On uniqueness of symmetric Navier-Stokes flows around a body in the plane

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Ω: Exterior domain in $\mathbb{R}^2$ with Lipschitz boundary

$$
\text{(NSt)} \begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = f & \text{in } \Omega, \\ \text{div } u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}
$$

$u = (u_1, u_2)$: unknown velocity vector

p: unknown pressure

$f = (f_1, f_2)$: given external force

Difficulty of the problem

- **Stokes paradox (Linear theory is not useful)**
- $\nabla u \in L^2(\Omega)$ **does not imply** $u(x) \to 0$ $(|x| \to \infty)$

We have no general theory of the existence for (NSt)
in spite of important contributions:

Leray(1933), Fujita(1961), Gilbarg-Weinberger(1974,78), Amick(1988), etc.

Galdi(2004), Yamazaki(2011), Pileckas-Russo(2012)

Symmetry $\Rightarrow$ **Existence of solutions to** (NSt)

Assumption:

Ω satisfies

$$(x_1, x_2) \in \Omega \ \Rightarrow \ (x_1, -x_2),\ (-x_1, x_2) \in \Omega$$

Symmetry condition on $u = (u_1, u_2)$:

For each $i = 1, 2,$ either $u_i(x_1, x_2) = -u_i(-x_1, x_2)$
or $u_i(x_1, x_2) = -u_i(x_1, -x_2)$ holds.

Function spaces

- $C^\infty_{0,\sigma}(\Omega) := \{\varphi \in C^\infty_0(\Omega);$ div $\varphi = 0$ in $\Omega\}$
- $\dot{H}^1_{0,\sigma}(\Omega)$: the completion of $C^\infty_{0,\sigma}(\Omega)$ in the norm $\|\nabla \cdot\|_2$
- $\dot{H}^{1,S}_{0,\sigma}(\Omega) := \{u \in \dot{H}^1_{0,\sigma}(\Omega);\ u$ is symmetric$\}$

Definition. Given $\boldsymbol{f} \in \dot{\boldsymbol{H}}^{1}_{\boldsymbol{0},\boldsymbol{\sigma}}(\boldsymbol{\Omega})^*$, we say that a function $\boldsymbol{u} \in \dot{\boldsymbol{H}}^{1,S}_{\boldsymbol{0},\boldsymbol{\sigma}}(\boldsymbol{\Omega})$ is a symmetric weak solution of $(\boldsymbol{NSt})$ if $\boldsymbol{u}$ satisfies

$$(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi}) \quad \text{for } \forall \boldsymbol{\varphi} \in \boldsymbol{C}^{\infty}_{\boldsymbol{0},\boldsymbol{\sigma}}(\boldsymbol{\Omega}).$$

Remark 1 (Galdi(2004), Russo(2009))*.*

$$\boldsymbol{u} \in \dot{\boldsymbol{H}}^{1,S}_{\boldsymbol{0},\boldsymbol{\sigma}}(\boldsymbol{\Omega}) \Rightarrow \lim_{r\to\infty} \int_0^{2\pi} |\boldsymbol{u}(r,\theta)|^2 \, d\theta = 0$$

Known results

- **Galdi(2004), Pileckas-Russo(2012)**
  Under symmetry assumption on $f \in \dot{H}^1_{0,\sigma}(\Omega)^*$
  $\exists$ **symmetric weak solution** $u \in \dot{H}^{1,S}_{0,\sigma}(\Omega)$ **with**
  $$u_1(x_1, x_2) = u_1(x_1, -x_2) = -u_1(-x_1, x_2),$$
  $$u_2(x_1, x_2) = -u_2(x_1, -x_2) = u_2(-x_1, x_2)$$
  and the energy inequality
  $$\|\nabla u\|_2^2 \leq (f, u)$$

- **Yamazaki(2011)**
  Under stronger symmetry assumption on Ω **and** f
  $\exists$ **unique symmetric solution** u **with**
  $$\sup_{x \in \Omega} (|x| + 1)^\alpha |u(x)| : \text{small} \quad (\alpha \in [1, 2]),$$
  $$\nabla u \in L^r(\Omega) \text{ for } \forall r \in (1, q] \quad (q > 2)$$

Theorem. *Suppose* $\boldsymbol{u}, \boldsymbol{v} \in \dot{\boldsymbol{H}}^{1,S}_{0,\sigma}(\Omega)$, *having the same symmetry property, are symmetric weak solutions of* $(\boldsymbol{NSt})$. *There exists a constant* $\delta = \delta(\Omega)$ *such that if*

$$\|\nabla \boldsymbol{u}\|_2^2 \leq (\boldsymbol{f}, \boldsymbol{u}) \quad \textit{and} \quad \sup_{\boldsymbol{x}\in\Omega}(|\boldsymbol{x}|+1)|\boldsymbol{v}(\boldsymbol{x})| \leq \delta,$$

then $\boldsymbol{u} = \boldsymbol{v}$.

Remark 2. As an application, we can give information on the asymptotic behavior of a symmetric weak solution such as

$$|\boldsymbol{u}(\boldsymbol{x})| = \boldsymbol{O}(|\boldsymbol{x}|^{-1}) \quad \text{as} \quad |\boldsymbol{x}| \to \infty$$

provided that $\boldsymbol{f}$ satisfies the conditons imposed by Yamazaki(2011).

Important step in the proof

Take u and v as test functions in the weak form of (NSt).

Difficulty

Little information on the class of $u \cdot \nabla u$

Lemma 1 (Galdi(2004)). *Let $u \in \dot{H}^{1,S}_{0,\sigma}(\Omega)$. There exists a constant $C = C(\Omega)$ such that*

$$\int_\Omega \frac{|u(x)|^2}{|x|^2}\,dx \le C\|\nabla u\|_2^2.$$

By Lemma 1, we see

$$\left\|\frac{u}{|x|+1} \cdot \nabla u\right\|_1 \le \left\|\frac{u}{|x|+1}\right\|_2 \|\nabla u\|_2 \le C\|\nabla u\|_2^2$$

Lemma 2. *For every* $v \in \dot{H}^1_{0,\sigma}(\Omega)$ *with*

$$\sup_{x\in\Omega}(|x|+1)|v(x)| < \infty,$$

there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset C^{\infty}_{0,\sigma}(\Omega)$ *such that*

$$\begin{aligned} &\nabla v_n \to \nabla v && \text{in } L^2(\Omega), \\ &(|x|+1)v_n \to (|x|+1)v && \text{weakly } * \text{ in } L^{\infty}(\Omega) \end{aligned}$$

as $n \to \infty$.

Lemmas 1 and 2 yield

$$\begin{aligned} (u\cdot\nabla u, v_n) &= \left(\frac{u}{|x|+1}\cdot\nabla u, (|x|+1)v_n\right) \\ &\to \left(\frac{u}{|x|+1}\cdot\nabla u, (|x|+1)v\right) \\ &= (u\cdot\nabla u, v). \end{aligned}$$