Dissipative structure for the Timoshenko system with Cattaneo's type heat conduction

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Introduction

Timoshenko system with the Cattaneo law:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + b\theta_x = 0, \\ \theta_t + q_x + b\psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa\theta_x = 0, \end{cases}$$

where $a > 0$, $b > 0$, $\kappa > 0$ and $0 < \tau_0 \leq 1$ are constants. $t \in [0, \infty)$ is a time variable, and $x \in \mathbb{R}$ is the spacial variable which denotes the point on the center line of the beam.

- $\varphi(x,t)$: the transversal displacement.
- $\psi(x,t)$: the rotation angle of the beam.
- $\theta(x,t)$: the temperature.
- $q(x,t)$: the heat flow.

Previous results

Timoshenko system with the Cattaneo law:

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- Timoshenko system
  - S.P. Timoshenko (1921)
  - S.P. Timoshenko (1922)
- Timoshenko system with friction
  - J.E.M. Rivera & R. Racke (2003)
  - K. Ide, K. Haramoto & S. Kawashima (2008)
  - K. Ide & S. Kawashima (2008)

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- Timoshenko system with heat conduction
  - J.E.M. Rivera & R. Racke (2002)
  - H.D.F. Sare & R. Racke (2009)
  - M.L. Santos, D.S.A. Jùnior & J.E.M. Rivera (2012)

Main results

The equivalent system is

$$\begin{cases} v_t - u_x + y = 0, \\ y_t - az_x - v + b\theta_x = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ \theta_t + \sqrt{\kappa}\tilde{q}_x + by_x = 0, \\ \tau_0\tilde{q}_t + \tilde{q} + \sqrt{\kappa}\theta_x = 0, \end{cases}$$

where

$$v = \varphi_x - \psi, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad \tilde{q} = \frac{1}{\sqrt{\kappa}}q.$$

Main results

The system is written as

$$A^0 U_t + A U_x + L U = 0,$$

Claim:

(a) A^0 is real symmetric and positive definite.
(b) A is real symmetric.
(c) L is nonnegative definite but <u>not real symmetric</u>.

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Claim:

(a) A^0 is real symmetric and positive definite.
(b) A is real symmetric.
(c) L is nonnegative definite but not real symmetric.
  $\rightarrow$ The general theory is not applicable.

Main results

where $U = (v, y, u, z, \theta)^T$,

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & b & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & \sqrt{\kappa} \\ 0 & 0 & 0 & 0 & \sqrt{\kappa} & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem (Decay estimate)

When $P := \tau_0(1 - a^2 - b^2) + \kappa(a^2 - 1) = 0$, we have

$$\|\partial_x^k U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}(\frac{1}{2}-\frac{1}{p})-\frac{k}{4}}\|U_0\|_{L^p} + Ce^{-ct}\|\partial_x^k U_0\|_{L^2}, \quad (1)$$

while in the case of $P \neq 0$, we have

$$\|\partial_x^k U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}(\frac{1}{2}-\frac{1}{p})-\frac{k}{4}}\|U_0\|_{L^p} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} U_0\|_{L^2}, \quad (2)$$

where $1 \le p \le 2$, and $k,\ l \ge 0$.

- Decay estimate of the regularity-loss type

Pointwise estimate

Lemma (Pointwise estimate)

When $P = 0$, we have

$$|\hat{U}(\xi,t)| \le Ce^{-c\rho_1(\xi)t}|\hat{U}_0(\xi)|, \tag{3}$$

while in the case of $P \neq 0$, we have

$$|\hat{U}(\xi,t)| \le Ce^{-c\rho_2(\xi)t}|\hat{U}_0(\xi)|, \tag{4}$$

where $\rho_1(\xi) = \xi^4/(1+\xi^2)^2$ and $\rho_2(\xi) = \xi^4/(1+\xi^2)^3$.

Dissipative structure:

- If $P = 0$, then $\mathrm{Re}\,\lambda(i\xi) \le -c\xi^4/(1+\xi^2)^2$.
  $\to$ Standard type.
- If $P \neq 0$, then $\mathrm{Re}\,\lambda(i\xi) \le -c\xi^4/(1+\xi^2)^3$.
  $\to$ Regularity-loss type.

When $P \neq 0$, calculating the asymptotic expansion of $\lambda(i\xi)$ for $|\xi| \to \infty$, we have

$$\mathrm{Re}\;\lambda_j(i\xi) = \begin{cases} -\frac{b^2\kappa}{2P^2}\,\xi^{-2} + \mathcal{O}(|\xi|^{-3}) & (j=1,2), \\ -\frac{\delta_j}{2} + \mathcal{O}(|\xi|^{-1}) & (j=3,4,5,6), \end{cases}$$

where $P := \tau_0(1-a^2-b^2) + \kappa(a^2-1)$ and $\delta_j > 0$ for any $j = 3,4,5,6$.

Proof of pointwise estimate: The system in the Fourier space:

$$\begin{aligned}
&\hat{v}_t - i\xi\hat{u} + \hat{y} = 0,\\
&\hat{y}_t - ai\xi\hat{z} - \hat{v} + bi\xi\hat{\theta} = 0,\\
&\hat{u}_t - i\xi\hat{v} = 0,\\
&\hat{z}_t - ai\xi\hat{y} = 0,\\
&\hat{\theta}_t + \sqrt{\kappa}i\xi\hat{q} + bi\xi\hat{y} = 0,\\
&\tau_0\hat{q}_t + \hat{q} + \sqrt{\kappa}i\xi\hat{\theta} = 0.
\end{aligned}$$

Using the energy method in the Fourier space, we construct the Lyapunov function for $P = 0$ and $P \neq 0$, respectively.

Proof of pointwise estimate

Lyapunov function: When $P \neq 0$,

$$E = \frac{1}{2}|\hat{U}|^2 + \alpha_3 \frac{\xi}{1+\xi^2} E_0 + \alpha_3\alpha_2\alpha_1 \frac{\xi^3}{(1+\xi^2)^3} \left\{ E_1 + (1+\xi^2)E_2 \right\}$$
$$+ \alpha_3\alpha_2 \frac{1}{(1+\xi^2)^2} \left\{ (\xi^2 \mathbb{E}_3 + \xi \mathbf{E}_4 + E_5) + (1+\xi^2)(\xi \mathbf{E}_6 + \mathbf{E}_7) \right\},$$

where α_1, $\alpha_2 > 0$ and $\alpha_3 > 0$ are small constants. Also, we put

$$|\hat{U}|^2 = |\hat{v}|^2 + |\hat{y}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{\theta}|^2 + \tau_0 |\hat{q}|^2,$$

$$E_0 = \tau_0 \mathrm{Re}\,(i\hat{\theta}\bar{\hat{q}}), \quad E_1 = \mathrm{Re}\,(i\hat{v}\bar{\hat{u}}), \quad E_2 = \mathrm{Re}\,(i\hat{z}\bar{\hat{y}}), \quad E_3 = -\mathrm{Re}\,(\hat{v}\bar{\hat{y}}),$$
$$E_4 = -\mathrm{Re}\,(\hat{u}\bar{\hat{z}}), \quad E_5 = -\mathrm{Re}\,(\hat{u}\bar{\hat{\theta}}), \quad E_6 = \mathrm{Re}\,(i\hat{y}\bar{\hat{\theta}}), \quad E_7 = \tau_0 \mathrm{Re}\,(\hat{v}\bar{\hat{q}}).$$

Moreover,

$$\mathbb{E}_3 = bE_3 + abE_4 + (a^2 - 1)E_5 - \frac{1 - a^2 - b^2}{\sqrt{\kappa}} E_7,$$

$$\mathbf{E}_4 = -bE_1 + E_6, \quad \mathbf{E}_6 = -bE_2 + aE_6, \quad \mathbf{E}_7 = bE_4 + aE_5.$$

Proof of pointwise estimate

We have

$$\frac{\partial}{\partial t}E + cF \le 0,$$

where

$$F = \frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2 + \frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{z}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{\theta}|^2 + |\hat{q}|^2.$$

Therefore we obtain

$$\frac{\partial}{\partial t}E + c\rho_2(\xi)E \le 0,$$

where $\rho_2(\xi) = \xi^4/(1+\xi^2)^3$. This yields the desired pointwise estimate (4).

Energy estimate: When $P \neq 0$, as a simple corollary of

$$\frac{\partial}{\partial t}E + cF \leq 0,$$

we have the following energy estimate:

$$\begin{aligned}
&\|\partial_x^k U(t)\|_{H^{s-k}}^2 + \int_0^t \big(\|\partial_x^{k+2} v(\tau)\|_{H^{s-k-3}}^2 + \|\partial_x^{k+2} y(\tau)\|_{H^{s-k-2}}^2 \\
&+ \|\partial_x^{k+1} u(\tau)\|_{H^{s-k-2}}^2 + \|\partial_x^{k+1} z(\tau)\|_{H^{s-k-1}}^2 \\
&+ \|\partial_x^{k+1} \theta(\tau)\|_{H^{s-k-1}}^2 + \|\partial_x^{k} q(\tau)\|_{H^{s-k}}^2 \big) d\tau \\
&\leq C \|\partial_x^k U(0)\|_{H^{s-k}}^2
\end{aligned}$$

where $s \geq 0$ and $0 \leq k \leq s$.

• Energy estimate of the regularity-loss type: In the dissipation part, we have the regularity-loss for the component (v, u).

Proof of decay estimate

Proof of decay estimate: When $P = 0$, we have

$$\begin{aligned} \|\partial_x^k U(t)\|_{L^2(\mathbb{R})}^2 &= C \int_{\mathbb{R}} \xi^{2k} |\hat{U}(\xi,t)|^2 d\xi \\ &\leq C \int_{\mathbb{R}} \xi^{2k} e^{-\rho_1(\xi)t} \, |\hat{U}(\xi,0)|^2 d\xi \\ &= C \left(\int_{|\xi|\leq 1} + \int_{|\xi|\geq 1} \right) \\ &=: I_1 + I_2. \end{aligned}$$

Here we have $\rho_1(\xi) = \xi^4/(1+\xi^2)^2 \geq c\xi^4$ for $|\xi| \leq 1$ so that low frequency term I_1 is estimated as

$$\begin{aligned} I_1 &= C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} |\hat{U}(\xi,0)|^2 d\xi \\ &\leq C\int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\xi^4 t} |\hat{U}(\xi,0)|^2 d\xi. \end{aligned}$$

We choose p' such that $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq 2$, and take r such that $\frac{1}{r} + \frac{2}{p'} = 1$. Then applying the Hölder inequality and the Hausdorff-Young inequality, we have

$$\begin{aligned} I_1 &\leq C\| |\xi|^{2k} e^{-c\xi^4 t} \|_{L^r(|\xi|\leq 1)} \|\hat{U}_0\|_{L^{p'}}^2 \\ &\leq C(1+t)^{-\frac{1}{4r}-\frac{k}{2}} \|\hat{U}_0\|_{L^{p'}}^2 \\ &\leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|U_0\|_{L^p}^2. \end{aligned}$$

On the other hand, in high frequency region $|\xi| \geq 1$, we have $\rho_1(\xi) = \xi^4/(1+\xi^2)^2 \geq c$ and hence

$$\begin{aligned} I_2 &= C \int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} |\hat{U}(\xi,0)|^2 d\xi \\ &\leq Ce^{-ct} \int_{|\xi|\geq 1} |\xi|^{2k} |\hat{U}(\xi,0)|^2 d\xi \\ &\leq Ce^{-ct} \|\partial_x^k U_0\|_{L^2}^2. \end{aligned}$$

This shows the desired decay estimate (1).

Next we consider the case of $P \neq 0$.

$$\begin{aligned}\|\partial_x^k U(t)\|_{L^2(\mathbb{R})}^2 &= C\int_{\mathbb{R}} \xi^{2k}|\hat{U}(\xi,t)|^2 d\xi \\ &\le C\int_{\mathbb{R}} \xi^{2k} e^{-\rho_2(\xi)t}\, |\hat{U}(\xi,0)|^2 d\xi \\ &= C\left(\int_{|\xi|\le 1} + \int_{|\xi|\ge 1}\right) \\ &=: J_1 + J_2.\end{aligned}$$

Since $\rho_2(\xi) = \xi^4/(1+\xi^2)^3 \ge c\xi^4$ for $|\xi| \le 1$, the low frequency part J_1 is estimated just in the same way as in the case of $P = 0$.

Proof of decay estimate

On the other hand, in high frequency region $|\xi| \geq 1$, we see that $\rho_2(\xi) = \xi^4/(1+\xi^2)^3 \geq c|\xi|^{-2}$. Thus we have

$$\begin{aligned} J_2 &= C \int_{|\xi|\geq 1} \xi^{2k} e^{-c\rho_2(\xi)t} \, |\hat{U}(\xi,0)|^2 d\xi \\ &\leq C \int_{|\xi|\geq 1} \xi^{2k} e^{-c\xi^{-2}t} \, |\hat{U}(\xi,0)|^2 d\xi \\ &\leq C \sup_{|\xi|\geq 1} (\xi^{-2l} e^{-c\xi^{-2}t}) \int_{|\xi|\geq 1} \xi^{2(k+l)} |\hat{U}(\xi,0)|^2 d\xi \\ &\leq C \, (1+t)^{-l} \, \|\partial_x^{k+l} U_0\|_{L^2(\mathbb{R})}^2 . \end{aligned}$$

This shows the desired decay estimate (2).

Summary

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- Timoshenko system with the Cattaneo law
  - ◦ Decay estimate
  - ◦ Pointwise estimate
  - ◦ Decay structure
  - ◦ Energy estimate

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Thank You for Your Attention