p}{p-2}} \, d\mathbf{x} \right)^{\frac{p-2}{2p}} \left(\int_\Omega |\mathbf{C}(t) - \mathbf{C}_m(t)|^p \, d\mathbf{x} \right)^{1/p} dt \right) \leq$$

$$\leq c \|\nabla \mathbf{w}_j\|_{\mathbf{L}^2(\Omega)} \Big(L_\psi \|\text{tr }\mathbf{C} - \text{tr }\mathbf{C}_m\|_{L^2(0,T;L^p(\Omega))} \|\mathbf{C}_m\|_{\mathbf{L}^2(0,T;\mathbf{L}^p(\Omega))} +$$

$$+ \|\psi(\text{tr }\mathbf{C})\|_{L^2(0,T;L^p(\Omega))} \|\mathbf{C} - \mathbf{C}_m\|_{\mathbf{L}^2(0,T;\mathbf{L}^p(\Omega))} \Big) \quad \to 0$$

as m $\to \infty$

The Galerkin approximations

Passing to the limit:

$$\mathbf{v}_m \rightharpoonup^* \mathbf{v} \text{ in } \mathbf{L}^\infty(0,T;\mathbf{L}^2_{div}(\Omega)) \qquad \mathbf{C}_m \rightharpoonup^* \mathbf{C} \text{ in } \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$$

$$\mathbf{v}_m \rightharpoonup \mathbf{v} \text{ in } \mathbf{L}^2(0,T;\mathbf{H}^1_{0,div}(\Omega)) \qquad \mathbf{C}_m \rightharpoonup \mathbf{C} \text{ in } \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega))$$

$$\mathbf{v}_m \to \mathbf{v} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^p_{div}(\Omega)) \qquad \mathbf{C}_m \to \mathbf{C} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^p(\Omega))$$

Initial values:

By comparison:

$$\rho(\mathbf{v}_0 - \mathbf{v}(0), \mathbf{w}) = 0 \quad \forall\ \mathbf{w} \in \mathbf{H}^1_{0,div}(\Omega)$$

$$(\mathbf{C}_0 - \mathbf{C}(0), \mathbf{D}) = 0 \quad \forall\ \mathbf{D} \in \mathbf{H}^1(\Omega)$$

□

Numerical analysis

Model for numerical simulation (M)

Let us consider

$$\psi(\text{tr } \mathbf{C}) = \text{tr } \mathbf{C}, \ \chi(\text{tr } \mathbf{C}) = \text{tr } \mathbf{C} \ \text{ and } \ \phi(\text{tr } \mathbf{C}) = (\text{tr } \mathbf{C})^2.$$

Then, the assumptions (A) are satisfied in both dimensions with

$$\alpha = 2, \ \beta = \gamma = 1 \text{ and } p = 4.$$

Thus, our model now reads as follows:

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \eta\Delta\mathbf{v} + \text{div } \mathbf{T} - \nabla p$$

$$\text{div } \mathbf{v} = 0$$

$$\mathbf{T} = \text{tr } \mathbf{C}.\mathbf{C}$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{C} = (\nabla\mathbf{v})\mathbf{C} + \mathbf{C}(\nabla\mathbf{v})^T + \text{tr } \mathbf{C}.\mathbf{I} - (\text{tr } \mathbf{C})^2\,\mathbf{C} + \varepsilon\Delta\mathbf{C}$$

$$\text{in } \Omega\times(0,T)$$

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial\Omega \qquad\qquad \mathbf{v}(0) = \mathbf{v}_0 \quad \text{in} \quad \Omega$$

$$\frac{\partial \mathbf{C}}{\partial \mathbf{n}} = \mathbf{0} \quad \text{on} \quad \partial\Omega \qquad\qquad \mathbf{C}(0) = \mathbf{C}_0 \quad \text{in} \quad \Omega.$$

$\mathbf{v} \in \mathbf{L}^2(\mathbf{H}^1_{0,div}) \cap \mathbf{L}^\infty(\mathbf{L}^2_{div})$ and $\mathbf{C} \in \mathbf{L}^2(\mathbf{H}^1) \cap \mathbf{L}^\infty(\mathbf{L}^2) \cap \mathbf{L}^4(\mathbf{L}^4)$

Weak formulation

Definition

The triple $(\mathbf{v}, p, \mathbf{C}) : (0, T) \to \mathbf{V} \times Q \times \mathbf{W}$ satisfying

$$\left(\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}, \mathbf{w}\right) + a_v(\mathbf{v}, \mathbf{w}) + b(\mathbf{w}, p) = (\nabla \mathbf{w}, \mathrm{tr}\ \mathbf{C}.\mathbf{C})$$

$$b(\mathbf{v}, q) = 0$$

$$\left(\frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t}, \mathbf{D}\right) + a_C(\mathbf{C}, \mathbf{D}) = o(\mathbf{v}, \mathbf{C}, \mathbf{D}) + h(\mathbf{C}, \mathbf{D})$$

$$\forall\ (\mathbf{w}, q, \mathbf{D}) \in \mathbf{V} \times Q \times \mathbf{W}, \quad \text{a.e. } t \in (0, T)$$

and $\mathbf{v}(0) = \mathbf{v}_0$, $\mathbf{C}(0) = \mathbf{C}_0$ is called a weak solution to (M).

The following notation is adopted:

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \mathbf{v}, \qquad \frac{\mathrm{D}\mathbf{C}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \mathbf{C}$$

$$a_v(\mathbf{v}, \mathbf{w}) = \eta \int_\Omega 2D(\mathbf{v}) : D(\mathbf{w})\ d\mathbf{x}, \quad a_C(\mathbf{C}, \mathbf{D}) = \varepsilon \int_\Omega \nabla \mathbf{C} : \nabla \mathbf{D}\ d\mathbf{x}, \quad b(\mathbf{v}, q) = \int_\Omega \mathrm{div}\ \mathbf{v}\ q\ d\mathbf{x}$$

$$o(\mathbf{v}, \mathbf{C}, \mathbf{D}) = \int_\Omega \left((\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T\right) : \mathbf{D}\ d\mathbf{x}, \quad h(\mathbf{C}, \mathbf{D}) = \int_\Omega \left(\mathrm{tr}\ \mathbf{C}.\mathbf{I} - (\mathrm{tr}\ \mathbf{C})^2 \mathbf{C}\right) : \mathbf{D}\ d\mathbf{x}$$

and for the spaces $\mathbf{V} = \mathbf{H}^1_0(\Omega)$, $Q = L^2_0(\Omega)$ and $\mathbf{W} = \mathbf{H}^1(\Omega)$.

Theorem

Let $d = 2$. *Let* $(\mathbf{v}, p, \mathbf{C})$ *be the weak solution to (M) satisfying additionally the following assumptions*

$$\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{W}^{2,2}(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{W}^{1,2}(\Omega)), \quad \mathbf{v}_t \in \mathbf{L}^2(0,T;\mathbf{W}^{1,2}(\Omega))$$
$$\mathbf{C} \in \mathbf{L}^2(0,T;\mathbf{W}^{2,2}(\Omega)) \cap \mathbf{L}^\infty(0,T;\mathbf{W}^{1,2}(\Omega)), \quad \mathbf{C}_t \in \mathbf{L}^2(0,T;\mathbf{W}^{1,2}(\Omega))$$
$$p \in L^2(0,T;W^{1,2}(\Omega)).$$

Let us consider $\mathbf{P}_2/\mathbf{P}_1$ *finite element for the velocity and the pressure and* $\mathbf{P}_1$ *finite element for the conformation tensor.*
Then there is a constant $C > 0$ *independent of* h *such that*

$$\sup_{\tau\in(0,T)} \left(\|e_v(\tau,\cdot)\|_2^2 + \|e_C(\tau,\cdot)\|_2^2\right) + $$
$$+ \int_0^T \|\nabla e_v\|_2^2 + \|\nabla e_C\|_2^2 \, dt \; + \int_0^T \|e_p\|_2^2 \, dt \; \leq Ch^2,$$

where $(e_v, e_p, e_C) = (\mathbf{v} - \mathbf{v}_h, p - p_h, \mathbf{C} - \mathbf{C}_h)$ *denotes the error.*

Method of characteristics

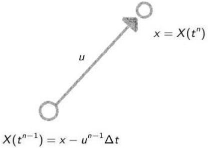
Having $u : \Omega \times (0, T) \to \mathbb{R}^d$ as a given function
and $f : \Omega \times (0, T) \to \mathbb{R}^d$ as an unknown function,

the material derivative $\frac{\mathrm{D}f}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + u \cdot \nabla \right) f$ can be discretized as follows:

$$\frac{\mathrm{D}f}{\mathrm{D}t}(X(t^n), t^n) = \frac{\mathrm{d}}{\mathrm{d}t} f(X(t), t)|_{t=t^n} = \frac{f(X(t^n), t^n) - f(X(t^{n-1}), t^{n-1})}{\Delta t},$$

where $X(.; x) : (0, T) \to \mathbb{R}^d$ is a solution to

$$X'(t) = u(x, t) \quad \text{in} \quad (t^{n-1}, t^n)$$
$$X(t^n) = x.$$



Thus,

$$\frac{\mathrm{D}f}{\mathrm{D}t}(X(t^n), t^n) = \frac{f(X(t^n), t^n) - f(x - u^{n-1}(x)\Delta t, t^{n-1})}{\Delta t}.$$

Numerical scheme (I)

- $\mathbf{P}_2/\mathbf{P}_1$ for veocity $\mathbf{v}$ and pressure p ... $\mathbf{V}_h$, Q_h
- $\mathbf{P}_1$ for conformation tensor $\mathbf{C}$ ... $\mathbf{W}_h$
- the method of characteristics ... $t^n = n\Delta t$

Find $\{(\mathbf{v}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset \mathbf{V}_h \times Q_h \times \mathbf{W}_h$ s.t. for $n = 1, \dots, N_T$

$$\left(\frac{\mathbf{v}_h^n - \mathbf{v}_h^{n-1} \circ X^{n-1}}{\Delta t}, \mathbf{w}_h\right) + a_v(\mathbf{v}_h^n, \mathbf{w}_h) + b(\mathbf{w}_h, p_h^n) + b(\mathbf{v}_h^n, q_h) =$$

$$= (\nabla \mathbf{w}_h, \mathrm{tr}\ \mathbf{C}_h^{n-1}.\mathbf{C}_h^{n-1})$$

$$\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X^{n-1}}{\Delta t}, \mathbf{D}_h\right) + a_C(\mathbf{C}_h^n, \mathbf{D}_h) =$$

$$= o(\mathbf{v}_h^{n-1}, \mathbf{C}_h^{n-1}, \mathbf{D}_h) + h(\mathbf{C}_h^{n-1}, \mathbf{D}_h)$$

$$\forall\ (\mathbf{w}_h, q_h, \mathbf{D}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{W}_h$$

where $X^{n-1}(x) = (x - \mathbf{v}^{n-1}(x)\Delta t)$.

Numerical scheme (II)

- $\mathbf{P}_1/\mathbf{P}_1$ for veocity $\mathbf{v}$ and pressure p $\quad \dots \tilde{\mathbf{V}}_h,\ Q_h$
  $+$ pressure stabilization
- $\mathbf{P}_1$ for conformation tensor $\mathbf{C}$ $\quad \dots \mathbf{W}_h$
- the method of characteristics $\quad \dots t^n = n\Delta t$

Find $\{(\mathbf{v}_h^n, p_h^n, \mathbf{C}_h^n)\}_{n=1}^{N_T} \subset \tilde{\mathbf{V}}_h \times Q_h \times \mathbf{W}_h$ s.t. for $n = 1, \dots, N_T$

$$\left(\frac{\mathbf{v}_h^n - \mathbf{v}_h^{n-1} \circ X^{n-1}}{\Delta t}, \mathbf{w}_h\right) + a_v(\mathbf{v}_h^n, \mathbf{w}_h) + b(\mathbf{w}_h, p_h^n) + b(\mathbf{v}_h^n, q_h) + \frac{1}{\eta} s(p_h, q_h) =$$

$$= (\nabla \mathbf{w}_h, \mathrm{tr}\ \mathbf{C}_h^{n-1}.\mathbf{C}_h^{n-1})$$

$$\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1} \circ X^{n-1}}{\Delta t}, \mathbf{D}_h\right) + a_C(\mathbf{C}_h^n, \mathbf{D}_h) =$$

$$= o(\mathbf{v}_h^{n-1}, \mathbf{C}_h^{n-1}, \mathbf{D}_h) + h(\mathbf{C}_h^{n-1}, \mathbf{D}_h)$$

$$\forall\ (\mathbf{w}_h, q_h, \mathbf{D}_h) \in \tilde{\mathbf{V}}_h \times Q_h \times \mathbf{W}_h,$$

where the stabilization term is

$$s(p, q) = -\delta_0 \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K, \quad \delta_0 > 0 \text{ constant.}$$

H.Notsu, M.Tabata: Error estimates of a Pressure-Stabilized Characteristics Finite Element Scheme for the Navier - Stokes equations

Future goals

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \eta\Delta\mathbf{v} + \mathrm{div}\ (\mathrm{tr}\ \mathbf{C}.\mathbf{C}) - \nabla p$$

$$\mathrm{div}\ \mathbf{v} = 0$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{C} = (\nabla\mathbf{v})\mathbf{C} + \mathbf{C}(\nabla\mathbf{v})^T + \mathrm{tr}\ \mathbf{C}.\mathbf{I} - (\mathrm{tr}\ \mathbf{C})^2\,\mathbf{C} + \varepsilon\Delta\mathbf{C}$$

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(1) error estimates of a (pressure-stabilized) characteristics finite element scheme

(2) numerical simulations by using the schemes (I) and (II)

Future goals

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \eta \Delta \mathbf{v} + \mathrm{div}\ (\mathrm{tr}\ \mathbf{C}.\mathbf{C}) - \nabla p$$

$$\mathrm{div}\ \mathbf{v} = 0$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C} = (\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T + \mathrm{tr}\ \mathbf{C}.\mathbf{I} - (\mathrm{tr}\ \mathbf{C})^2\, \mathbf{C} + \varepsilon \Delta \mathbf{C}$$

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(1) error estimates of a (pressure-stabilized) characteristics finite element scheme

(2) numerical simulations by using the schemes (I) and (II)

Thank you for your attention!