Variable Exponents Spaces and Their Applications to Fluid Dynamics

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Overview

1. Variable Exponent Spaces

2. Applications to Fluid Dynamics

Variable Exponent Lebesgue Spaces

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- $L^{p(\cdot)}(\Omega) := \{f : \varrho_{p(\cdot)}(f) < \infty\}$ with norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\}$$

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- Let $W_0^{k,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ w.r.t. $\|\cdot\|_{k,p(\cdot)}$

Properties of Variable Exponent Spaces

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- Hölder inequality
  $\int_\Omega |f(x)g(x)|\,dx \leq 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ a.e.

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- $\left(L^{p(\cdot)}(\Omega)\right)' = L^{p'(\cdot)}(\Omega)$
- If Ω is bounded and $p \geq q$ a.e., then
  $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$

log-Hölder Continuity

An exponent p on a bounded domain Ω is called **log-Hölder continuous** if

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For $d \geq 2$ and an exponent p define the Sobolev-conjugate exponent

$$p^*(x) := \begin{cases} \frac{dp(x)}{d-p(x)} & : p(x) < d \\ \infty & : p(x) \geq d. \end{cases}$$

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- $p^+ < d \Rightarrow W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p^*(\cdot)-\epsilon}(\Omega)$

From now on we assume

- $\Omega \subset \mathbb{R}^d$ bounded domain
- p is log-Hölder continuous
- $f \in L^{p'(\cdot)}(\Omega)$ or $\mathbf{f} \in L^{p'(\cdot)}(\Omega)^d$
- $\delta \geq 0$

Navier-Stokes equations

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$$\begin{aligned} -\operatorname{div}(\delta + |D\mathbf{u}|)^{p(\cdot)-2} D\mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla \pi &= \mathbf{f} \quad \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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- Classical theory of monotone operators:
  Existence of weak solution $\mathbf{u} \in W_0^{1,p(\cdot)}(\Omega)^d$ for $p^- > \frac{3d}{d+2}$.

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- Classical theory of monotone operators:
  Existence of weak solution $\mathbf{u} \in W_0^{1,p(\cdot)}(\Omega)^d$ for $p^- > \frac{3d}{d+2}$.
- Method of Lipschitz-Truncations:
  Existence of weak solution $\mathbf{u} \in W_0^{1,p(\cdot)}(\Omega)^d$ for $p^- > \frac{2d}{d+2}$.
  (Diening, Málek, Steinhauer 2008 and Diening, Růžička 2010)

convection-diffusion equations

$$\begin{aligned} -\operatorname{div}(\delta + |\nabla u|)^{p(\cdot)-2}\nabla u + \mathbf{a}\cdot\nabla u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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Theorem

For $1 < p^- \leq p^+ < d$ or $d < p^- \leq p^+ < \infty$. Let $\mathbf{a} \in L_\sigma^{r(\cdot)}(\Omega)^d$, where r is continuous and $r > (p^*)'$ a.e. or $r = 1$ respectively. Then there is a weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$ of the equation.

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- Solve by classical theory of monotone operators for every $n \in \mathbb{N}$

$$\int_\Omega (\delta + |\nabla u_n|)^{p(\cdot)-2} \nabla u_n \cdot \nabla \varphi \, dx - \int_\Omega u_n \mathbf{a}_n \cdot \nabla \varphi \, dx + \frac{1}{n} \int_\Omega |u_n|^{q-2} u_n \varphi \, dx = (f, \varphi)$$

in $W_0^{1,p(\cdot)}(\Omega) \cap L^q(\Omega)$ where the $\mathbf{a}_n$ are smooth divergence free functions with $\mathbf{a}_n \to \mathbf{a}$ in $L^{r(\cdot)}(\Omega)$

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- For limit in nonlinear $p(\cdot)$-Laplacian use method of Lipschitz-Truncations

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- u solves then

$$\int_\Omega (\delta + \nabla u)^{p(\cdot)-2} \nabla u \cdot \nabla \varphi \, dx - \int_\Omega u \mathbf{a}_n \cdot \nabla \varphi \, dx = (f, \varphi)$$

for all $\varphi \in W_0^{1,\infty}(\Omega)$

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Thank you for your attention!