Dissipative Structure for Symmetric Hyperbolic Systems with Relaxation

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Outline

1. Introduction

Consider linear partial differential equations:

$$P(\partial_t, \partial_x)u = 0.$$

The characteristic equation is: $P(\lambda, i\xi) = 0.$

Let $\lambda = \lambda(i\xi)$ be the dispersion relation.

Dissipative structure: Key to the stability for $t \to \infty$.

- Dissipativity: $\mathrm{Re}\,\lambda(i\xi) \le 0$ for any ξ.
- Strict dissipativity: $\mathrm{Re}\,\lambda(i\xi) < 0$ for any $\xi \neq 0$.

Type (I): $\mathrm{Re}\,\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)$

Type (II): $\mathrm{Re}\,\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)^2$

Strict dissipativity

Strict dissipativity of Type (p, q):

$$\operatorname{Re}\lambda(i\xi) \le -c\eta(\xi), \qquad \eta(\xi) = |\xi|^{2p}/(1+|\xi|^2)^q$$

Regularity-loss index: $\sigma = 2(q-p)$

- Type (I): Type $(1,1)$, $\sigma = 0$
- Type (II): Type $(1,2)$, $\sigma = 2$
- Type $(2,2)$: $\sigma = 0$
- Type $(2,3)$: $\sigma = 2$

Strict dissipativity

Type (I): $\mathrm{Re}\,\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)$

$$\mathrm{Re}\,\lambda(i\xi) \sim -c|\xi|^2 \ \text{ for } \ |\xi| \to 0,$$
$$\mathrm{Re}\,\lambda(i\xi) \sim -c \ \text{ for } \ |\xi| \to \infty.$$

General framework:

- Symmetric hyperbolic-parabolic systems
- Symmetric hyperbolic systems
  - T. Umeda, S.K & Y. Shizuta (1984): Condition (K)
  - Y. Shizuta & S.K (1985): (SK) stability condition
  - K. Beauchard & E. Zuazua (2010): Kalman rank condition
- Symmetric hyperbolic-elliptic systems
  - S.K, Y. Nikkuni & S. Nishibata (1998)

Strict dissipativity

Type (II): $\operatorname{Re}\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)^2$

$$\operatorname{Re}\lambda(i\xi) \sim -c|\xi|^2 \quad \text{for} \quad |\xi| \to 0,$$
$$\operatorname{Re}\lambda(i\xi) \sim -c|\xi|^{-2} \quad \text{for} \quad |\xi| \to \infty.$$

$\lambda(i\xi)$ may approach the imaginary axis $\operatorname{Re}\lambda = 0$ for $|\xi| \to \infty$.

Decay property of regularity-loss type:

- Dissipative Timoshenko system
  - ◦ J.E.M. Rivera & R. Racke (2003)
  - ◦ K. Ide, K. Haramoto & S.K (2008)
  - ◦ N. Mori & S.K (2014)
- Euler-Maxwell system
  - ◦ R. Duan (2011)
  - ◦ Y. Ueda & S.K (2011)

Strict dissipativity

Type (2,3): $\operatorname{Re}\lambda(i\xi) \le -c|\xi|^4/(1+|\xi|^2)^3$

$$\operatorname{Re}\lambda(i\xi) \sim -c|\xi|^4 \quad \text{for} \quad |\xi| \to 0,$$
$$\operatorname{Re}\lambda(i\xi) \sim -c|\xi|^{-2} \quad \text{for} \quad |\xi| \to \infty.$$

Decay property of regularity-loss type:

- Timoshenko-Fourier system
  - ◦ J.E.M. Rivera & R. Racke (2002)
  - ◦ N. Mori & S.K (2014)
- Timoshenko-Cattaneo system
  - ◦ H.D.F. Sare & R. Racke (2009)
  - ◦ M.L. Santos, D.S.A. Júnior & J.E.M. Rivera (2012)

    Stability number
  - ◦ N. Mori & S.K (2014 preprint)

Aim

- To survey the general theory for Type (I).
- To study the **dissipative Timoshenko system** as an example of Type (II).

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\,\psi_{xx} - (\varphi_x - \psi) + \gamma\psi_t = 0, \end{cases}$$

  where a, $\gamma > 0$ are constants.
- To study the **Euler-Maxwell system** as an example of Type (II).
- To report a general framework for Type (II).
  ◦ A joint work with Y. Ueda and R. Duan (2012)
- To study the **Timoshenko-Cattaneo system** as an example of Type (2,3).

Compressible Navier-Stokes equation: The simplest model is given by

$$\begin{cases} \rho_t + u_x = 0, \\ u_t + \rho_x = u_{xx}. \end{cases}$$

Standard energy: Multiply the first and the second equations by ρ and u, respectively, and add them. This yields

$$\frac{1}{2}(\rho^2 + u^2)_t + (\rho u - u u_x)_x + u_x^2 = 0.$$

Similarly, we have

$$\frac{1}{2}(\rho_x^2 + u_x^2)_t + (\rho_x u_x - u_x u_{xx})_x + u_{xx}^2 = 0.$$

Special technique: The technique due to Kanal' (1968) and Matsumura & Nishida (1981) is as follows. Substitute $u_x = -\rho_t$ to the second equaion:

$$\rho_{xt} + \rho_x + u_t = 0.$$

Multiply by ρ_x:

$$\frac{1}{2}(\rho_x^2)_t + \rho_x^2 + \rho_x u_t = 0.$$

Here the last term can be rewritten as

$$\begin{aligned} \rho_x u_t &= (\rho_x u)_t - \rho_{tx} u \\ &= (\rho_x u)_t - (\rho_t u)_x + \rho_t u_x \qquad \Leftarrow \quad \rho_t = -u_x \\ &= (\rho_x u)_t - (\rho_t u)_x - u_x^2. \end{aligned}$$

Thus we obtain

$$\frac{1}{2}(\rho_x^2 + 2\rho_x u)_t - (\rho_t u)_x + \rho_x^2 - u_x^2 = 0.$$

Simpler technique

Simpler technique: The above computation is perfect. But it is too technical and not suitable for the general framework. A simpler computation is as follows. Multiply the first and the second equations by $-u_x$ and ρ_x, and add them. This yields

$$(\rho_x u_t - \rho_t u_x) + \rho_x^2 - \rho_x u_{xx} - u_x^2 = 0$$

with $\rho_x u_t - \rho_t u_x = (\rho_x u)_t - (\rho_t u)_x$. This computation can be expressed as

$$\begin{pmatrix} -u_x \\ \rho_x \end{pmatrix} \cdot \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} \cdot \begin{pmatrix} \rho_t \\ u_t \end{pmatrix} = \begin{pmatrix} \rho_x \\ u_x \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \rho_t \\ u_t \end{pmatrix},$$

which suggests a formulation in terms of a skew-symmetric matrix.

Symmetric hyperbolic systems:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \tag{1}$$

where $u = u(x,t)$: m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$.

(a) A^0 is symmetric and $A^0 > 0$ on $\mathbb{C}^m$,

(b) A^j is symmetric for each j,

(c) L is symmetric and $L \geq 0$ on $\mathbb{C}^m$.

(c) $L \geq 0$ on $\mathbb{C}^m$ (not necessarily symmetric) such that

$$\mathrm{Ker}(L) = \mathrm{Ker}(L_1),$$

where L_1 is the symmetric part of L.

Condition (K)

Symmetric hyperbolic systems:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0,$$

Apply the Fourier transform:

$$A^0 \hat{u}_t + i|\xi| A(\omega)\hat{u} + L\hat{u} = 0, \qquad (2)$$

where $A(\omega) = \sum_{j=1}^{n} A^j \omega_j, \quad \omega = \xi/|\xi| \in S^{n-1}$.

• Dispersion relation $\lambda = \lambda(i\xi)$:

$$\det(\lambda A^0 + i|\xi| A(\omega) + L) = 0.$$

Condition (K)

Condition (K): Umeda, S.K. & Shizuta (1984)

There exists $K(\omega)$ with the following properties:

(i) $K(\omega)A^0$ is skew-symmetric.

(ii) $(K(\omega)A(\omega))_1 > 0$ on $\mathrm{Ker}(L)$,

where X_1 is the symmetric part of X.

Claim

When L is symmetric, the following two conditions are equivalent:

(a) $(K(\omega)A(\omega))_1 > 0$ on $\mathrm{Ker}(L)$,

(b) $\alpha(K(\omega)A(\omega))_1 + L > 0$ on $\mathbb{C}^m$,

where $\alpha > 0$ is a suitably small constant.

Decay property

Theorem 1 (Pointwise estimate) Umeda, S.K. & Shizuta (1984)

Under the condition (K), we have

$$|\hat{u}(\xi,t)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \tag{3}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

Corollary (Decay estimate) Umeda, S.K. & Shizuta (1984)

Under the condition (K), we have

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2}, \tag{4}$$

where $k \ge 0$.

- Decay estimate of the standard type (without loss of regularity)

Lyapunov function:

$$E[\hat{u}] = \langle A^0\hat{u},\, \hat{u}\rangle - \frac{\alpha|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\, \hat{u}\rangle,$$

where $\alpha > 0$ is a small constant. We have

$$\frac{\partial}{\partial t}E[\hat{u}] + \frac{c|\xi|^2}{1+|\xi|^2}\,|\hat{u}|^2 + c|(I-P)\hat{u}|^2 \le 0,$$

where P is the orthogonal projection onto $\mathrm{Ker}(L)$. Therefore we have

$$\frac{\partial}{\partial t}E[\hat{u}] + c\rho(\xi)E[\hat{u}] \le 0,$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$. This is solved as

$$E[\hat{u}](\xi,t) \le e^{-c\rho(\xi)t}E[\hat{u}_0](\xi),$$

which gives the desired pointwise estimate (3).

Energy estimate: As a simple corollary of

$$\frac{\partial}{\partial t}E[\hat{u}] + \frac{c|\xi|^2}{1+|\xi|^2}\,|\hat{u}|^2 + c|(I-P)\hat{u}|^2 \le 0,$$

we have the energy estimate of the form

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}^2 + \|(I-P)u(\tau)\|_{H^s}^2\,d\tau \le C\|u_0\|_{H^s}^2,$$

where $s \ge 0$.

- Energy estimate of the standard type (without loss of regularity)

Dissipative structure

(SK) stability condition: Shizuta & S.K. (1985)

Let $\varphi \in \mathbb{R}^m$, $\mu \in \mathbb{R}$ and $\omega \in S^{n-1}$.

$$\text{If} \quad L\varphi = 0 \quad \text{and} \quad \mu A^0\varphi + A(\omega)\varphi = 0, \quad \text{then} \quad \varphi = 0.$$

Theorem 2 (Characterization of dissipativity)
Shizuta & S.K. (1985), Beauchard & Zuazua (2010)

The following five conditions are equivalent.

(a) (SK) stability condition.

(b) Condition (K).

(c) $\mathrm{Re}\,\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)$ for any $\xi \in \mathbb{R}^n$. Type (I)

(d) $\mathrm{Re}\,\lambda(i\xi) < 0$ for any $\xi \ne 0$. Strict dissipativity

(e) Kalman rank condition.

Dissipative Timoshenko system:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi) + \gamma \psi_t = 0, \end{cases}$$

where $a > 0$, $\gamma > 0$ are constants. The equivalent 1st order system is

$$\begin{cases} v_t - u_x + y = 0, \\ y_t - a z_x - v + \gamma y = 0, \\ u_t - v_x = 0, \\ z_t - a y_x = 0, \end{cases} \tag{5}$$

where

$$u = \varphi_t, \ v = \varphi_x - \psi, \ y = \psi_t, \ z = a\psi_x.$$

Dissipative Timoshenko system

The system is written as

$$U_t + AU_x + LU = 0,$$

where

$$U = \begin{pmatrix} v \\ y \\ u \\ z \end{pmatrix}, \quad A = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Claim:

$L \geq 0$ is not symmetric but satisfies the (SK) stability condition:

$$\text{If} \quad L\varphi = 0 \quad \text{and} \quad \mu\varphi + A\varphi = 0, \quad \text{then} \quad \varphi = 0.$$

In this case, however, $\ker(L) \neq \ker(L_1)$.

Dissipative structure:

- If $a = 1$, then $\mathrm{Re}\,\lambda(i\xi) \le -c\,\xi^2/(1+\xi^2)$. Type (I)
- If $a \neq 1$, then $\mathrm{Re}\,\lambda(i\xi) \le -c\,\xi^2/(1+\xi^2)^2$. Type (II)

When $a \neq 1$, for $|\xi| \to \infty$, the eigenvalues satisfy

$$\mathrm{Re}\,\lambda_j(i\xi) = \begin{cases} -\frac{\gamma}{P^2}\,\xi^{-2} + O(|\xi|^{-3}), & j = 1, 2, \\ -\frac{\gamma}{2} + O(|\xi|^{-1}), & j = 3, 4, \end{cases}$$

where $P = a^2 - 1$.

Decay property

Theorem 3 (Pointwise estimate) Ide, Haramoto & S.K (2008)

When $a \neq 1$, we have

$$|\hat{U}(\xi, t)| \leq Ce^{-c\eta(\xi)t}|\hat{U}_0(\xi)|, \quad (6)$$

where $\eta(\xi) = \xi^2/(1+\xi^2)^2$.

Corollary (Decay estimate) Ide, Haramoto & S.K (2008)

When $a \neq 1$, we have

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-1/4-k/2}\|U_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l} U_0\|_{L^2}, \quad (7)$$

where k, $l \geq 0$.

- Decay estimate of the regularity-loss type

Lyapunov function

Lyapunov function: Ide, Haramoto & S.K (2008)

When $a \neq 1$,

$$E = |\hat{U}|^2 + \frac{\alpha_1}{1+\xi^2}\Big\{(\tilde{E}_1 + a\tilde{E}_2) + \frac{\alpha_2 \xi}{1+\xi^2}(E_1 + E_2)\Big\},$$

where α_1, $\alpha_2 > 0$ are small constants, and

$$E_1 = \mathrm{Re}(i\hat{v}\bar{\hat{u}}), \quad E_2 = \mathrm{Re}(i\hat{y}\bar{\hat{z}}), \qquad \text{Skew symmetric}$$

$$\tilde{E}_1 = -\mathrm{Re}(\hat{v}\bar{\hat{y}}), \quad \tilde{E}_2 = -\mathrm{Re}(\hat{u}\bar{\hat{z}}). \quad \text{Symmetric}$$

We have

$$\frac{\partial}{\partial t}E + cD \leq 0,$$

where

$$D = \frac{1}{1+\xi^2}|\hat{v}|^2 + |\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}(|\hat{u}|^2 + |\hat{z}|^2).$$

Energy estimate

Therefore we obtain

$$\frac{\partial}{\partial t}E + c\eta(\xi)E \leq 0,$$

where $\eta(\xi) = \xi^2/(1+\xi^2)^2$. This yields the desired pointwise estimate (6).

Energy estimate: The corresponding energy estimate is

$$\|U(t)\|_{H^s}^2 + \int_0^t \|v(\tau)\|_{H^{s-1}}^2 + $$
$$+ \|y(\tau)\|_{H^s}^2 + \|\partial_x(u,z)(\tau)\|_{H^{s-2}}^2 \, d\tau \leq C\|U_0\|_{H^s}^2,$$

where $s \geq 0$.

• Energy estimate of the regularity-loss type: In the dissipation part, we have the regularity loss for (v, u, z). Not optimal

Lyapunov function (Refinement)

Lyapunov function (Refinement): Mori & S.K (2014)

When $a \neq 1$,

$$E = |\hat{U}|^2 + \frac{\alpha_1}{1+\xi^2}\Big\{(\tilde{E}_1 + a\tilde{E}_2) + \frac{\alpha_2 \xi}{1+\xi^2}\{E_1 + (1+\xi^2)E_2\}\Big\},$$

where α_1, $\alpha_2 > 0$ are small constants, and

$$E_1 = \mathrm{Re}(i\hat{v}\bar{\hat{u}}), \quad E_2 = \mathrm{Re}(i\hat{y}\bar{\hat{z}}), \qquad \text{Skew symmetric}$$

$$\tilde{E}_1 = -\mathrm{Re}(\hat{v}\bar{\hat{y}}), \quad \tilde{E}_2 = -\mathrm{Re}(\hat{u}\bar{\hat{z}}). \qquad \text{Symmetric}$$

We have

$$\frac{\partial}{\partial t}E + cD \leq 0,$$

where

$$D = \frac{1}{1+\xi^2}|\hat{v}|^2 + |\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + \frac{\xi^2}{1+\xi^2}|\hat{z}|^2.$$

Energy estimate (Refinement): The corresponding energy estimate is

$$\|U(t)\|_{H^s}^2 + \int_0^t \|v(\tau)\|_{H^{s-1}}^2 + \\ + \|y(\tau)\|_{H^s}^2 + \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|\partial_x z(\tau)\|_{H^{s-1}}^2 \, d\tau \le C\|U_0\|_{H^s}^2,$$

where $s \ge 0$.

• Energy estimate of the regularity-loss type: In the dissipation part, we have the regularity loss only for (v, u). Optimal

Proof of decay estimate

Proof of Corollary: We have

$$\|\partial_x^k U(t)\|_{L^2}^2 = \int |\xi|^{2k} |\hat{U}(\xi,t)|^2 \, d\xi \le C \int |\xi|^{2k} e^{-c\eta(\xi)t} \, |\hat{U}_0(\xi)|^2 \, d\xi$$
$$= \int_{|\xi|\le 1} + \int_{|\xi|\ge 1} := I_1 + I_2.$$

Low frequency term I_1 is estimated as

$$I_1 \le C \int_{|\xi|\le 1} |\xi|^{2k} e^{-c|\xi|^2 t} \, |\hat{U}_0(\xi)|^2 \, d\xi$$
$$\le C \sup_{|\xi|\le 1} |\hat{U}_0(\xi)|^2 \int_{|\xi|\le 1} |\xi|^{2k} e^{-c|\xi|^2 t} \, d\xi$$
$$\le C(1+t)^{-1/2-k} \|U_0\|_{L^1}^2.$$

Proof of decay estimate

High frequency term I_2 can be estimated as

$$\begin{aligned} I_2 &\le C\int_{|\xi|\ge 1} |\xi|^{2k} e^{-ct/|\xi|^2}\, |\widehat{U}_0(\xi)|^2\, d\xi \\ &\le C \sup_{|\xi|\ge 1} \frac{e^{-ct/|\xi|^2}}{|\xi|^{2l}} \int_{|\xi|\ge 1} |\xi|^{2(k+l)}\, |\hat{U}_0(\xi)|^2\, d\xi \\ &\le C(1+t)^{-l} \|\partial_x^{k+l} U_0\|_{L^2}^2. \end{aligned}$$

This shows the desired decay estimate (7).

Euler-Maxwell system in $\mathbb{R}^3$:

$$\begin{cases} n_t + \operatorname{div}(nu) = 0, \\ (nu)_t + \operatorname{div}(nu \otimes u) + \nabla p(n) = -n(E + u \times B) - nu, \\ E_t - \operatorname{rot} B = nu, \\ B_t + \operatorname{rot} E = 0, \end{cases} \tag{8}$$

$$\operatorname{div} E = n_\infty - n, \qquad \operatorname{div} B = 0, \tag{9}$$

Here $n > 0$: mass density, $u \in \mathbb{R}^3$: velocity, $E \in \mathbb{R}^3$: electric field, and $B \in \mathbb{R}^3$: magnetic induction; $p(n)$: pressure satisfying $p'(n) > 0$ for $n > 0$, and $n_\infty > 0$: a constant.

Solutions of (8) satisfy (9) for $t > 0$ if the initial data verify (9).

Euler-Maxwell system

The system (8) is written as

$$A^0(w)w_t + \sum_{j=1}^{3} A^j(w)w_{x_j} + L(w)w = 0,$$

where $w = (n, u, E, B)^T$ and

$$A^0(w) = \begin{pmatrix} p'(n)/n & & & \\ & nI & & \\ & & I & \\ & & & I \end{pmatrix}, \quad L(w) = \begin{pmatrix} 0 & & & \\ & n(I - \Omega_B) & nI & \\ & -nI & O & \\ & & & O \end{pmatrix},$$

$$\sum_{j=1}^{3} A^j(w)\xi_j = \begin{pmatrix} (p'(n)/n)(u \cdot \xi) & p'(n)\xi & & \\ p'(n)\xi^T & n(u \cdot \xi)I & & \\ & & O & -\Omega_\xi \\ & & \Omega_\xi & O \end{pmatrix}.$$

Euler-Maxwell system

Here I and O denotes the 3×3 identity matrix and the zero matrix, respectively, and Ω_ξ is the skew-symmetric matrix defined by

$$\Omega_\xi = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

for $\xi \in \mathbb{R}^3$. Note that $\Omega_\xi E = \xi \times E$ as a column vector in $\mathbb{R}^3$.

Constant equilibrium: The Euler-Maxwell system admits a constant equilibrium state

$$w_\infty = (n_\infty, 0, 0, B_\infty)^T,$$

where $n_\infty > 0$ and $B_\infty \in \mathbb{R}^3$ are constant states. Note that $L(w)w_\infty = 0$ for each w.

Linearized Euler-Maxwell system:

$$A^0 U_t + \sum_{j=1}^{3} A^j U_{x_j} + LU = 0, \tag{10}$$

$$\operatorname{div} E = -\rho, \qquad \operatorname{div} h = 0, \tag{11}$$

where $U = (\rho, u, E, h)^T$ with $\rho = n - n_\infty$ and $h = B - B_\infty$, and $A^0 = A^0(w_\infty)$, $A^j = A^j(w_\infty)$ and $L = L(w_\infty)$.

Apply the Fourier transform:

$$A^0 \hat{U}_t + i|\xi| A(\omega) \hat{U} + L\hat{U} = 0, \tag{12}$$

$$i|\xi| \hat{E} \cdot \omega = -\hat{\rho}, \qquad i|\xi| \hat{h} \cdot \omega = 0, \tag{13}$$

where $A(\omega) = \sum_{j=1}^{3} A^j \omega_j, \quad \omega = \xi/|\xi| \in S^2$.

Claim: Ueda, Wang & S.K (2012)

$L \geq 0$ is not symmetric but satisfies the following modified (SK) stability condition: Let $\varphi = (\rho, u, E, h)^T \in \mathbb{R}^{10}$.

$$\text{If} \quad L\varphi = 0, \quad \mu A^0 \varphi + A(\omega)\varphi = 0 \quad \text{and} \quad h \cdot \omega = 0, \quad \text{then} \quad \varphi = 0.$$

In this case, however, $\ker(L) \neq \ker(L_1)$.

Let $\varphi = (\rho, u, E, h)^T$. Then $L\varphi = 0$ gives

$$n_\infty(u - B_\infty \times u) + n_\infty E = 0, \qquad -n_\infty u = 0,$$

which shows that $u = E = 0$. Then $\varphi = (\rho, 0, 0, h)^T$. For this φ, we suppose that $\mu A^0 \varphi + A(\omega)\varphi = 0$ and $h \cdot \omega = 0$. This implies

$$\mu a_\infty \rho = 0, \quad b_\infty \rho\, \omega = 0, \quad h \times \omega = 0, \quad \mu h = 0, \quad h \cdot \omega = 0,$$

where $a_\infty = p'(n_\infty)/n_\infty$ and $b_\infty = p'(n_\infty)$ are positive constants. This shows that $\rho = h = 0$ and hence $\varphi = 0$.

Decay property

Theorem 4 (Pointwise estimate) Ueda & S.K (2011), Duan (2011)

We have

$$|\hat{U}(\xi,t)| \leq Ce^{-c\eta(\xi)t}|\hat{U}_0(\xi)|, \tag{14}$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$.

Corollary (Decay estimate) Ueda & S.K (2011), Duan (2011)

We have

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-3/4-k/2}\|U_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l}U_0\|_{L^2}, \tag{15}$$

where k, $l \geq 0$.

- Decay estimate of the regularity-loss type

Lyapunov function:

$$E[\hat{U}] = E_0 + \frac{\alpha_1}{1+|\xi|^2}\Big\{\tilde{E}_1 + a_\infty|\xi|E_1 + \frac{\alpha_2|\xi|}{1+|\xi|^2}E_2\Big\},$$

where

$$E_0 = \langle A^0\hat{U},\, \hat{U}\rangle = a_\infty|\hat{\rho}|^2 + n_\infty|\hat{u}|^2 + |\hat{E}|^2 + |\hat{h}|^2,$$

$$E_1 = \mathrm{Re}\langle \hat{\rho} i\omega \,|\, \hat{u}\rangle, \quad E_2 = \mathrm{Re}\langle \hat{E} \,|\, \hat{h}\times i\omega\rangle, \qquad \text{Skew symmetric}$$

$$\tilde{E}_1 = \mathrm{Re}\langle \hat{u} \,|\, \hat{E}\rangle. \qquad \text{Symmetric}$$

Here α_1, $\alpha_2 > 0$ are small constants, $a_\infty = p'(n_\infty)/n_\infty$, and $\langle\cdot\,|\,\cdot\rangle$ is the inner product of $\mathbb{C}^3$.

Lyapunov function

We have

$$\frac{\partial}{\partial t}E[\hat{U}] + cD[\hat{U}] \le 0,$$

where

$$D[\hat{U}] = |\hat{\rho}|^2 + |\hat{u}|^2 + \frac{1}{1+|\xi|^2}|\hat{E}|^2 + \frac{|\xi|^2}{(1+|\xi|^2)^2}|\hat{h}|^2.$$

Therefore we obtain

$$\frac{\partial}{\partial t}E[\hat{U}] + c\eta(\xi)E[\hat{U}] \le 0,$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$. This yields the desired pointwise estimate (14).

Energy estimate: As a simple corollary of

$$\frac{\partial}{\partial t}E[\hat{U}] + cD[\hat{U}] \le 0,$$

we have the following energy estimate:

$$\|U(t)\|_{H^s}^2 + \int_0^t \|(\rho,u)(\tau)\|_{H^s}^2 + \|E(\tau)\|_{H^{s-1}}^2 + \|\partial_x h(\tau)\|_{H^{s-2}}^2 \, d\tau \le C\|U_0\|_{H^s}^2,$$

where $s \ge 0$.

• Energy estimate of the regularity-loss type: In the dissipation part, we have the regularity loss for (E, h).

Symmetric hyperbolic systems:

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \qquad (16)$$

where $u = u(x,t)$: m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t > 0$.

(a) A^0 is symmetric and $A^0 > 0$ on $\mathbb{C}^m$,

(b) A^j is symmetric for each j,

(c) $L \geq 0$ on $\mathbb{C}^m$ (not symmetric) such that

$$\ker(L) \neq \ker(L_1),$$

where L_1 is the symmetric part of L.

Structural conditions

Apply the Fourier transform:

$$A^0 \hat{u}_t + i|\xi| A(\omega)\hat{u} + L\hat{u} = 0. \tag{17}$$

Condition (S): Ueda, Duan & S.K (2012)

There exits S with the following properties:

(i) SA^0 is symmetric.

(ii) $(SL)_1 + L_1 \geq 0$ on $\mathbb{C}^m$ and $\mathrm{Ker}((SL)_1 + L_1) = \mathrm{Ker}(L)$.

When we use the condition (S), we additionaly assume one of the following conditions:

(a) $i(SA(\omega))_2 \geq 0$ on $\mathrm{Ker}(L_1)$,

(b) $i(SA(\omega))_2 \geq 0$ on $\mathbb{C}^m$,

where X_2 is the skew-symmetric part of X.

Structural conditions

Condition (K): Umeda, S.K. & Shizuta (1984)

There exists $K(\omega)$ with the following properties:

(i) $K(\omega)A^0$ is skew-symmetric.

(ii) $(K(\omega)A(\omega))_1 > 0$ on $\mathrm{Ker}(L)$,

where X_1 is the symmetric part of X.

Claim

When L is not symmetric, under the condition (S), the following two statements are equivalent:

(a) $(K(\omega)A(\omega))_1 > 0$ on $\mathrm{Ker}(L)$,

(b) $\alpha(K(\omega)A(\omega))_1 + (SL)_1 + L_1 > 0$ on $\mathbb{C}^m$,

where $\alpha > 0$ is a suitably small constant.

Decay property

Theorem 5 (Pointwise estimate) Ueda, Duan & S.K (2012)

Under the conditions (K) and (S) with $i(SA(\omega))_2 \geq 0$ on $\mathrm{Ker}(L_1)$, we have

$$|\hat{u}(\xi,t)| \leq Ce^{-c\eta(\xi)t}|\hat{u}_0(\xi)|, \tag{18}$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$.

Corollary (Decay estimate) Ueda, Duan & S.K (2012)

Under the conditions (K) and (S) with $i(SA(\omega))_2 \geq 0$ on $\mathrm{Ker}(L_1)$, we have

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l}u_0\|_{L^2}, \tag{19}$$

where k, $l \geq 0$.

- Decay estimate of the regularity-loss type

Lyapunov function:

$$E[\hat{u}] = \langle A^0\hat{u},\, \hat{u}\rangle + \frac{\alpha_1}{1+|\xi|^2}\Big\{\langle SA^0\hat{u},\, \hat{u}\rangle - \frac{\alpha_2|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\, \hat{u}\rangle\Big\},$$

where $\alpha_1,\, \alpha_2 > 0$ are small constants. We have

$$\frac{\partial}{\partial t}E[\hat{u}] + \frac{c|\xi|^2}{(1+|\xi|^2)^2}\,|\hat{u}|^2 + \frac{c}{1+|\xi|^2}\,|(I-P)\hat{u}|^2 + c|(I-P_1)\hat{u}|^2 \le 0,$$

where P and P_1 are the orthogonal projections onto $\mathrm{Ker}(L)$ and $\mathrm{Ker}(L_1)$, respectively. Therefore we have

$$\frac{\partial}{\partial t}E[\hat{u}] + c\eta(\xi)E[\hat{u}] \le 0,$$

where $\eta(\xi) = |\xi|^2/(1+|\xi|^2)^2$. This gives the desired pointwise estimate (18).

Energy estimate: As a simple corollary of

$$\frac{\partial}{\partial t}E[\hat{u}] + \frac{c|\xi|^2}{(1+|\xi|^2)^2}|\hat{u}|^2 + \frac{c}{1+|\xi|^2}|(I-P)\hat{u}|^2 + c|(I-P_1)\hat{u}|^2 \le 0,$$

we have the energy estimate of the form

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x u(\tau)\|_{H^{s-2}}^2 + \|(I-P)u(\tau)\|_{H^{s-1}}^2 + \|(I-P_1)u(\tau)\|_{H^s}^2\, d\tau \le C\|u_0\|_{H^s}^2,$$

where $s \ge 0$.

- Energy estimate of the regularity-loss type: In the dissipation part, we have the regularity loss for the component $P_1 u$.

Decay property in a special case

Theorem 6 (Pointwise estimate) Ueda, Duan & S.K (2012)

Under the conditions (K) and (S) with $i(SA(\omega))_2 \geq 0$ on $\mathbb{C}^m$, we have

$$|\hat{u}(\xi,t)| \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|, \tag{20}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

Corollary (Decay estimate) Ueda, Duan & S.K (2012)

Under the conditions (K) and (S) with $i(SA(\omega))_2 \geq 0$ on $\mathbb{C}^m$, we have

$$\|\partial_x^k u(t)\|_{L^2} \leq C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2}, \tag{21}$$

where $k \geq 0$.

- Decay estimate of the standard type

Lyapunov function in a special case

Lyapunov function: When $i(SA(\omega))_2 \geq 0$ on $\mathbb{C}^m$,

$$E[\hat{u}] = \langle A^0\hat{u},\, \hat{u}\rangle + \alpha_1\Big\{\langle SA^0\hat{u},\, \hat{u}\rangle - \frac{\alpha_2|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\, \hat{u}\rangle\Big\},$$

where α_1, $\alpha_2 > 0$ are small constants. We have

$$\frac{\partial}{\partial t}E[\hat{u}] + \frac{c|\xi|^2}{1+|\xi|^2}\,|\hat{u}|^2 + c|(I-P)\hat{u}|^2 \leq 0,$$

where P is the orthogonal projection onto $\mathrm{Ker}(L)$. Therefore we have

$$\frac{\partial}{\partial t}E[\hat{u}] + c\rho(\xi)E[\hat{u}] \leq 0,$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$. This gives the desired pointwise estimate (20).

Timoshenko-Cattaneo system:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + b\theta_x = 0, \\ \theta_t + \tilde{q}_x + b\psi_{tx} = 0, \\ \tau_0\tilde{q}_t + \tilde{q} + \kappa\theta_x = 0, \end{cases}$$

where a, b, κ, $\tau_0 > 0$ are constants.

Timoshenko-Fourier system: When $\tau_0 = 0$, we have formally

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi) + b\theta_x = 0, \\ \theta_t + b\psi_{tx} = \kappa\theta_{xx}. \end{cases}$$

Timoshenko-Cattaneo system: The equaivalent 1st order system is

$$\begin{cases} v_t - u_x + y = 0, \\ y_t - az_x + b\theta_x - v = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ \theta_t + by_x + \sqrt{\kappa} q_x = 0, \\ \tau_0 q_t + \sqrt{\kappa}\theta_x + q = 0, \end{cases}$$

where $u = \varphi_t, \quad v = \varphi_x - \psi, \quad y = \psi_t, \quad z = a\psi_x, \quad q = \frac{1}{\sqrt{\kappa}}\tilde{q}$.

The system is written as

$$A^0 U_t + AU_x + LU = 0.$$

Timoshenko-Cattaneo system

$$U = \begin{pmatrix} v \\ y \\ u \\ z \\ \theta \\ q \end{pmatrix}, \qquad A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & b & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & \sqrt{\kappa} \\ 0 & 0 & 0 & 0 & \sqrt{\kappa} & 0 \end{pmatrix}, \ L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Stability number: Santos, Júnior & Rivera (2012)

$$P = \frac{\tau_0}{\kappa}(1 - a^2 - b^2) + (a^2 - 1).$$

Dissipative structure:

- If $P = 0$, then $\mathrm{Re}\,\lambda(i\xi) \le -c\,\xi^4/(1+\xi^2)^2$. Type (2,2)
- If $P \neq 0$, then $\mathrm{Re}\,\lambda(i\xi) \le -c\,\xi^4/(1+\xi^2)^3$. Type (2,3)

When $P \neq 0$, for $|\xi| \to \infty$, the eigenvalues satisfy

$$\mathrm{Re}\,\lambda_j(i\xi) = \begin{cases} -\frac{b^2}{2\kappa P^2}\,\xi^{-2} + O(|\xi|^{-3}), & j = 1, 2, \\ -\frac{1}{2}\delta_j + O(|\xi|^{-1}), & j = 3, 4, 5, 6, \end{cases}$$

where $\delta_j > 0$.

Decay property

Theorem 7 (Pointwise estimate) Mori & S.K (2014 preprint)

When $P \neq 0$, we have

$$|\hat{U}(\xi,t)| \leq Ce^{-c\eta(\xi)t}|\hat{U}_0(\xi)|, \tag{22}$$

where $\eta(\xi) = \xi^4/(1+\xi^2)^3$.

Corollary (Decay estimate) Mori & S.K (2014 preprint)

When $P \neq 0$, we have

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-1/8-k/2}\|U_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l}U_0\|_{L^2}, \tag{23}$$

where k, $l \geq 0$.

- Decay estimate of the regularity-loss type
- Weaker decay $t^{-1/8}$

Lyapunov function: Mori & S.K (2014 preprint)

When $P \neq 0$,

$$E = |\hat{U}|^2 + \frac{\alpha_0 \xi}{1+\xi^2}\tau_0 E_0 + \frac{\alpha_1}{(1+\xi^2)^2}\Big[\big\{(H_1 - P_* \xi^2 \tau_0 \tilde{E}_0) + + (1+\xi^2)H_2\big\} + \frac{\alpha_2 \xi^3}{1+\xi^2}\{E_1 - (1+\xi^2)E_2\}\Big],$$

where $\alpha_0,\ \alpha_1,\ \alpha_2 > 0$ are small constants,

$$H_1 = b\{\xi^2(\tilde{E}_1 + a\tilde{E}_2) - \xi E_1\} + (\xi E_3 + \tilde{E}_3) + (a^2-1)\xi^2 \tilde{E}_3,$$

$$H_2 = b(\xi E_2 + \tilde{E}_2) + a(\xi E_3 + \tilde{E}_3).$$

and $P_* = \frac{1}{\sqrt{\kappa}}(1 - a^2 - b^2)$. Compare

$$E = |\hat{U}|^2 + \frac{\alpha_1}{1+\xi^2}\Big\{(\tilde{E}_1 + a\tilde{E}_2) + \frac{\alpha_2 \xi}{1+\xi^2}\{E_1 + (1+\xi^2)E_2\}\Big\}.$$

Lyapunov function

Here

$$E_1 = \mathrm{Re}(i\hat{v}\bar{\hat{u}}),\ E_2 = \mathrm{Re}(i\hat{y}\bar{\hat{z}}),\ E_3 = \mathrm{Re}(i\hat{y}\bar{\hat{\theta}}),\ E_0 = \mathrm{Re}(i\hat{\theta}\bar{\hat{q}}),$$

$$\tilde{E}_1 = -\mathrm{Re}(\hat{v}\bar{\hat{y}}),\ \tilde{E}_2 = -\mathrm{Re}(\hat{u}\bar{\hat{z}}),\ \tilde{E}_3 = -\mathrm{Re}(\hat{u}\bar{\hat{\theta}}),\ \tilde{E}_0 = \mathrm{Re}(\hat{v}\bar{\hat{q}}).$$

We have

$$\frac{\partial}{\partial t}E + cD \le 0,$$

where

$$D = \frac{\xi^4}{(1+\xi^2)^3}|\hat{v}|^2 + \frac{\xi^4}{(1+\xi^2)^2}|\hat{y}|^2 + \frac{\xi^2}{(1+\xi^2)^2}|\hat{u}|^2 + \frac{\xi^2}{1+\xi^2}(|\hat{z}|^2 + |\hat{\theta}|^2) + |\hat{q}|^2.$$

- Regularity-loss for (v, u).

8. Open questions

Strict dissipativity of Type (p, q):

$$\mathrm{Re}\,\lambda(i\xi) \leq -c|\xi|^{2p}/(1+|\xi|^2)^q,$$

Open questions:

- General framework (Structural conditions) for Type (p, q)

Open questions for Type $(1, 2)$:

- Characterization of (S) + (K)
- Relation with Kalman rank condition (in progress)

  Kalman rank condition $\Longrightarrow$ (K) Ueda, Duan & S.K (2012)
- General framework for nonlinear problems

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Thank You for Your Attention