Numerical tests on some viscoelastic flows

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An overview of different models

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Navier Stokes

$$\begin{cases} Re(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla p + \nabla\cdot(2\alpha\mathbf{D}(\mathbf{u}) + \boldsymbol{\sigma}) \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

$\mathbf{u}$: velocity,
p: pressure,
Re: Reynolds number,
α: ratio of Newtonian viscosity,
$\boldsymbol{\sigma}$: elastic viscosity ,
$\mathbf{D}(\mathbf{u}) = \frac{\nabla\mathbf{u} - \nabla\mathbf{u}^T}{2}$.

Oldroyd-B model

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\tau} - \nabla \mathbf{u} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot (\nabla \mathbf{u})^T = \frac{1}{We}(\mathbf{I} - \boldsymbol{\tau})$$

where $\boldsymbol{\sigma} = \frac{1-\alpha}{We}(\boldsymbol{\tau} - \mathbf{I})$.

$\boldsymbol{\tau}$: conformation tensor, symmetric positive definite

We: Weissenberg number, relaxation time over characteristic time

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Perterlin model

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where $\boldsymbol{\sigma} = \frac{1}{We}(tr\boldsymbol{\tau})\boldsymbol{\tau}$.

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model	global weak solution	numerics
Oldroyd-B	no	no
Rotational	yes	yes
Perterlin	yes	yes

some stability results

- $\psi = sqrt(\tau)$ Square root(SR)
- $\psi = log(\tau)$ Logarithm conformation representation(LCR)

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$$\nabla\mathbf{u} = \mathbf{B} + \boldsymbol{\Omega} + \mathbf{N}\boldsymbol{\tau}^{-1}.$$

where $\mathbf{N}, \boldsymbol{\Omega}$ anti-symmetric, $\mathbf{B}$ symmetric and commutes with $\boldsymbol{\tau}$.

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$$R^T \boldsymbol{\tau} R = diag(\lambda_1, \lambda_2)$$

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = R^T(\nabla \mathbf{u})R.$$

$$\mathbf{N} = R \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix} R^T, \ \mathbf{B} = R \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} R^T, \ \mathbf{\Omega} = R \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} R^T$$

with $n = (m_{12} + m_{21})/(\lambda_2^{-1} - \lambda_1^{-1}), \omega = (\lambda_2 m_{12} + \lambda_1 m_{21})/(\lambda_2 - \lambda_1)$.

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$$\nabla\mathbf{u}\cdot\boldsymbol{\tau} + \boldsymbol{\tau}\cdot(\nabla\mathbf{u})^T = \boldsymbol{\Omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\Omega} + 2\mathbf{B}\boldsymbol{\tau}$$

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Apply LCR: $\boldsymbol{\psi} = log(\tau)$

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(4)Extension: $\frac{\partial \boldsymbol{\tau}}{\partial t} = 2\mathbf{B}\boldsymbol{\tau} \implies \boldsymbol{\tau}(t) = e^{2\mathbf{B}t}\boldsymbol{\tau}_0,$
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Note:
If $\mathbf{Y}$ is skew-symmetric, $e^{\mathbf{Y}}$ is orthogonal,
If Y is invertible, then $e^{\mathbf{YXY}^{-1}} = \mathbf{Y}e^{\mathbf{X}}\mathbf{Y}^{-1}$,
If $\mathbf{XY} = \mathbf{YX}$, then $e^{\mathbf{X}}e^{\mathbf{Y}} = e^{\mathbf{X}+\mathbf{Y}}$.

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- SR stable for relative high Weissenberg number, LCR stable for arbitrary high Weissenberg number

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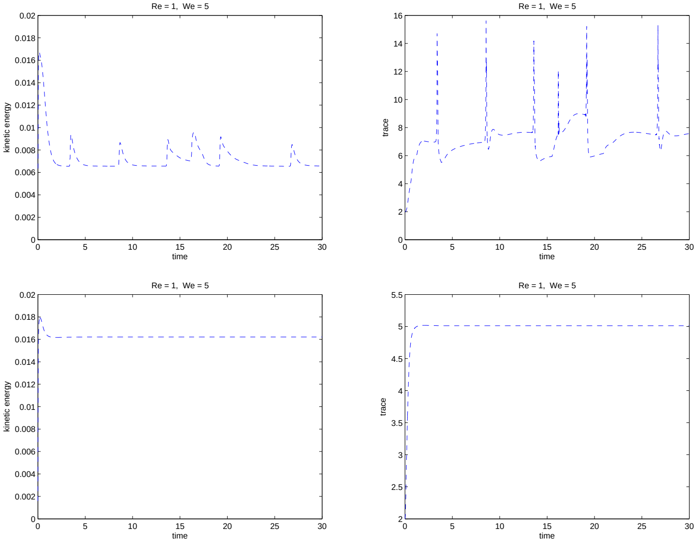

EOC for Oldroyd-B model with LCR

$$e(\phi(N)) = ||\phi(N) - \phi(256)||_2$$

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1. for low We

Table : We=0.5

t	$e(\psi_{11}^{64})$	$e(\psi_{11}^{128})$	EOC	$e(\tau_{11}^{64})$	$e(\tau_{11}^{128})$	EOC
1	0.0722	0.0310	1.2225	0.6398	0.3212	0.9942
2	0.0864	0.0381	1.1808	1.0305	0.5773	0.8358
4	0.0864	0.0381	1.1664	1.3045	0.7836	0.7354
8	0.0928	0.0414	1.1657	1.3245	0.8001	0.7273

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2. for high We

Table : We=2

t	$e(\psi_{11}^{64})$	$e(\psi_{11}^{128})$	EOC	$e(\tau_{11}^{64})$	$e(\tau_{11}^{128})$	EOC
1	0.0989	0.0428	1.2095	1.2414	0.6129	1.0182
2	0.1425	0.0630	1.1767	6.8478	4.1627	0.7181
4	0.1961	0.0882	1.1525	20.5848	15.0224	0.4545

characteristic method

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material derivative

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot f \to \frac{Df}{Dt} = \frac{f^n - f^{n-1} \circ X_1(u^{n-1}, \Delta t)}{\Delta t} + O(\Delta t)$$

$$X = X_1 + u^{n-1} \Delta t$$

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- finite difference
- finite element

Finite difference

Part I.

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\alpha}{Re} \Delta \mathbf{u} + \frac{1-\alpha}{Re} \frac{1}{We} \nabla \cdot \boldsymbol{\tau}$$

$$\mathbf{u}_t = -\nabla p \qquad (\nabla \cdot u = 0)$$

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$$U^* - \Delta t \tfrac{\alpha}{Re}(U^*_{xx} + U^*_{yy}) = U^{k-1} \circ X_1(\mathbf{u}^{k-1}, \Delta t) + \Delta t \tfrac{1-\alpha}{ReWe}(\tau^{k-1}_{11x} + \tau^{k-1}_{12y})$$
$$V^* - \Delta t \tfrac{\alpha}{Re}(V^*_{xx} + V^*_{yy}) = V^{k-1} \circ X_1(\mathbf{u}^{k-1}, \Delta t) + \Delta t \tfrac{1-\alpha}{ReWe}(\tau^{k-1}_{21x} + \tau^{k-1}_{22y})$$

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Chorin's projection

$$\begin{aligned} \tfrac{U^k - U^*}{\Delta t} &= -(P^k)_x \\ \tfrac{V^k - V^*}{\Delta t} &= -(P^k)_y \end{aligned}$$

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Part II.

$$\frac{\psi^k - \psi^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t)}{\Delta t} = \boldsymbol{\Omega}^{k-1}\psi^{k-1} - \psi^{k-1}\boldsymbol{\Omega}^{k-1} + 2\mathbf{B}^{k-1} + \frac{e^{-\psi^{k-1}} - \mathbf{I}}{We}$$

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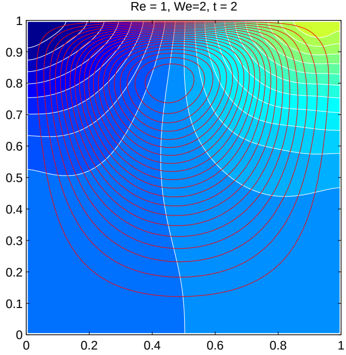
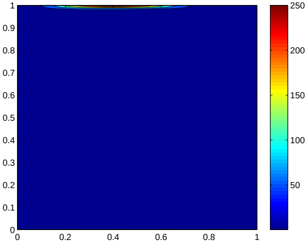
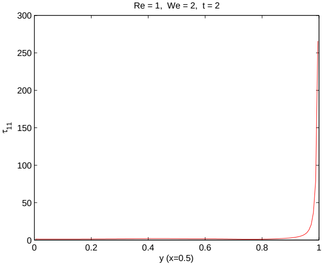
Weakly coupled, iteration over $\mathbf{u}, p, \boldsymbol{\tau}$.



Re = 1, We=2, t = 2





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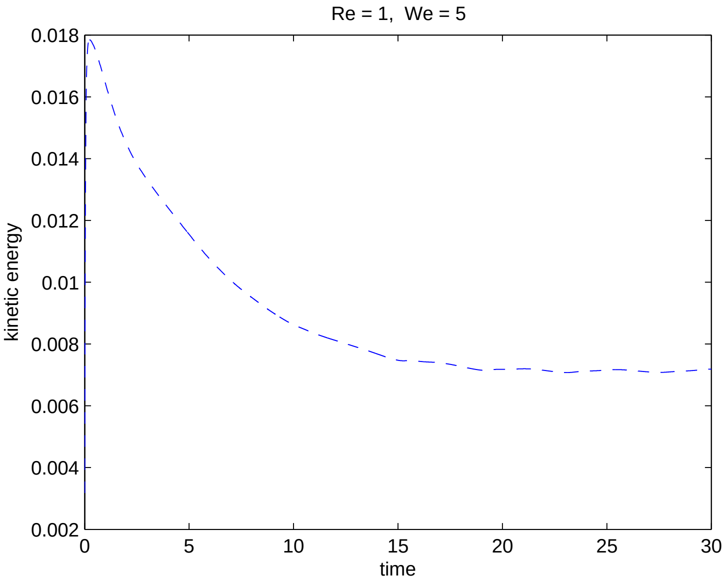
τ_{11}
y (x=0.5)


Re = 1, We = 5
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
kinetic energy
0
5
10
15
20
25
30
time

Finite element

Weak formula:

$$\begin{cases} Re\frac{\mathbf{u}^k-\mathbf{u}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t)}{\Delta t} + \nabla p^k - 2\alpha\nabla\cdot\mathbf{D}(\mathbf{u}^k) = \frac{1-\alpha}{We}\nabla\cdot(\boldsymbol{\tau}^{k-1}-\mathbf{I}) \\ \nabla\cdot\mathbf{u}^k = 0 \\ \frac{\boldsymbol{\tau}^k-\boldsymbol{\tau}^{k-1}\circ X_1(\mathbf{u}^k,\Delta t)}{\Delta t} = \nabla\mathbf{u}^k\boldsymbol{\tau}^{k-1} + \boldsymbol{\tau}^{k-1}(\nabla\mathbf{u}^k)^T + \frac{1}{We}(\mathbf{I}-\boldsymbol{\tau}^k) \end{cases}$$

Weak formula:

$$\begin{cases} Re\frac{\mathbf{u}^k-\mathbf{u}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t)}{\Delta t}+\nabla p^k-2\alpha\nabla\cdot\mathbf{D}(\mathbf{u}^k)=\frac{1-\alpha}{We}\nabla\cdot(\boldsymbol{\tau}^{k-1}-\mathbf{I}) \\ \nabla\cdot\mathbf{u}^k=0 \\ \frac{\boldsymbol{\tau}^k-\boldsymbol{\tau}^{k-1}\circ X_1(\mathbf{u}^k,\Delta t)}{\Delta t}=\nabla\mathbf{u}^k\boldsymbol{\tau}^{k-1}+\boldsymbol{\tau}^{k-1}(\nabla\mathbf{u}^k)^T+\frac{1}{We}(\mathbf{I}-\boldsymbol{\tau}^k) \end{cases}$$

$$\frac{Re}{\Delta t}(v,u)+2\alpha(\mathbf{D}(v),\mathbf{D}(u))-(\nabla\cdot v,p)-(\nabla\cdot u,q)$$

$$=(v,\mathbf{u}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t))+\frac{1-\alpha}{We}(v,\nabla\cdot\boldsymbol{\tau}^{k-1})$$

$$(\frac{1}{\Delta t}+\frac{1}{We})(\phi,\boldsymbol{\tau}^k)$$

$$=(\phi,\nabla\mathbf{u}^k\boldsymbol{\tau}^{k-1}+\boldsymbol{\tau}^{k-1}(\nabla\mathbf{u}^k)^T+\frac{1}{We}\mathbf{I})+(\phi,\frac{1}{\Delta t}\boldsymbol{\tau}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t)$$

Weak formula:

$$\begin{cases} Re\frac{\mathbf{u}^k-\mathbf{u}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t)}{\Delta t}+\nabla p^k-2\alpha\nabla\cdot\mathbf{D}(\mathbf{u}^k)=\frac{1-\alpha}{We}\nabla\cdot(\boldsymbol{\tau}^{k-1}-\mathbf{I}) \\ \nabla\cdot\mathbf{u}^k=0 \\ \frac{\boldsymbol{\tau}^k-\boldsymbol{\tau}^{k-1}\circ X_1(\mathbf{u}^k,\Delta t)}{\Delta t}=\nabla\mathbf{u}^k\boldsymbol{\tau}^{k-1}+\boldsymbol{\tau}^{k-1}(\nabla\mathbf{u}^k)^T+\frac{1}{We}(\mathbf{I}-\boldsymbol{\tau}^k) \end{cases}$$

$$\frac{Re}{\Delta t}(v,u)+2\alpha(\mathbf{D}(v),\mathbf{D}(u))-(\nabla\cdot v,p)-(\nabla\cdot u,q)$$
$$=(v,\mathbf{u}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t))+\frac{1-\alpha}{We}(v,\nabla\cdot\boldsymbol{\tau}^{k-1})$$
$$(\frac{1}{\Delta t}+\frac{1}{We})(\phi,\boldsymbol{\tau}^k)$$
$$=(\phi,\nabla\mathbf{u}^k\boldsymbol{\tau}^{k-1}+\boldsymbol{\tau}^{k-1}(\nabla\mathbf{u}^k)^T+\frac{1}{We}\mathbf{I})+(\phi,\frac{1}{\Delta t}\boldsymbol{\tau}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t)$$

$P2/P1/(P1)^3$ elements,
$(P1/P1+\mathit{pressurestablization})/(P1)^3$ elements.

Weak formula:

$$\begin{cases} Re\frac{\mathbf{u}^k-\mathbf{u}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t)}{\Delta t}+\nabla p^k-2\alpha\nabla\cdot\mathbf{D}(\mathbf{u}^k)=\frac{1-\alpha}{We}\nabla\cdot(\boldsymbol{\tau}^{k-1}-\mathbf{I}) \\ \nabla\cdot\mathbf{u}^k=0 \\ \frac{\boldsymbol{\tau}^k-\boldsymbol{\tau}^{k-1}\circ X_1(\mathbf{u}^k,\Delta t)}{\Delta t}=\nabla\mathbf{u}^k\boldsymbol{\tau}^{k-1}+\boldsymbol{\tau}^{k-1}(\nabla\mathbf{u}^k)^T+\frac{1}{We}(\mathbf{I}-\boldsymbol{\tau}^k) \end{cases}$$

$$\frac{Re}{\Delta t}(v,u)+2\alpha(\mathbf{D}(v),\mathbf{D}(u))-(\nabla\cdot v,p)-(\nabla\cdot u,q)$$
$$=(v,\mathbf{u}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t))+\frac{1-\alpha}{We}(v,\nabla\cdot\boldsymbol{\tau}^{k-1})$$
$$(\frac{1}{\Delta t}+\frac{1}{We})(\phi,\boldsymbol{\tau}^k)$$
$$=(\phi,\nabla\mathbf{u}^k\boldsymbol{\tau}^{k-1}+\boldsymbol{\tau}^{k-1}(\nabla\mathbf{u}^k)^T+\frac{1}{We}\mathbf{I})+(\phi,\frac{1}{\Delta t}\boldsymbol{\tau}^{k-1}\circ X_1(\mathbf{u}^{k-1},\Delta t)$$

$P2/P1/(P1)^3$ elements,
$(P1/P1+\mathit{pressurestablization})/(P1)^3$ elements.

$$\frac{\psi^k-\psi^{k-1}\circ X_1(\mathbf{u}^k,\Delta t)}{\Delta t}=\Omega^{k-1}\psi^{k-1}+\psi^{k-1}\Omega+\frac{1}{We}(e^{-\psi^{k-1(\nabla\mathbf{u}^k)^T a}}-\mathbf{I})$$

Thank you for your attention!