Navier-Stokes Flow in Spatially Periodic Domains

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The Group Setup

- Stokes resolvent problem

$$(R)\begin{cases} \lambda u - \Delta u + \nabla p = f & \text{in } H, \\ \operatorname{div} u = g & \text{in } H, \end{cases} \qquad \text{in } L^q_\omega(H)\text{-setting}$$

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Muckenhoupt weights

Definition

$1 < q < \infty$. $\omega \geq 0$ is in Muckenhoupt class $A_q(\mathbb{R}^n)$, if

$$\mathcal{A}_q(\omega) := \sup_{U \subset \mathbb{R}^n} \left(\frac{1}{|U|} \int_U \omega \, \mathrm{d}x \right) \left(\frac{1}{|U|} \int_U \omega^{-\frac{q'}{q}} \, \mathrm{d}x \right)^{\frac{q}{q'}} < \infty,$$

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Properties of Muckenhoupt weights

Proposition

1. *Open-ended property*: If $\omega \in A_q(\mathbb{R}^n)$, then there is $p < q$ such that $\omega \in A_p(\mathbb{R}^n)$ (note that this is trivial for $p > q$)

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2. *Boundedness of maximal operator*: $\omega \in A_q(\mathbb{R}^n)$ if and only if the maximal operator

$$\mathcal{M}_{\mathbb{R}^n} f(x) := \sup_{\mathbb{R}^n \supset U \ni x} \frac{1}{|U|} \int_U |f| \, dx$$

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Properties of Muckenhoupt weights

Theorem 1 (S. 2013)

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2. *Boundedness of maximal operator*: $\omega \in A_q(G)$ if and only if the maximal operator

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$\Rightarrow$ unique solution $(u, p) \in W^{2,q}_\omega(H) \times \hat{W}^{1,q}_\omega(H)$ to (R) satisfying

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Theorem 3, S. 2013

The Stokes operator $A_{q,\omega} : W^{2,q}_\omega(H) \cap L^q_{\omega,\sigma}(H) \to L^q_{\omega,\sigma}(H)$ has maximal L^p-regularity.

Proof of Theorem 2 ($\omega = 1$)

Ingredients

- Solution formula (choose $\eta = (\xi', 2\pi k)$ as phase variable)

$$u = \mathcal{F}^{-1}\left(\frac{1}{\lambda + |\eta|^2}\left(id - \frac{\eta \otimes \eta}{|\eta|^2}\right)\hat{f} + \frac{i\eta}{|\eta|^2}\hat{g}\right)$$

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- $f = 0$: Split function spaces with projection

$$\mathcal{P}g(x) := \frac{1}{L}\int_0^L g(x', x_n)\,\mathrm{d}x_n = \left(\mathcal{F}^{-1}\chi_{\{k=0\}}\hat{g}\right)(x)$$

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- $L^q(H) = L^q(\mathbb{R}^{n-1}) \oplus (id - \mathcal{P})L^q(H)$ and similar for Sobolev spaces

Thank you very much for your attention!

Questions?