The fundamental solution of compressible and incompressible fluid flow past a rotating obstacle

JSPS-DFG Japanese-German Graduate Externship
Kickoff Meeting
Waseda University, Tokyo, June 17-18, 2014

Reinhard Farwig, TU Darmstadt
Milan Pokorný, Charles University Prague
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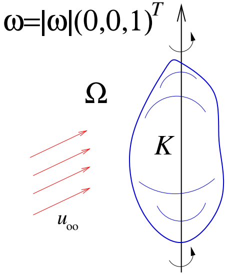

The Incompressible Case

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p &= \tilde{f} && \text{in } \Omega(t) \\ \operatorname{div} u &= 0 && \text{in } \Omega(t) \\ u &= \omega \wedge x && \text{on } \partial\Omega(t) \\ u &\to u_\infty && \text{at } \infty \end{aligned}$$

- The exterior domain $\Omega(t)$ depends on t
- General assumption: $u_\infty = ke_3, k = |u_\infty| \geq 0$ and $\omega = e_3$

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- The exterior domain $\Omega(t)$ depends on t
- General assumption: $u_\infty = ke_3, k = |u_\infty| \geq 0$ and $\omega = e_3$
- Work in a t-independent domain $\Rightarrow$ coordinate transform

$$y = O(t)^\top x,\; v(y,t) = O(t)^\top\big(u(x,t) - u_\infty\big), \ldots$$

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Linearize v w.r.t. $v = 0$
- Consider the stationary problem in $\mathbb{R}^3 \Leftrightarrow t$-periodic solutions of the original problem in $\mathbb{R}^3$:

The Linear Problem

$$
\begin{aligned}
-\nu\Delta v - (\omega \wedge y)\cdot\nabla v + u_\infty \cdot \nabla v + \omega \wedge v + \nabla p &= f \text{ in } \mathbb{R}^3 \\
\operatorname{div} v &= 0 \text{ in } \mathbb{R}^3 \\
v &\to 0 \text{ at } \infty
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- Apply the Helmholtz projection $P \Rightarrow$
Modified Stokes/Oseen operator

$$A_{\omega,u_\infty}(v) = -\nu P\Delta v - (\omega\wedge y)\cdot\nabla v + \omega\wedge v + u_\infty\cdot\nabla v$$

- T. Hishida (1999, 2000): A_{ω,u_∞} does not (!) generate an analytic semigroup

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- Analysis of the fundamental solution in L^q-spaces:

R. F., T. Hishida, D. Müller: Pacific J. Math. (2004)
R. F.: Tôhoku Math. J. (2005)
G.P. Galdi, M. Kyed: Proc. Amer. Math. Soc. (2013), (2013)
R. F., M. Krbec, Š. Nečasová: Ann. Univ. Ferrara (2008)
R. F., M. Krbec, Š. Nečasová: Math. Methods Appl. Sci. (2008)

Further Literature

• Spectral properties of A_{ω,u_∞}:
R. F., J. Neustupa: Integral Equations Operator Theory (2008)
R. F., Š. Nečasová, J. Neustupa: J. Math. Soc. Japan (2011)

• Explicit representation of the fundamental solution:
R. F., R.B. Guenther, Š. Nečasová, E.A. Thomann: DCDS-A (2014)

• Semigroup estimates:
M. Geissert, H. Heck, M. Hieber: J. reine angew. Math. (2006)
T. Hishida, Y. Shibata: Arch. Ration. Mech. Anal. (2009)

• Numerous results on the nonlinear problem, e.g. by
F.-Hishida, F.-Galdi-Kyed, Galdi-Kyed, Galdi-Silvestre, Geissert-Heck-Hieber, Goetze, Hansel, Heck-Kim-Kozono, Hieber-Shibata, Hieber-Sawada, Hishida-Shibata, et al.

L^q-estimates for the Fundamental Solution

Determine the fundamental solution, prove L^q-estimates

- Use cylindrical coordinates $y \sim (r, \theta, y_3)$, $r = |(y_1, y_2)| \geq 0 \Rightarrow$ $(\omega \wedge y) \cdot \nabla = \partial_\theta$
- Use Fourier transform and cylindrical coordinates $\xi \sim (s, \varphi, \xi_3) \Rightarrow \widehat{\partial_\theta u}(\xi) = \partial_\varphi \hat{u}(\xi) \Rightarrow$

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- We get a first order ODE w.r.t. φ:

$$\nu|\xi|^2\hat{v} + ik\xi_3\hat{v} - \partial_\varphi\hat{v} + \omega \wedge \hat{v} = \hat{P}f, \quad i\xi \cdot \hat{v} = 0,$$

where $\hat{v}$ has to be 2π-periodic in φ

The solution of the First Order ODE ($\nu = 1$)

$$\hat{v}(\xi) = \frac{1}{1 - e^{-2\pi|\xi|^2}} \int_0^{2\pi} e^{-|\xi|^2 t - ikt\xi_3} O(t)^\top \widehat{Pf}(O(t)\xi)\, dt$$

$$= \int_0^{\infty} e^{-|\xi|^2 t} O(t)^\top \mathcal{F}((Pf)(O(t) \cdot -kte_3))\, dt$$

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- If $k = 0$ and $O(t)^\top((Pf)(O(t)\cdot)$ is t-independent $\Rightarrow$
  $\hat{v}(\xi) = \frac{1}{|\xi|^2}\widehat{Pf}(\xi) \Rightarrow \|\nabla^2 v\|_q \leq c\|f\|_q$ for $1 < q < \infty$

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- For $q = 2$: L^2-estimates of $\nabla^2 v$ by Plancherel's Theorem

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- For $q = 2$: L^2-estimates of $\nabla^2 v$ by Plancherel's Theorem
- Evidence that the integral kernel Γ satisfies

$$|\Gamma(x, y)| \ge \frac{\log|x - y|}{|x - y|}$$

Partition of Unity and Littlewood-Paley Theory ($k = 0$)

- To control $\nabla^2 v \sim \Delta v$ let

$$\hat{\psi}(\xi) = |\xi|^2 e^{-|\xi|^2}, \quad \psi_t(x) = t^{-n/2}\psi(\frac{x}{\sqrt{t}}),\ t > 0$$

- Decompose $\psi = \sum_j \chi_j * \psi =: \sum_j \psi^j$
- Use Littlewood-Paley theory for $\|\cdot\|_q$:

$$\|f\|_q \sim \Big\| \Big(\int_0^\infty |\varphi_s * f(\cdot)|^2 \frac{ds}{s} \Big)^{1/2} \Big\|_q$$

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- Decompose operator

$$T : f \mapsto \Delta v(x) = \int_{\mathbb{R}^3} K(x,y) Pf(y)\, dy,$$

into pieces T_j via ψ^j, i.e. $T = \sum T_j$

Estimate of T_j (I)

- $$\|T_j f\|_q \leq c 2^{-|j|} \|\mathcal{M}^j\|^{1/2}_{\mathcal{SL}(L^{(q/2)'})} \|f\|_q$$

  where

  $$\mathcal{M}^j g(x) = \sup_{r>0} \int_{r/16}^{16r} (|\psi_t^j| * |g|)(O(t)^\top x) \frac{dt}{t}$$

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- Prove that
  $$\mathcal{M}^j g(x) \leq c2^{-2|j|}\mathcal{M}(\mathcal{M}_\theta g)(x)$$

- $\mathcal{M}$ is the classical Hardy-Littlewood maximal operator, bounded on L^q for each $1 < q < \infty$

Estimate of T_j (II)

$$(M_\theta g)(x) = \sup_{r>0} \int_{r/16}^{16r} |g(O(t)^\top x| \frac{dt}{t}$$

behaves like a 1D maximal operator in θ for each $r = |(x_1, x_2)|, x_3$ and is bounded on L^q for each $1 < q < \infty$

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End of Proof - Incompressible Case

$$\|T_j f\|_q \leq c 2^{-2|j|} \|f\|_q \;\Rightarrow\; T \text{ is bounded on } L^q,\; q \geq 2$$

Complex interpolation $\Rightarrow$ T is bounded on L^q, $1 < q < \infty$

The Compressible Case

- Flow of a viscous, compressible fluid around ($u_\infty = 0$) or past ($0 \neq u_\infty // e_3$) a rotating obstacle with angular velocity $\omega = e_3 // u_\infty$:

$$\begin{aligned} \rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u - \mu \Delta u - (\mu + \nu) \nabla \operatorname{div} u + \nabla p(\rho) &= \rho f \\ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) &= 0 \end{aligned}$$

together with the boundary conditions

$$\begin{aligned} u(x,t) = \omega \wedge x \quad & \text{at} \quad \bigcup_{t \in (0,\infty)} \partial\Omega(t) \times \{t\} \\ u(x,t) \to u_\infty \quad & \text{as} \quad |x| \to \infty \\ \rho(x,t) \to \rho_\infty \quad & \text{as} \quad |x| \to \infty. \end{aligned}$$

The new system (I) in v, σ

- The exterior domain $\Omega(t)$ depends on t
- Use a coordinate transform as before to work in a t-independent domain:

$$y = O(t)^\top x,\ v(y,t) = O(t)^\top\big(u(x,t) - u_\infty\big),\ \tilde{\rho}(y,t) = \rho(x,t)$$

- Linearize v w.r.t. $v = 0$, linearize $\tilde{\rho}$ w.r.t. to $\rho_\infty = 1$: $\tilde{\rho} = 1 + \sigma$
- Consider the stationary problem in $\mathbb{R}^3 \Leftrightarrow t$-periodic solutions of the original problem in $\mathbb{R}^3$

The new system (II) in v, σ

- Determine the fundamental solution, prove L^q-estimates

$$
\begin{aligned}
-\mu\Delta v - (\mu+\nu)\nabla \operatorname{div} v + (u_\infty - \omega\wedge y)\cdot\nabla v + \omega\wedge v + \nabla\sigma &= F\\
\operatorname{div} v + \operatorname{div}\big(\sigma(u_\infty - \omega\wedge y)\big) &= G
\end{aligned}
$$

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\end{aligned}
$$

Apply div to get for $g := \operatorname{div} v$ and σ the system

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\begin{aligned}
-(2\mu+\nu)\Delta g + (u_\infty - \omega\wedge y)\cdot\nabla g + \Delta\sigma &= \operatorname{div} F \\
g + (u_\infty - \omega\wedge y)\cdot\nabla\sigma &= G.
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\end{aligned}
$$

Simplify to get for $g = \mathrm{div}\, v$

$$
\begin{aligned}
&\Delta g-\big((u_\infty-\omega\wedge y)\cdot\nabla\big)^2 g+(u_\infty-\omega\wedge y)\cdot\nabla(2\mu+\nu)\Delta g\\
&= H\\
&:= \Delta G-(u_\infty-\omega\wedge y)\cdot\nabla(\mathrm{div}\, F).
\end{aligned}
$$

The new system (III) in g

- Use cylindrical coordinates $y \sim (r, \theta, y_3)$, $r = |(y_1, y_2)| \geq 0 \Rightarrow (\vec{\omega} \wedge y) \cdot \nabla = \partial_\theta$
- Use Fourier transform and cylindrical coordinates $\xi \sim (s, \varphi, \xi_3) \Rightarrow \widehat{\partial_\theta u}(\xi) = \partial_\varphi \hat{u}(\xi) \Rightarrow$

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- With $k = |u_\infty|$, $u_\infty = k e_3$ we get a 2^{nd} order ODE w.r.t. φ:

$$\partial_\varphi^2 \hat{g} + \partial_\varphi \hat{g}\big(-(2\mu+\nu)|\xi|^2 - 2ik\xi_3 \big) + \hat{g}\big(|\xi|^2 + (2\mu+\nu)ik\xi_3|\xi|^2 - k^2\xi_3^2\big) = \hat{H}$$

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$$\partial_\varphi^2 \hat{g} + \partial_\varphi \hat{g}\big(-(2\mu+\nu)|\xi|^2 - 2ik\xi_3\big) + \hat{g}\big(|\xi|^2 + (2\mu+\nu)ik\xi_3|\xi|^2 - k^2\xi_3^2\big) = \hat{H}$$

- Characteristic polynomial

$$\chi(\lambda) = \lambda^2 - \big((2\mu+\nu)|\xi|^2 + 2ik\xi_3\big)\lambda + \big(|\xi|^2 + (2\mu+\nu)ik\xi_3|\xi|^2 - k^2\xi_3^2\big)$$

- Characteristic zeros

$$\lambda_{1,2}(\xi) = \frac{1}{2}\Big((2\mu+\nu)|\xi|^2 + 2ik\xi_3 \pm \sqrt{(2\mu+\nu)^2|\xi|^4 - 4|\xi|^2}\Big).$$

The solution g

With - for simplicity $2\mu + \nu = 2$ - and the zeros

$$\lambda_{1,2}(\xi) = |\xi|^2 + ik\xi_3 \pm \sqrt{|\xi|^4 - |\xi|^2}$$

we get a unique 2π-periodic solution $\hat{g}$:

$$\hat{g}(\xi) = \frac{1}{\lambda_1 - \lambda_2} \int_0^{2\pi} \left(\frac{e^{-\lambda_2 t}}{1 - e^{-2\pi\lambda_2}} - \frac{e^{-\lambda_1 t}}{1 - e^{-2\pi\lambda_1}} \right) \hat{H}\big(O(t)\xi\big) \, \mathrm{d}t.$$

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If H and $\hat{H}$ are independent of θ and ϕ, resp., then

$$\hat{g}(\xi) = 2\pi \hat{H}(\xi) \big(|\xi|^2 + 2ik\xi_3 |\xi|^2 - k^2 \xi_3^2 \big)^{-1}.$$

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$$\hat{g}(\xi) = 2\pi \hat{H}(\xi)\big(|\xi|^2 + 2ik\xi_3|\xi|^2 - k^2\xi_3^2\big)^{-1}.$$

We also have, with $\mu_{1,2}(\xi) = |\xi|^2 \pm \sqrt{|\xi|^4 - |\xi|^2}$,

$$\hat{g}(\xi) = \int_0^{\infty} \frac{e^{-\mu_2 t} - e^{-\mu_1 t}}{\mu_1 - \mu_2} \mathcal{F}\big(H(O(t) \cdot -kte_3)\big)(\xi)\, \mathrm{d}t$$

Fourier Analysis

Note: $\mu_{1,2}(\xi)$ is a smooth multiplier function on $\mathbb{R}^3 \setminus \{0\} \setminus \partial B_1$
Use a partition of unity $1 = \eta_0 + \eta_1 + \eta_2$ where

$$\eta_0 \in C_0^\infty(B_{1/2}), \quad \eta_1(\xi) = 1 \text{ for } \frac{1}{2} \leq |\xi| \leq 2, \quad \eta_2 = 1 \text{ in } (B_4)^c$$

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For $|\xi| > 4$ all multiplier functions are smooth, Hörmander-Mikhlin multiplier theory $\Rightarrow$ L^q-estimates, $1 < q < \infty$

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Analysis near $|\xi| = 1$: multiplication of smooth multiplier functions (okay) with the crucial multiplier functions

$$\eta_1(\xi)\sqrt{1 - |\xi|^2}\chi_{B_1}(\xi) \sim (1 - |\xi|^2)_+^{1/2},$$
$$\eta_1(\xi)\sqrt{|\xi|^2 - 1}\chi_{B_1^c}(\xi) \sim \eta_1(\xi)(|\xi|^2 - 1|)_+^{1/2}$$

Theorem: The Bochner-Riesz multiplier operator $B^{1/2}$ defined by the multiplier function $(1-|\xi|^2)_+^{1/2}$ on $\mathbb{R}^3$ can be represented by convolution with the kernel $cJ_2(|x|)|x|^{-2}$ and is $L^q(\mathbb{R}^3)$-bounded if and only if $\frac{6}{5} < q < 6$.

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Let the Bochner-Riesz multiplier operator $\tilde{B}^\lambda$ on $\mathbb{R}^3$ be defined by the multiplier function $(|\xi|^2-1)_+^\lambda \eta_1(\xi)$ and let $K_\lambda(x)$ denote the corresponding integral kernel:

$$K_\lambda(x) = \frac{2}{|x|} \int_1^4 \sin(|x|\tau)(\tau^2-1)^{\Re\lambda} e^{i\,\Im\lambda \ln(\tau^2-1)}\, \tau\eta_1(\tau)\, \mathrm{d}\tau$$

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Theorem: The Bochner-Riesz multiplier operator $B^{1/2}$ defined by the multiplier function $(1-|\xi|^2)_+^{1/2}$ on $\mathbb{R}^3$ can be represented by convolution with the kernel $cJ_2(|x|)|x|^{-2}$ and is $L^q(\mathbb{R}^3)$-bounded if and only if $\frac{6}{5} < q < 6$.

Let the Bochner-Riesz multiplier operator $\tilde{B}^\lambda$ on $\mathbb{R}^3$ be defined by the multiplier function $(|\xi|^2-1)_+^\lambda \eta_1(\xi)$ and let $K_\lambda(x)$ denote the corresponding integral kernel:

$$K_\lambda(x) = \frac{2}{|x|}\int_1^4 \sin(|x|\tau)(\tau^2-1)^{\Re\lambda} e^{i\,\Im\lambda \ln(\tau^2-1)}\,\tau\eta_1(\tau)\,\mathrm{d}\tau$$

Theorem: For $\lambda \in \mathbb{C},\ \Re\lambda > -1$

$$|K_\lambda(x)| \le C\frac{\big(1+|\Im\lambda|\big)^{[\Re\lambda]+2}}{(1+|x|)^{\Re\lambda+2}}\ \big(\times\ \log(2+|x|)\quad \text{when } \Re\lambda \in \mathbb{N}_0\big).$$

Proof: Integration by parts (with $\sin(|x|\tau)$ to get $|x|^{-(\Re\lambda+2)}$)

Details of Proof

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- K_λ^0 is a bounded convolution operator on each L^q.
- **Claim**: For $p \geq 1$ satisfying $\frac{3}{4} \leq \frac{1}{p} < \frac{1}{3}(2 + \Re\lambda)$ ($\Rightarrow \Re\lambda > \frac{1}{4}$)

$$\|K_\lambda^j * f\|_p \leq C(\lambda) j\, 2^{-j\delta} \|f\|_p;$$

  here $\delta = \frac{1}{2}(1 + 2\Re\lambda) - 3(\frac{1}{p} - \frac{1}{2}) > 0$.
- **Consequence**: K_λ is a bounded convolution operator on L^p with p as above

Proof of Claim, $j \in \mathbb{N}$ (Part I)

- It suffices to consider f with supp $f \subset$ cube of side length 2^j $\Rightarrow$ supp $K_\lambda^j * f$ lies in a cube of side length $3 \cdot 2^j$

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- K_λ^j and $\hat{K}_\lambda^j$ are radial $\Rightarrow$ for such f

$$\|K_\lambda^j * f\|_p^2 \leq c\, 2^{6j(\frac{1}{p}-\frac{1}{2})} \|K_\lambda^j * f\|_2^2 = c\, 2^{6j(\frac{1}{p}-\frac{1}{2})} \|\hat{K}_\lambda^j \hat{f}\|_2^2,$$

$$\|\hat{K}_\lambda^j \hat{f}\|_2^2 = \int_0^\infty |\hat{K}_\lambda^j(r,0,0)|^2 \Big(\int_{\mathbb{S}^2} |\hat{f}(r\theta)|^2 \, \mathrm{d}\theta \Big) r^2 \, \mathrm{d}r.$$

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Fourier Restriction Theorem, $1 \leq p \leq \frac{4}{3}$ $\Rightarrow$

$$\int_{\mathbb{S}^2} |\hat{f}(r\theta)|^2 \, \mathrm{d}\theta \leq C r^{-6(p-1)/p} \|f\|_p^2$$

Proof of Claim, $j \in \mathbb{N}$ (Part II)

$$\|\hat{K}_\lambda^j \hat{f}\|_2^2 \le C_p^2 \, \|f\|_p^2 \int_0^\infty |\hat{K}_\lambda^j(r,0,0)|^2 \, r^{2-6(p-1)/p} \, \mathrm{d}r$$

$$= C_p^2 \, \|f\|_p^2 \int_{\mathbb{R}^3} |\hat{K}_\lambda^j(\zeta)|^2 \, |\zeta|^{-6/p'} \, \mathrm{d}\zeta$$

(decompose integral into parts on $B_{1/2}$ and $B_{1/2}^c$,

use $\psi_j, \hat{\psi}_j \in \mathcal{S}(\mathbb{R}^3)$ or pointwise estimate of $K_\lambda(x)$)

$$\le C j^2 2^{-(1+2\Re\lambda)j} \|f\|_p^2$$

Summary: it holds

$$\|K_\lambda^j * f\|_p \le C j \, 2^{\{3(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}(1+2\Re\lambda)\}j} \, \|f\|_p$$

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Theorem I

K_λ defines a bounded convolution operator on each L^p, $p \geq 1$ satisfying $\frac{3}{4} \leq \frac{1}{p} < \frac{1}{3}(2 + \Re\lambda)$, i.e. for $\frac{3}{2+\Re\lambda} < p \leq \frac{4}{3}$ and for $p = 2$.

Complex interpolation $\Rightarrow \frac{3}{2+\Re\lambda} < p \leq 2$
Duality $\Rightarrow$

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{6}(1 + 2\Re\lambda)$$

For $\lambda = \frac{1}{2}$ this condition reduces to $\frac{6}{5} < p < 6$.

Theorem II

Let $\frac{6}{5} < q < 6$ and $|u_\infty| \leq \frac{1}{2}$. Furthermore, let $G \in L^q(\mathbb{R}^3)$ and let $F = \operatorname{div} \Phi$ where $\Phi, (\omega \wedge y) \cdot \nabla\Phi$, $u_\infty \cdot \nabla\Phi \in L^q(\mathbb{R}^3)$

$\Rightarrow$ the linear problem has a unique solution $g \in L^q(\mathbb{R}^3)$ such that

$$\|g\|_q \leq c\big(\|G\|_q + \|\omega\Phi\|_q + \|(\omega \wedge y) \cdot \nabla\Phi\|_q + \|u_\infty \cdot \nabla\Phi\|_q\big)$$

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Get v, σ from the system

$$\begin{aligned} -\mu\Delta v + (u_\infty - \omega\wedge y)\cdot\nabla v + \omega\wedge v + \nabla\sigma &= F + (\mu+\nu)\nabla g \\ \operatorname{div} v &= g \end{aligned}$$

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Theorem III

Let $F = \operatorname{div} \Phi$, g as before, let $\frac{6}{5} < q < 6$. Then v, σ satisfy

$$\begin{aligned} &\|\nabla^2 v\|_q + \|\nabla\sigma\|_q \\ &\qquad \leq c\big(\|F + (\mu + \nu)\nabla g\|_q + \|\mu\nabla g + (\omega \wedge y)g - u_\infty g\|_q\big) \end{aligned}$$

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From Incompressibility to Compressibility

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Standard estimates for incompressible fluid flow around/past a rotating obstacle:

$$1 < q < \infty$$

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Standard estimates for incompressible fluid flow around/past a rotating obstacle:

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Estimates for compressible fluid flow around/past a rotating obstacle:

$$\text{Only for} \quad \frac{6}{5} < q < 6$$

Bochner-Riesz Means

Let B^λ denote the Bochner-Riesz Operator defined by the multiplier function $(1-|\xi|^2)_+^\lambda$, $\lambda \geq 0$, on $\mathbb{R}^n$

- If $\lambda > \frac{n-1}{2}$, then B^λ is bounded on L^q, $1 \leq q \leq \infty$
- If $\lambda > \frac{n-1}{2(n+1)}$ and $\left|\frac{1}{q} - \frac{1}{2}\right| < \frac{1}{2n} + \frac{\lambda}{n}$, then B^λ is L^q-bounded
- If $\lambda > 0$ and $\left|\frac{1}{q} - \frac{1}{2}\right| \geq \frac{1}{2n} + \frac{\lambda}{n}$, then B^λ is L^q-unbounded