Logical Extraction of Effective Bounds in Nonlinear Analysis

Metastability and Abstract Cauchy Problems

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Kickoff Meeting JSPS-DFG Graduate School
Mathematical Fluid Dynamics
Waseda University
June 17-18, 2014
Proof Theory and Consistency Proofs

Proof Theory started with D. Hilbert's program to prove the consistency of mathematics by finitistic means.

Not possible in the strict sense due to Gödel's 2nd incompleteness theorem. Program is still alive: partial realizations possible, relative consistency proofs. Important tools (so-called proof interpretation) were developed in this program which are now used in mathematical practice. G. Kreisel (since 50's): use proof interpretations to extract new information from (prima facie noneffective) proofs.
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**Central Method:** Modern extensions of Gödel’s 1958 (developed as part of modified Hilbert program) **Functional (‘Dialectica’) Interpretation**!
‘Proof Mining’ in core mathematics

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- General logical metatheorems explain this as instances of logical phenomena (K. 2005, Gerhardy/K. 2008, TAMS).

- Some of the logical tools used have recently been rediscovered in special cases by Terence Tao prompted by concrete mathematical needs “finitary analysis”!
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- **independent** from parameters $y \in K$ (**compactness necessary** for **separable** spaces);
- **independent** from parameters $z_1, z_2, z_3$ provided that appropriate norm-bounds $b_1, b_2, b_3$ on $z_1, z_2, z_3$ are available.
Abstract (nonseparable) structures

Examples of such spaces $X$: metric, hyperbolic, CAT(0), normed, their completions, Hilbert, uniformly convex, uniformly smooth, uniformly nonsquare, abstract $L^p$ and $C(K)$ spaces (Günzel/K.)... (not: separable, strictly convex or smooth spaces).
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- Theorems for **abstract** spaces $X$ (not assumed to be separable!): uniform bounds depending only on norm bounds on the $X$-data.
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- Theorems for abstract spaces $X$ (not assumed to be separable!): uniform bounds depending only on norm bounds on the $X$-data.

- Crucially used for this that the proof treats $X$ as abstract structure that is not represented as separable space (via $\mathbb{N}^{\mathbb{N}}$).
Formal systems for analysis with abstract spaces $X$

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$PA^{\omega,X}$ is the extension of Peano Arithmetic to all types.

$A^{\omega,X} := PA^{\omega,X} + DC$, where

**DC**: axiom of dependent choice for all types

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Equality **defined** notion: $x^X =_X y^X : \equiv d_X(x, y) =_{\mathbb{R}} 0_{\mathbb{R}}$. In general only **rule**

If $s =_X t$ has been proved, then $f(s) =_X f(t)$. 
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$A^\omega[X, \| \cdot \|, \ldots]$ results by adding constants with axioms expressing e.g. that $(X, \| \cdot \|, \ldots)$ is a normmed, uniformly convex, Hilbert $\ldots$ space.
Keeping track of uniform bounds: majorization

\[ y, x \text{ functionals of types } \rho, \hat{\rho} := \rho[\mathbb{N}/X] : \]

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x^{\mathbb{N}} \gtrsim_{\mathbb{N}} y^{\mathbb{N}} : \equiv x \geq y
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Example:

\[f^* \gtrsim_{x\rightarrow x} f \equiv \forall n \in \mathbb{N}, x \in X[n \geq \|x\| \rightarrow f^*(n) \geq \|f(x)\|].\]
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If $f : X \rightarrow X$ is nonexpansive (n.e.), i.e. $d(f(x), f(y)) \leq d(x, y).$
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Then for $\|a\|, \|a - f(a)\| \leq b$ and $f^*(n) := n + 3b$: $f^* \succcurlyeq_{X \to X} f$. 

Let $P, K$ be Polish resp. compact metric spaces, $A_\exists \exists$-formula, $z := z_1, \ldots z_k$ variables ranging over $X$, $\IN \to X$ or $X \to X$. 

Interesting connections to uniformity in ultraproducts (continuous model theory; Keisler, Henson, ...).

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$$\forall x \in P \forall y \in K \forall z \exists^\tau \exists^\infty v \exists^\infty A_\exists(x, y, z, v),$$

then one can extract a computable $\Phi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ s.t. the following holds in every nontrivial normed space: for all representatives $r x \in \mathbb{N} \mathbb{N}$ of $x \in P$ and all $z^\tau$ and $z^* \in \mathbb{N} \mathbb{N}$ s.t. $z^* \gtrsim \tau z^\tau$:

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The running theme: convergence statements in analysis

Let \((x_n)\) be a Cauchy sequence in a metric space \((X, \rho)\), i.e.

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is noneffectively equivalent to its Gödel functional interpretation

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A bound \(\Phi(k, g)\) on ‘\(\exists n\)’ in the latter formula is a **rate of metastability** (introduced by **Kreisel** in 1951 as **no-counterexample interpretation**).
Effective full rates of convergence?

In general impossible: There exists a computable decreasing sequence \((a_n)\) of rationals in \([0, 1]\) with no computable rate of convergence (Specker 1949).

Usually possible for asymptotic regularity results \(\rho(x_n, f(x_n)) \to 0\), even when \((x_n)\) may not converge to a fixed point of \(f\).

Possible for \((x_n)\) if sequence converges to unique fixed point/solution. Possible if proof is 'semi-constructive'.

Effective and uniform rates metastability: always possible.

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Ineffective convergence proofs in fluid dynamics

Specialized weak and strong compactness arguments are ubiquitous in fluid dynamics to establish weak or strong solutions of NSE. E.g., Temam 1995 shows that a sequence \((u_k)\) based on a suitable space/time discretization schema for NSE is bounded and so has a weak in \(L^2\) (as well as weak-star in \(L^\infty\)) convergent subsequence; using a specialized Ascoli-type compactness argument, the subsequence even converges strongly in \(L^2\) (and even in \(L^q\) for \(1 \leq q < \infty\)), needed for the passage to the limit in the nonlinear term, and so converges towards a solution \(u\) of NSE; by the uniqueness of \(u\) (in the situation at hand) already the entire sequence \((u_k)\) converges to \(u\).

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- E.g. Temam 1995 shows that a sequence \((u_k)\) based on a suitable space/time discretization schema for NSE is bounded and so has a weak in \(L^2\) (as well as weak-star in \(L^\infty\)) convergent subsequence;
- Using a specialized Ascoli-type compactness argument, the subsequence even converges strongly in \(L^2\) (and even in \(L^q\) for \(1 \leq q < \infty\)), needed for the passage to the limit in the nonlinear term, and so converges towards a solution \(u\) of NSE;
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- By the uniqueness of \(u\) (in the situation at hand) already the entire sequence \((u_k)\) converges to \(u\).
- Many noneffective convergence proofs exist in the context of the abstract nonlinear semigroup approach to abstract Cauchy problems (see below).
Rates of Asymptotic Regularity
Accretive operators (F.E. Browder, T. Kato)

$X$ Banach space, $C \subset X$ convex, $f : C \to C$ pseudocontraction if

$\forall u, v \in C \forall \lambda > 1 \left( (\lambda - 1)\|u - v\| \leq \|(\lambda I - f)(u) - (\lambda I - f)(v)\| \right)$. 

Accretive (pseudocontractive, dissipative) operators are used (often set-valued) in the nonlinear semigroup approach to PDE's (Brezis, Crandall, Liggett, Lions..., Barbu, Bothe). See application further below!
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\]

\( f \) is a pseudocontraction iff \( A := \text{Id} - f \) is accretive (monotone), i.e.

\[
\forall u, v \in C \forall s > 0 \quad \|u - v\| \leq \| u - v + s(Au - Av) \|, \text{ i.e.}
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$$\forall u, v \in C \left( \langle Au - Av, u - v \rangle \geq 0 \right) \text{ in Hilbert space.}$$
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Accretive (pseudocontractive, dissipative) operators are used (often set-valued) in the nonlinear semigroup approach to PDE’s (Brezis, Crandall, Liggett, Lions..., Barbu, Bothe).

See application further below!
Connection to Fluid Dynamics

That $y : [0, T] \to V'$ with $V := \{ y \in (H^1_0(\Omega))^N : \nabla y = 0 \}$ is a weak solution to the classical Navier-Stokes equations can be written as

$$\frac{dy}{dt}(t) + v_0Ay(t) + By(t) = f(t), \quad \text{a.e. } t \in (0, T),$$

where $dy/dt$ is the strong derivative of $y : [0, T] \to V'$.
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where \( \frac{dy}{dt} \) is the strong derivative of \( y : [0, T] \rightarrow V' \), \( A := -P\Delta \), \( B := P(y \cdot \nabla)y \) with the Helmholtz-Leray operator \( P \).
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Define for each \( M > 0 \)

\[
B_M y := \begin{cases} 
By, & \text{if } \|y\| \leq M, \\
\frac{M^2}{\|y\|^2} By, & \text{if } \|y\| > M.
\end{cases}
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Connection to Fluid Dynamics

That \( y : [0, T] \rightarrow V' \) with \( V := \{ y \in (H_0^1(\Omega))^N : \nabla y = 0 \} \) is a weak solution to the classical Navier-Stokes equations can be written as

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\]

Then for a suitable \( \alpha_M > 0 \) the operator \( v_0A + B_M + \alpha_M \cdot I \) is \((m)\)-accretive (e.g. V. Barbu: Nonlinear Differential Equations of Monotone Types in Banach spaces. Springer 2010).
Asymptotic regularity for pseudocontractions

Let $X$ be a Banach space, $C \subset X$ a bounded convex subset and $f : C \to C$ a (Lipschitzian) pseudocontraction.
Asymptotic regularity for pseudocontractions

Let $X$ be a Banach space, $C \subset X$ a bounded convex subset and $f : C \to C$ a (Lipschitzian) pseudocontraction.

In 1974 Bruck considered the following iteration schema

$$x_{n+1} := (1 - \lambda_n)x_n + \lambda_n f(x_n) - \lambda_n \theta_n(x_n - x_1),$$

for suitable $(\lambda_n), (\theta_n)$ in $(0, 1]$ and showed asymptotic regularity and strong convergence (towards a fixed point) results in Hilbert space.
Asymptotic regularity for Lipschitzian pseudocontractions in arbitrary Banach spaces

**Theorem (Chidume, Zegeye 2004):** \( \lim_{n \to \infty} \| x_n - f(x_n) \| = 0 \), where

(i) \( \lim \theta_n = 0 \),

(ii) \( \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty \),

(iii) \( \lim \frac{\lambda_n}{\theta_n} = 0 \),

(iv) \( \lim \frac{\theta_{n-1} - \theta_n}{\lambda_n \theta_n} = 0 \),

(v) \( \lambda_n (1 + \theta_n) \leq 1 \).
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(iv) \( \lim \frac{\theta_n - 1}{\lambda_n \theta_n} = 0 \), (v) \( \lambda_n (1 + \theta_n) \leq 1 \).

Let \( D \geq \text{diam}(C) \), \( L \)-Lipschitz constant and \((\lambda_n), (\theta_n) \subset (0, 1]\) with rates of conv./div. \( R_i : (0, \infty) \to \mathbb{N} \)

1. \( \forall \varepsilon > 0 \forall n \geq R_1(\varepsilon) (\theta_n \leq \varepsilon) \),
2. \( \forall x \in (0, \infty) \left( \sum_{n=1}^{R_2(x)} \lambda_n \theta_n \geq x \right) \),
3. \( \forall \varepsilon > 0 \forall n \geq R_3(\varepsilon) (\lambda_n \leq \theta_n \varepsilon) \),
4. \( \forall \varepsilon > 0 \forall n \geq R_4(\varepsilon) \left( \frac{\left| \frac{\theta_n - 1}{\theta_n} - 1 \right|}{\lambda_n \theta_n} \leq \varepsilon \right) \).
Theorem (D. Körnlein/K. Nonlinear Analysis 2011)

∀ε > 0 ∀n ≥ \(\Psi (D, L, R_1, R_2, R_3, R_4, \varepsilon)\) \(||x_n - fx_n|| < \varepsilon\)
Theorem (D. Körnlein/K. Nonlinear Analysis 2011)

∀ε > 0 ∀n ≥ \( \Psi (D, L, R_1, R_2, R_3, R_4, \varepsilon) (\|x_n - fx_n\| < \varepsilon) \)

where

\[ \Psi (D, L, R_1, R_2, R_3, R_4, \varepsilon) = \max \left\{ N_2 (C) + 1, R_1 \left( \frac{\varepsilon}{2D} \right) + 1 \right\} \]

and

\[ N_1 (\varepsilon) := \max \left\{ R_3 \left( \frac{\varepsilon}{4D^2 (2 + L)} \right), R_4 \left( \sqrt{\frac{\varepsilon}{D^2}} + 1 - 1 \right) \right\}, \]

\[ N_2 (x) := R_2 \left( \frac{x}{2} \right) + 1, \]

\[ C := \frac{8 (1 + L)^2 D^2}{\varepsilon^2} + 2 \left( N_1 \left( \frac{\varepsilon^2}{8 (1 + L)^2} \right) - 1 \right). \]
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Exponential bound for unbounded $C$ if $\text{Fix}(f) \neq \emptyset$. 
Bounds on Metastability
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Tao used a rate of metastability for the von Neumann Mean Ergodic Theorem as base step for a generalization to commuting families of operators.
Bounds on Metastability

Tao used a rate of metastability for the von Neumann Mean Ergodic Theorem as base step for a generalization to commuting families of operators.

‘We shall establish Theorem 1.6 by “finitary ergodic theory” techniques, reminiscent of those used in [Green-Tao]...’ ‘The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit’...‘In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation’

(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)
Let \((x_n)\) be the Bruck iteration of an \(L\)-Lipschitzian pseudo-contraction \(f : C \rightarrow C\), where \(C\) is a \(D\)-bounded closed and convex subset of a real Hilbert space \(X\).
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Then \((x_n)\) converges to a fixed point of \(f\) (Bruck, Chidume, Zegeye).


\[
\forall \varepsilon > 0 \forall g : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \chi_M(g(\lceil 64D^2/\varepsilon^2 \rceil)) h, \chi(1) + \Psi(\varepsilon) + 1
\forall i, j \in \{n; n + g(n)\} \forall k \geq n (\|x_i - x_j\| \leq \varepsilon \land \|Tx_k - x_k\| \leq \varepsilon),
\]

where \(h : \mathbb{N} \rightarrow \mathbb{N}\) such that \(h(n) \geq 1/\theta n\) for all \(n \in \mathbb{N}\) and \(\chi(n) := R_1(1/n)\), \(g^{\prime}(n) := g(n + 1 + \Psi(\varepsilon)) + \Psi(\varepsilon) + 1\), \(g^h,\chi(n) := \max\{h(i) : i \leq \chi(n) + g^h(\chi(n))\}\).

Here \(R_1\) and \(\Psi\) are as in the previous theorem.
Let \((x_n)\) be the Bruck iteration of an \(L\)-Lipschitzian pseudo-contraction \(f : C \to C\), where \(C\) is a \(D\)-bounded closed and convex subset of a real Hilbert space \(X\).

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\[
\begin{align*}
\forall \varepsilon > 0 \forall g : \mathbb{N} &\to \mathbb{N} \exists n \leq \chi^M \left(g^{\left\lceil \frac{64D^2}{\varepsilon^2} \right\rceil}(1)\right) + \Psi(\varepsilon) + 1 \\
\forall i, j \in [n; n + g(n)] \forall k \geq n \left(\|x_i - x_j\| \leq \varepsilon \land \|Tx_k - x_k\| \leq \varepsilon\right),
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and \(\chi(n) := \mathcal{R}_1(1/n), \quad g'(n) := g(n + 1 + \Psi(\varepsilon)) + \Psi(\varepsilon) + 1,
\]

\(g_{h,\chi}(n) := \max \{h(i) : i \leq \chi(n) + g'(\chi(n))\}\)

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Cauchy problems and set-valued accretive operators

A set-valued operator $A : D(A) \to 2^X$ is accretive if

$$\forall (x, u), (y, v) \in A \ (\langle u - v, x - y \rangle_+ \geq 0),$$

where $\langle y, x \rangle_+ := \max\{\langle y, j \rangle : j \in J(x)\}$ for the normalized duality map $J$ of the Banach space $X$. 

E.g. this holds for $m$-ψ-strongly accretive operators or even for $\varphi$-accretive operators in the sense of García-Falset if $\varphi$ has some normal form (which is the case in many applications).
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A with $0 \in Az$ is **uniformly accretive at zero with modulus** $\Theta : \mathbb{N}^2 \to \mathbb{N}$ if, moreover,

$$\forall k, K \in \mathbb{N} \forall (x, u) \in A \ (\|x - z\| \in [2^{-k}, K] \to \langle u, x - z \rangle_+ \geq 2^{-\Theta_k(k)}),$$

(Koutsoukou-Argyraki/K. 2014).
Cauchy problems and set-valued accretive operators

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Consider the following homogeneous Cauchy problem for an accretive $A$
(with range condition):

\[
\begin{aligned}
(1) \quad & \begin{cases}
    u'(t) + A(u(t)) \ni 0, & t \in [0, \infty) \\
    u(0) = x_0,
\end{cases}
\end{aligned}
\]

which has a unique integral solution for $x_0 \in D(A)$ given by the
Crandall-Ligget formula

\[
\begin{aligned}
    u(t) := S(t)(x_0) := \lim_{n \to \infty} (I + \frac{t}{n} A)^{-n}(x_0).
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A continuous $v : [0, \infty) \to \overline{D(A)}$ is an almost-orbit of the nonexpansive semigroup $S$ if

$$\lim_{s \to \infty} \left( \sup_{t \in [0, \infty)} \|v(t + s) - S(t)v(s)\| \right) = 0.$$
**Theorem (García-Falset 2005)**

Let $A$ be a $\phi$-accretive (at zero) operator with range condition s.t. (1) has a strong solution for each $x_0 \in D(A)$. Then every almost-orbit (for the semigroup generated by $-A$) strongly converges to the zero $z$ of $A$. 

**Theorem (Koutsoukou-Argyraki/K. 2014)**

Same as above but $A$ uniformly accretive at zero with modulus $\Theta$. Then $\forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists n \leq \Psi \forall x \in [n, n + g(n)] (\|v(x) - z\| < 2^{-k}),$ where

$$\Psi(k, g, B, \Phi, \Theta) := \Phi(k + 1, g) + h(\Phi(k + 1, g)),$$

with

$$h(n) := (B(n) + 2) \cdot 2 \cdot K(n)^{k+2} + 1, \quad g(n) := g(n + h(n)) + h(n),$$

$$K(n) := \lceil \sqrt{2(B(n) + 1)} \rceil, \quad B(n) \geq \frac{1}{2} \|v(n) - z\|^2,$$

and $\Phi$ is rate of metastability for $v$, i.e. $\forall k, g \exists n \leq \Phi(k, g) \forall t \in [0, g(n)] (\|v(t + n) - S(t)v(n)\| \leq 2^{-k})$. 

Logical Extraction of Bounds
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$$\forall k \in \mathbb{N} \forall g : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \Psi \forall x \in [n, n + g(n)] \left( ||v(x) - z|| < 2^{-k} \right),$$

where $\Psi(k, g, B, \Phi, \Theta) := \Phi(k + 1, g) + h(\Phi(k + 1, g))$

$h(n) := (B(n) + 2) \cdot 2^{\Theta K(n)} (k + 2 + 1)$,

g(n) := g(n + h(n)) + h(n),

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where

$$
\Psi(k, g, B, \Phi, \Theta) := \Phi(k + 1, g) + h(\Phi(k + 1, g)), \text{ with }
$$

$$
h(n) := (B(n) + 2) \cdot 2^{\Theta K(n)(k+2)+1}, \ g(n) := g(n + h(n)) + h(n), \text{ with }
$$

$$
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$$

and $\Phi$ is rate of metastability for $v$, i.e.

$\forall k, g \exists n \leq \Phi(k, g) \forall t \in [0, g(n)] \left( \| v(t + n) - S(t)v(n) \| \leq 2^{-k} \right).$
Consider now the inhomogeneous Cauchy problem ($A$ as before):

\[
\begin{align*}
(2) \quad \begin{cases} 
 u'(t) + A(u(t)) \ni f(t), & t \in [0, \infty) \\
 u(0) = x,
\end{cases}
\end{align*}
\]

where $f \in L^1(0, \infty, X)$. 
Consider now the inhomogeneous Cauchy problem \((A\text{ as before})\):

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\end{cases}
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where \(f \in L^1(0, \infty, X)\).

Then for each \(x \in D(A)\) the \textbf{integral solution} \(u(\cdot)\) of (2) is an \textbf{almost-orbit} (Miyadera-Kobayasi 1982)
Consider now the inhomogeneous Cauchy problem ($A$ as before):

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(2) \begin{cases}
    u'(t) + A(u(t)) \in f(t), & t \in [0, \infty) \\
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\end{cases}
\]

where $f \in L^1(0, \infty, X)$.

Then for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of (2) is an almost-orbit (Miyadera-Kobayasi 1982) and

Proposition (Koutsoukou-Argyraki/K., 2014)

\[
\Phi_M(k, g) := \tilde{g}^M \cdot 2^{k+1}(0) \text{ with } \tilde{g}(n) := n + g(n), \quad M \geq \int_0^\infty \|f(\xi)\| d\xi
\]

is a rate of metastability of $u$ (and so can be used as $\Phi$ in the previous theorem).
A concrete Cauchy problem

Consider the following Cauchy problem (compare Andreu, Mazón, Moll 2005):

\[ u_t - \text{div}(|Du|^{p-2}Du) + \varphi(x, u) = f, \text{ on } (0, \infty) \times \Omega, \]

\[ -\frac{\partial u}{\partial \eta} \in \beta(u) \text{ on } [0, \infty) \times \partial \Omega, \]

\[ u(0, x) = u_0 \in L^q(\Omega), \]

where \( \Omega \) is a bounded open domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( f \in L^1((0, \infty), L^q(\Omega)) \), \( 1 \leq p, q < \infty \), \( \frac{\partial u}{\partial \eta} = \langle |Du|^{p-2}Du, \eta \rangle \), \( \eta \) the unit outward normal on \( \partial \Omega \), \( Du \) the gradient of \( u \), \( \beta \) a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) with \( 0 \in \beta(0) \) and \( \varphi : \Omega \times \mathbb{R} \to \mathbb{R} \) satisfying the following conditions:
1. For almost all \( x \in \Omega \), \( r \to \varphi(x, r) \) is continuous and nondecreasing,

2. For every \( r \in \mathbb{R} \), \( x \to \varphi(x, r) \) is in \( L^1(\Omega) \),

3. \( \varphi(x, 0) = 0 \), \( \varphi(x, r) \neq 0 \) whenever \( r \neq 0 \) and there exist \( \lambda > 0 \), \( \alpha \geq 2 \) such that \( \varphi(x, r)r \geq \lambda|r|^\alpha \).
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Then the problem can be written in the form (2) s.t. (1) has a strong solution (García-Falset 2005) and \( A \) is even uniformly accretive at zero with modulus being any

\[
\Theta(k) \geq k \cdot \alpha - \log_2 C_{\alpha, \Omega, \lambda}
\]

for some constant \( C_{\alpha, \Omega, \lambda} \) depending only on the data indicated (Koutsoukou-Argyraki/K., 2014).
Other Recent Applications to Nonlinear Analysis

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- Metastability for Baillon's nonlinear ergodic theorem ([K., Comm.Contemp.Math. 2012]).

- Metastability for Wittmann's strong nonlinear ergodic theorem on Halpern iterations ([K., Adv.Math. 2011]).


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\[ \forall u, v \in C \left( \|f(u) + f(v)\| \leq \|u + v\| \right) \].

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