Logical Extraction of Effective Bounds in Nonlinear Analysis

Metastability and Abstract Cauchy Problems

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- Important tools (so-called **proof interpretation**) where developed in this program which are now used in mathematical practice.
- G. Kreisel (since 50's): use proof interpretations to **extract new information from** (prima facie noneffective) proofs.

Extractive Proof Theory: Proof Interpretations

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Central Method: Modern extensions of **Gödel**'s 1958 (developed as part of modified Hilbert program) **Functional ('Dialectica') Interpretation!**

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- General **logical metatheorems** **explain** this as instances of logical phenomena (K. 2005, Gerhardy/K. 2008, TAMS).
- Some of the logical tools used have recently been rediscovered in special cases by Terence Tao prompted by concrete mathematical needs **“finitary analysis”!**

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- **independent** from parameters $y \in K$ (**compactness necessary** for **separable** spaces);
- **independent** from parameters z_1, z_2, z_3 provided that appropriate norm-bounds b_1, b_2, b_3 on z_1, z_2, z_3 are available.

Abstract (nonseparable) structures

Examples of such spaces X : metric, hyperbolic, CAT(0), normed, their completions, Hilbert, uniformly convex, uniformly smooth, uniformly nonsquare, abstract L^p and $C(K)$ spaces (Günzel/K.)... (not: separable, strictly convex or smooth spaces).

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- Theorems for **abstract** spaces X (not assumed to be separable!): uniform bounds depending only on norm bounds on the X-data.
- Crucially used for this that the proof treats X as **abstract structure** that is **not represented as separable space** (via $\mathbb{N}^{\mathbb{N}}$).

Formal systems for analysis with abstract spaces X

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Equality **defined** notion: $\mathbf{x^X =_X y^X :\equiv d_X(x, y) =_{\mathbb{R}} 0_{\mathbb{R}}}$.. In general only **rule**

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$\mathcal{A}^{\omega}[X, \|\cdot\| \ldots]$ results by adding constants with axioms expressing e.g. that $(X, \|\cdot\|, \ldots)$ is a normmed, uniformly convex, Hilbert ... space.

Keeping track of uniform bounds: majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$:

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$$\mathbf{f}^* \gtrsim_{\mathbf{X}\to\mathbf{X}} \mathbf{f} \equiv \forall \mathbf{n} \in \mathbb{N}, \mathbf{x} \in \mathbf{X}[\mathbf{n} \geq \|\mathbf{x}\| \to \mathbf{f}^*(\mathbf{n}) \geq \|\mathbf{f}(\mathbf{x})\|].$$

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If $\mathbf{f} : \mathbf{X} \to \mathbf{X}$ is **nonexpansive (n.e.)**, i.e. $\mathbf{d}(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})) \leq \mathbf{d}(\mathbf{x}, \mathbf{y})$.

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Then for $\|\mathbf{a}\|, \|\mathbf{a} - \mathbf{f}(\mathbf{a})\| \leq \mathbf{b}$ and $\mathbf{f}^*(\mathbf{n}) := \mathbf{n} + 3\mathbf{b}$: $\mathbf{f}^* \gtrsim_{\mathbf{X}\to\mathbf{X}} \mathbf{f}$.

Theorem (K., Trans.AMS 2005, Gerhardy/K.,Trans.AMS 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{z} := z_1, \ldots z_k$ variables ranging over X, $\mathbb{N} \to X$ or $X \to X$.

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then one can extract a **computable** $\Phi : \mathbb{N}^\mathbb{N} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every nontrivial normed space: for all representatives $r_x \in \mathbb{N}^\mathbb{N}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\underline{z}^* \gtrsim_{\underline{\tau}} \underline{z}$:

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Interesting connections to uniformity in **ultraproducts** (continuous model theory; Keisler, Henson,...).

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A bound $\Phi(k, g)$ on '$\exists n$' in the latter formula is a **rate of metastability** (introduced by **Kreisel** in 1951 as **no-counterexample interpretation).**

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- Possible if proof is **'semi-constructive'**.

Effective full rates of convergence?

- **In general impossible:** There exists a computable decreasing sequence $(\mathbf{a_n})$ of rationals in $[0,1]$ $(\mathbf{a_n})$ with **no computable rate of convergence** (Specker 1949).
- Usually **possible for asymptotic regularity** results

$$\rho(\mathbf{x_n}, \mathbf{f}(\mathbf{x_n})) \to \mathbf{0},$$

even when (x_n) may not converge to a fixed point of f.

- Possible for (x_n) if sequence converges to **unique** fixed point/solution.
- Possible if proof is **'semi-constructive'**.
- **Effective and uniform rates metastability:** always possible.

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- By the uniqueness of u (in the situation at hand) already the entire sequence (u_k) converges to u.
- Many noneffective convergence proofs exist in the context of the abstract nonlinear semigroup approach to abstract Cauchy problems (see below).

Rates of Asymptotic Regularity

Accretive operators (F.E. Browder, T. Kato)

X Banach space, $C \subset X$ convex, $f : C \to C$ **pseudocontraction** if

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{C} \forall \boldsymbol{\lambda} > \mathbf{1} \left((\boldsymbol{\lambda} - \mathbf{1}) \|\mathbf{u} - \mathbf{v}\| \leq \|(\boldsymbol{\lambda} \mathbf{I} - \mathbf{f})(\mathbf{u}) - (\boldsymbol{\lambda} \mathbf{I} - \mathbf{f})(\mathbf{v})\| \right).$$

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f is a pseudocontraction iff $A := Id - f$ is **accretive (monotone)**, i.e

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{C} \forall \mathbf{s} > \mathbf{0} \left(\|\mathbf{u} - \mathbf{v}\| \leq \|\mathbf{u} - \mathbf{v} + \mathbf{s}(\mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{v})\|, \text{ i.e.} \right.$$

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$$\forall \mathbf{u},\mathbf{v} \in \mathbf{C} \; (\langle \mathbf{Au} - \mathbf{Av}, \mathbf{u} - \mathbf{v} \rangle \geq \mathbf{0}) \; \text{ in Hilbert space.}$$

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See application further below!

Connection to Fluid Dynamics

That $y : [0, T] \to V'$ with $V :=:= \{y \in (H_0^1(\Omega))^N : \nabla y = 0\}$ is a weak solution to the classical Navier-Stokes equations can be written as

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Define for each $M > 0$

$$\mathbf{B_M y} := \begin{cases} \mathbf{By}, & \text{if } \|\mathbf{y}\| \leq \mathbf{M}, \\ \frac{\mathbf{M^2}}{\|\mathbf{y}\|^2}\mathbf{By}, & \text{if } \|\mathbf{y}\| > \mathbf{M}. \end{cases}$$

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Then for a suitable $\alpha_M > 0$ the operator $\mathbf{v_0 A + B_M + \alpha_M \cdot I}$ is (m)-**accretive** (e.g. V. Barbu: Nonlinear Differential Equations of Monotone Types in Banach spaces. Springer 2010).

Asymptotic regularity for pseudocontractions

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In 1974 Bruck considered the following iteration schema

$$\mathbf{x_n + 1 := (1 - \lambda_n)x_n + \lambda_n f(x_n) - \lambda_n \theta_n (x_n - x_1),}$$

for suitable $(\lambda_n), (\theta_n)$ in $(0, 1]$ and showed asymptotic regularity and strong convergence (towards a fixed point) results in Hilbert space.

Asymptotic regularity for Lipschitzian pseudocontractions in arbitrary Banach spaces

Theorem (Chidume,Zegeye 2004): $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0$, where

$$\textbf{(i)}\ \lim \theta_n = 0,\ \textbf{(ii)}\ \sum_{n=1}^{\infty} \lambda_n \theta_n = \infty,\quad \textbf{(iii)}\ \lim \frac{\lambda_n}{\theta_n} = 0,$$

$$\textbf{(iv)}\ \lim \frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n \theta_n} = 0,\quad \textbf{(v)}\ \lambda_n(1+\theta_n) \le 1.$$

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Let $D \ge diam(C)$, L-Lipschitz constant and $(\lambda_n), (\theta_n) \subset (0,1]$ with rates of conv./div. $R_i : (0,\infty) \to \mathbb{N}$

1. $\forall \varepsilon > 0 \forall n \ge R_1(\varepsilon)\,(\theta_n \le \varepsilon)$,
2. $\forall x \in (0,\infty) \left(\sum\limits_{n=1}^{R_2(x)} \lambda_n\theta_n \ge x\right)$,
3. $\forall \varepsilon > 0 \forall n \ge R_3(\varepsilon)\,(\lambda_n \le \theta_n\varepsilon)$,
4. $\forall \varepsilon > 0 \forall n \ge R_4(\varepsilon) \left(\frac{\left|\frac{\theta_{n-1}}{\theta_n} - 1\right|}{\lambda_n\theta_n} \le \varepsilon\right)$.

Theorem (D. Körnlein/K. Nonlinear Analysis 2011)

$$\forall \varepsilon > 0 \forall \mathbf{n} \geq \mathbf{\Psi}\left(\mathbf{D}, \mathbf{L}, \mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}, \mathbf{R_4}, \varepsilon\right)\left(\|\mathbf{x_n} - \mathbf{f x_n}\| < \varepsilon\right)$$

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where

$$\Psi\left(D, L, R_1, R_2, R_3, R_4, \varepsilon\right) = \max\left\{N_2\left(C\right) + 1, R_1\left(\frac{\varepsilon}{2D}\right) + 1\right\}$$

and

$$N_1\left(\varepsilon\right) := \max\left\{R_3\left(\frac{\varepsilon}{4D^2\left(2+L\right)}\right), R_4\left(\sqrt{\frac{\varepsilon}{D^2} + 1} - 1\right)\right\},$$

$$N_2\left(x\right) := R_2\left(\frac{x}{2}\right) + 1,$$

$$C := \frac{8\left(1+L\right)^2 D^2}{\varepsilon^2} + 2\left(N_1\left(\frac{\varepsilon^2}{8\left(1+L\right)^2}\right) - 1\right).$$

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Exponential bound for **unbounded** C if $Fix(f) \neq \emptyset$.

Bounds on Metastability

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'We shall establish Theorem 1.6 by "finitary ergodic theory" techniques, reminiscent of those used in [Green-Tao]...' 'The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit'...'In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation'
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

Let (x_n) be the Bruck iteration of an L-Lipschitzian pseudo-contraction $f : C \to C$, where C is a D-bounded closed and convex subset of a real **Hilbert space** X.

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Theorem (Körnlein/K., Num. Funct. Anal. Opt. 2013)

$$\begin{cases} \forall \varepsilon > 0 \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \leq \chi^M \left(g_{h,\chi}^{(\lceil 64D^2/\varepsilon^2 \rceil)}(1) \right) + \Psi(\varepsilon) + 1 \\ \quad \forall i, j \in [n; n + g(n)] \, \forall k \geq n \left(\|x_i - x_j\| \leq \varepsilon \wedge \|Tx_k - x_k\| \leq \varepsilon \right), \end{cases}$$

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where $h : \mathbb{N} \to \mathbb{N}$ such that $h(n) \geq 1/\theta_n$ for all $n \in \mathbb{N}$
and $\chi(n) := R_1(1/n), \quad g'(n) := g(n + 1 + \Psi(\varepsilon)) + \Psi(\varepsilon) + 1,$
$g_{h,\chi}(n) := \max\{h(i) : i \leq \chi(n) + g'(\chi(n))\}$

Here R_1 and Ψ are as in the previous theorem.

Cauchy problems and set-valued accretive operators

A set-valued operator $A : D(A) \to 2^X$ is **accretive** if

$$\forall \mathbf{(x,u),(y,v)} \in \mathbf{A} \ \big(\langle \mathbf{u}-\mathbf{v}, \mathbf{x}-\mathbf{y}\rangle_+ \geq \mathbf{0}\big),$$

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A with $0 \in Az$ is **uniformly accretive at zero with modulus** $\Theta : \mathbb{N}^2 \to \mathbb{N}$ if, moreover,

$$\forall \mathbf{k}, \mathbf{K} \in \mathbb{N}\, \forall (\mathbf{x}, \mathbf{u}) \in \mathbf{A} \left(\|\mathbf{x} - \mathbf{z}\| \in [\mathbf{2}^{-\mathbf{k}}, \mathbf{K}] \to \langle \mathbf{u}, \mathbf{x} - \mathbf{z} \rangle_+ \geq \mathbf{2}^{-\Theta_{\mathbf{K}}(\mathbf{k})} \right)$$

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(Koutsoukou-Argyraki/K. 2014). E.g. this holds for m-ψ-strongly accretive operators or even for ϕ-accretive operators in the sense of García-Falset if ϕ has some normal form (which is the case in many applications).

Consider the following homogeneous Cauchy problem for an accretive A (with range condition):

$$\mathbf{(1)}\ \begin{cases} u'(t) + A(u(t)) \ni 0, & t \in [0, \infty) \\ u(0) = x_0, \end{cases}$$

which has a unique integral solution for $x_0 \in \overline{D(A)}$ given by the Crandall-Liggett formula

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A continuous $\mathbf{v : [0,\infty) \to \overline{D(A)}}$ is an **almost-orbit** of the nonexpansive semigroup $\mathbf{S}$ if

$$\mathbf{\lim_{s\to\infty} \Big(\sup_{t\in[0,\infty)} \|v(t+s) - S(t)v(s)\| \Big) = 0.}$$

Theorem (García-Falset 2005)

Let A be a ϕ-accretive (at zero) operator with range condition s.t. (1) *has a strong solution for each $x_o \in D(A)$. Then every almost-orbit (for the semigroup generated by $-A$) strongly converges to the zero z of A.*

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Same as above but A uniformly accretive at zero with modulus Θ. Then

$$\forall \mathbf{k} \in \mathbb{N} \forall \overline{\mathbf{g}} : \mathbb{N} \to \mathbb{N} \exists \overline{\mathbf{n}} \leq \boldsymbol{\Psi} \forall \mathbf{x} \in [\overline{\mathbf{n}}, \overline{\mathbf{n}} + \overline{\mathbf{g}}(\overline{\mathbf{n}})] \left(\|\mathbf{v}(\mathbf{x}) - \mathbf{z}\| < \mathbf{2}^{-\mathbf{k}} \right),$$

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where

$$\Psi(k, \overline{g}, B, \Phi, \Theta) := \Phi(k+1, g) + h(\Phi(k+1, g)), \text{ with}$$
$$h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1}, \; g(n) := \overline{g}(n + h(n)) + h(n),$$
$$K(n) := \left\lceil \sqrt{2(B(n)+1)} \right\rceil, \; B(n) \geq \frac{1}{2}\|v(n) - z\|^2,$$

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$$\begin{aligned}
&\mathbf{\Psi}(\mathbf{k}, \overline{\mathbf{g}}, \mathbf{B}, \mathbf{\Phi}, \mathbf{\Theta}) := \mathbf{\Phi}(\mathbf{k}+1, \mathbf{g}) + \mathbf{h}(\mathbf{\Phi}(\mathbf{k}+1, \mathbf{g})), \ \text{with} \\
&\mathbf{h}(\mathbf{n}) := (\mathbf{B}(\mathbf{n}) + 2) \cdot 2^{\Theta_{\mathbf{K}(\mathbf{n})}(\mathbf{k}+2)+1}, \ \mathbf{g}(\mathbf{n}) := \overline{\mathbf{g}}(\mathbf{n} + \mathbf{h}(\mathbf{n})) + \mathbf{h}(\mathbf{n}), \\
&\mathbf{K}(\mathbf{n}) := \left\lceil \sqrt{2(\mathbf{B}(\mathbf{n}) + 1)} \right\rceil, \ \mathbf{B}(\mathbf{n}) \geq \tfrac{1}{2} \|\mathbf{v}(\mathbf{n}) - \mathbf{z}\|^2,
\end{aligned}$$

and $\mathbf{\Phi}$ is rate of metastability for $\mathbf{v}$, i.e.

$$\forall \mathbf{k}, \mathbf{g} \exists \mathbf{n} \leq \mathbf{\Phi}(\mathbf{k}, \mathbf{g}) \forall \mathbf{t} \in [0, \mathbf{g}(\mathbf{n})] (\|\mathbf{v}(\mathbf{t}+\mathbf{n}) - \mathbf{S}(\mathbf{t})\mathbf{v}(\mathbf{n})\| \leq 2^{-\mathbf{k}}).$$

Application (compare García-Falset, 2005)

Consider now the inhomogeneous Cauchy problem (A as before):

$$\mathbf{(2)}\quad \begin{cases} \mathbf{u'(t) + A(u(t)) \ni f(t), \quad t \in [0, \infty)} \\ \mathbf{u(0) = x,} \end{cases}$$

where $\mathbf{f \in L^1(0, \infty, X)}$.

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Proposition (Koutsoukou-Argyraki/K., 2014)

$\mathbf{\Phi_M(k, g) := \tilde{g}^{M \cdot 2^{k+1}}(0)}$ *with* $\mathbf{\tilde{g}(n) := n + g(n)}$, $\mathbf{M} \geq \int_0^\infty \|\mathbf{f(\xi)}\|\mathbf{d}\xi$ *is a rate of metastability of* $\mathbf{u}$ *(and so can be used as* $\mathbf{\Phi}$ *in the previous theorem).*

A concrete Cauchy problem

Consider the following Cauchy problem (compare Andreu,Mazón, Moll 2005):

$$u_t - div(|Du|^{p-2}Du) + \varphi(x,u) = f, \ on \ (0,\infty) \times \Omega,$$

$$-\frac{\partial u}{\partial \eta} \in \beta(u) \ on \ [0,\infty) \times \partial\Omega,$$

$$u(0,x) = u_0 \in L^q(\Omega),$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $f \in L^1((0,\infty), L^q(\Omega))$, $1 \leq p, q < \infty$, $\frac{\partial u}{\partial \eta} = \langle |Du|^{p-2}Du, \eta \rangle$, η the unit outward normal on $\partial\Omega$, Du the gradient of u, β a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

1. for almost all $x \in \Omega$, $r \to \varphi(x,r)$ is continuous and nondecreasing,
2. for every $r \in \mathbb{R}$, $x \to \varphi(x,r)$ is in $L^1(\Omega)$,
3. $\varphi(x,0) = 0$, $\varphi(x,r) \neq 0$ whenever $r \neq 0$ and there exist $\lambda > 0$, $\alpha \geq 2$ such that $\varphi(x,r)r \geq \lambda |r|^\alpha$.

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Then the problem can be written in the form (2) s.t. (1) **has a strong solution** (García-Falset 2005) and A is even uniformly accretive at zero with modulus being any

$$\mathbf{\Theta(k) \geq k \cdot \alpha - \log_2 C_{\alpha,\Omega,\lambda}}$$

for some constant $\mathbf{C_{\alpha,\Omega,\lambda}}$ depending only on the data indicated (Koutsoukou-Argyraki/K., 2014).

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- **Generalization to CAT(κ) spaces** $\kappa > 0$ (**Leuştean, Nicolae (submitted)**).

- **Metastable version of strong nonlinear ergodic theorem** for maps
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(covers Baillon's result for n.e. odd operators): **Safarik, JMAA 2012**.

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