Stability of time-dependent Navier-Stokes flow and algebraic energy decay

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Weak Solution to:

$$\partial_t u + u \cdot \nabla u + V \cdot \nabla u + u \cdot \nabla V = \Delta u - \nabla p$$

$$\text{div } u = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty)$$

$$u(\cdot, 0) = u_0 \in L^2_\sigma \qquad (n = 3, 4)$$

$V = V(x, t)$ basic flow (Navier-Stokes flow)

Assumption

$$V = V(x,t) \quad \text{Navier-Stokes flow, small}$$

$$V \in L^\infty(0,\infty; L^{n,\infty}) \cap C_w([0,\infty); L^{n,\infty})$$

Example

Solution to the Cauchy problem
Forward Self-Similar Solution
Time-Periodic Solution

Assumption

$V = V(x,t)$ Navier-Stokes flow, small

$V \in L^\infty(0,\infty; L^{n,\infty}) \cap C_w([0,\infty); L^{n,\infty})$

Known Result (Karch-Pilarczyk-Schonbek)

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0$$

Question: Structure of decay rate

Case $V = 0$

Leray, Masuda, Kato

Schonbek, Wiegner, Kajikiya-Miyakawa, ...

$$\|e^{t\Delta}u_0\|_{L^2} = O(t^{-\alpha}) \quad \text{as } t \to \infty$$

$\Longrightarrow$ Every Weak Solution with (SEI)

$$\|u(t)\|_{L^2} = O(t^{-\beta}), \qquad \beta = \min\left\{\alpha,\ \frac{n}{4} + \frac{1}{2}\right\}$$

Some other stability of time-dependent flow

- Yamazaki (2000)

  $V \in BC(0, \infty; L^{n,\infty})$ small

  L^r stability ($n < r < \infty$) for small $u_0 \in L^{n,\infty}$

- Kozono (2000)

  $V \in L^q(0, \infty; L^r), \quad \frac{2}{q} + \frac{n}{r} = 1, \quad 2 \le q < \infty$

  L^2 stability for every $u_0 \in L^2$

Linearized system

$$\partial_t u - \Delta u + V \cdot \nabla u + u \cdot \nabla V + \nabla p = 0$$

$$\text{div } u = 0 \qquad\qquad \text{in } \mathbb{R}^n \times (s, \infty)$$

$$u(\cdot, s) = f$$

$$u(\cdot, t) = T(t, s) f \quad (t \geq s \geq 0)$$

What we need:

$$\|T(t,s)f\|_{L^r} \leq C(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_{L^q}$$

$$\|T(t,s)P\,\mathrm{div}\ F\|_{L^q} \leq C(t-s)^{-1/2}\|F\|_{L^q}$$

for $0 \leq s < t < \infty$

Theorem 1 $n = 3, 4$, $\quad u_0 \in L^2_\sigma$, $\quad \sup_t \|V(t)\|_{L^{n,\infty}}$ *small*

1. Every Weak Solution with (SEI)

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0$$

$$\|u(t) - T(t,0)u_0\|_{L^2} = O\Big(t^{-\frac{n}{4}+\frac{1}{2}}\Big) \quad \text{as } t \to \infty$$

2. $$\|T(t,0)u_0\|_{L^2} = O(t^{-\alpha}) \quad \textit{as } t \to \infty$$

$\Longrightarrow$ *Every Weak Solution with (SEI)*

$$\|u(t)\|_{L^2} = O(t^{-\beta}), \qquad \beta = \min\left\{\alpha,\ \frac{n}{4} + \frac{1}{2} - \varepsilon\right\}$$

$$\|u(t) - T(t,0)u_0\|_{L^2} = \begin{cases} O\Big(t^{-\frac{n}{4}+\frac{1}{2}-2\alpha+\varepsilon}\Big) & \alpha \le \frac{1}{2} \\ O(t^{-\frac{n}{4}-\frac{1}{2}+\varepsilon}) & \alpha > \frac{1}{2} \end{cases}$$

Corollary 1 *Let* $n = 3, 4$

1. $$u_0 \in L^2_\sigma \cap L^q \qquad \textit{for some } q \in [1, 2)$$
$$\Longrightarrow \qquad \|u(t)\|_{L^2} = O(t^{-\beta}), \qquad \beta = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right)$$

2. $$u_0 \in L^2_\sigma \cap L^1, \qquad \int |x||u_0(x)|dx < \infty$$
$$\Longrightarrow \qquad \|u(t)\|_{L^2} = O(t^{-\beta}), \qquad \beta = \frac{n}{4} + \frac{1}{2} - \varepsilon$$

Ingredients of the proof

(1) Generation of $\{T(t,s)\}$

(2) L^q-L^r estimate of $T(t,s)$

(3) Integral equation for weak solution

(4) Fourier splitting method

$$a(t; u, w) = \langle \nabla u, \nabla w \rangle + \langle V(t) \cdot \nabla u, w \rangle - \langle V(t) \otimes u, \nabla w \rangle$$

for $u, w \in H^1_\sigma$. Define $L(t) : H^1_\sigma \to (H^1_\sigma)^*$ by

$$\langle L(t)u, w \rangle = a(t; u, w)$$

J.L. Lions' theorem implies:

$$\begin{cases} \dfrac{du}{dt} + L(t)u = 0 \quad \text{a.e. } t \in (s,\infty) \text{ in } (H^1_\sigma)^* \\ u(s) = f \in L^2_\sigma \end{cases}$$

admits a unique solution

$$\begin{aligned} & u \in L^2(s,T;H^1_\sigma) \cap C([s,\infty);L^2_\sigma) \\ & \frac{du}{dt} \in L^2(s,T;(H^1_\sigma)^*) \qquad\qquad \forall\, T \in (s,\infty) \end{aligned}$$

Define $T(t,s) : L^2_\sigma \to L^2_\sigma$ by

$$T(t,s)f := u(t), \qquad t \in [s,\infty)$$

Another existence theorem

$$\forall\, r_1 \in (2,\infty) \quad \exists\, \delta > 0 \quad \text{such that} \quad \sup_t \|V(t)\|_{L^{n,\infty}} \le \delta$$

$\Longrightarrow$

$$u(t) = e^{(t-s)\Delta} f - \int_s^t e^{(t-\tau)\Delta} P\text{div }(u \otimes V + V \otimes u)(\tau)\, d\tau$$

admits a unique solution that satisfies

$$\|u(t)\|_{L^r} \le C(t-s)^{-\frac{n}{2}(\frac{1}{2}-\frac{1}{r})} \|f\|_{L^2}$$

for $\quad f \in L^2_\sigma, \;\; r \in [2, r_1), \;\; 0 \le s < t < \infty$

Yamazaki (2000)

$$\begin{cases} \int_0^\infty \|\nabla e^{t\Delta} f\|_{L^{r,1}} \leq C\|f\|_{L^{q,1}} \\ \text{provided} \quad 1 < q < r < \infty, \qquad \dfrac{1}{q} - \dfrac{1}{r} = \dfrac{1}{n} \end{cases}$$

from which one can estimate

$$\int_s^t \langle u \otimes V + V \otimes u, \nabla e^{(t-\tau)\Delta}\psi\rangle d\tau$$
$$\leq C \int_s^t \|V\|_{L^{n,\infty}} \|u\|_{L^{r_1,\infty}} \|\nabla e^{(t-\tau)\Delta}\psi\|_{L^{p,1}}\, d\tau$$
$$= \int_s^{(s+t)/2} + \int_{(s+t)/2}^t \qquad \text{where } \tfrac{1}{p} = 1 - \tfrac{1}{n} - \tfrac{1}{r_1}$$

Relation between both solutions

$w(t) := u(t) - T(t,s)f \in L^{\infty}(s,\infty; L^2_\sigma)$ obeys

$$\langle w(t), \psi\rangle = -\int_s^t \langle w\otimes V + V\otimes w, \nabla e^{(t-\tau)\Delta}\psi\rangle d\tau$$

$$\|w(t)\|_{L^{2,\infty}} \le C\|V\|_{L^\infty(0,\infty;L^{n,\infty})}\|w\|_{L^\infty(s,\infty;L^{2,\infty})}$$

we obtain $u(t) = T(t,s)f$

Proposition 1 $\quad q \in [1,\infty)$

$\forall\, r_1 \in (q,\infty) \quad \exists\, \delta > 0$ *such that if*

$$\sup_t \|V(t)\|_{L^{n,\infty}} \leq \delta$$

then

$$\|T(t,s)f\|_{L^r} \leq C(t-s)^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\|f\|_{L^q}$$

for $f \in L^q_\sigma$*,* $r \in [q,r_1)$ *(*$r > 1$ *when* $q = 1$*)*

Composite operator $T(t,s)P$**div on** L^q, $1 < q < \infty$

$$\max\{n', q\} < r_2 < \infty, \quad \frac{1}{q} - \frac{1}{r_2} < \frac{1}{n}, \quad \beta := \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r_2}\right) + \frac{1}{2}$$

$$E(t) = \sup_{\tau \in (s,t]} (\tau - s)^{\beta} \| T(\tau, s) P \mathrm{div}\ F \|_{L^{r_2,\infty}}, \qquad F \in C_0^{\infty}$$

Then

$$E(t) \le C \|F\|_{L^q} + C \|V\|_{L^{\infty}(0,\infty; L^{n,\infty})}\, E(t) \qquad \forall\, t \in (s, \infty)$$

Proposition 2 $\forall\, q \in (1,\infty) \quad \exists\, \delta' > 0$

such that if

$$\sup_t \|V(t)\|_{L^{n,\infty}} \le \delta'$$

then

$$\|T(t,s)P\mathrm{div}\ F\|_{L^q} \le C(t-s)^{-1/2}\|F\|_{L^q}$$

for $F \in L^q$ *and* $0 \le s < t < \infty$

Adjoint of generator $L(t)$

$$a(t; u, w) = \langle \nabla u, \nabla w \rangle + \langle V \cdot \nabla u, w \rangle - \langle V \otimes u, \nabla w \rangle$$

$$a^*(t; u, w) = \overline{a(t; w, u)}$$

for $u, w \in H^1_\sigma$. Define $M(t) : H^1_\sigma \to (H^1_\sigma)^*$ by

$$\langle M(t)u, w \rangle = a^*(t; u, w)$$

Then

$$L(t) = M(t)^*, \qquad M(t) = L(t)^*$$

Backward perturbed Stokes system

$$\begin{cases} -\dfrac{dv}{ds} + L(s)^* v = 0 & \text{a.e. } s \in [0, t) \text{ in } (H^1_\sigma)^* \\ v(t) = g \in L^2_\sigma & \text{final condition at } t > 0 \end{cases}$$

admits a unique solution

$$v \in L^2(0, t; H^1_\sigma) \cap C([0, t]; L^2_\sigma)$$
$$\frac{dv}{ds} \in L^2(0, t; (H^1_\sigma)^*)$$

Duality

Define $S(t,s) : L^2_\sigma \to L^2_\sigma$ by

$$S(t,s)g := v(s), \qquad s \in [0,t]$$

Then

$$T(t,s) = S(t,s)^*, \qquad S(t,s) = T(t,s)^*$$

in $\mathcal{L}(L^2_\sigma)$

Weak formulation of NS

$$\begin{aligned}\langle u(t), \varphi(t)\rangle + \int_s^t \Big[\langle \nabla u, \nabla \varphi\rangle + \langle u\cdot\nabla u, \varphi\rangle \\ + \langle V\cdot\nabla u, \varphi\rangle - \langle V\otimes u, \nabla\varphi\rangle\Big] d\tau \\ = \langle u(s), \varphi(s)\rangle + \int_s^t \langle u, \partial_\tau \varphi\rangle d\tau\end{aligned}$$

for all $t \geq s \geq 0$ and

$$\varphi \in C([0,\infty); L^2_\sigma) \cap L^\infty_{loc}([0,\infty); L^n)$$

$$\nabla\varphi \in L^2_{loc}([0,\infty); L^2), \qquad \partial_\tau\varphi \in L^2_{loc}([0,\infty); L^2)$$

Integral equation for weak solution

Set

$$\varphi(\tau) = S(t,\tau)\psi = T(t,\tau)^*\psi, \qquad \psi \in C_{0,\sigma}^\infty$$

to obtain

$$\langle u(t), \psi\rangle = \langle T(t,s)u(s), \psi\rangle - \int_s^t \langle T(t,\tau)P\mathrm{div}\ (u \otimes u), \psi\rangle\, d\tau$$

Idea of Fourier splitting (Schonbek 1985)

Suppose

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 \le 0$$
$$|\widehat{u}(\xi,t)| \le C|\xi|^\gamma, \qquad |\xi| \le 1$$

Then

$$\|u(t)\|_{L^2} \le C(1+t)^{-\frac{n}{4}-\frac{\gamma}{2}}$$

$$
\begin{aligned}
\|u(t)\|_{L^2}^2 &= \int_{|\xi|<\rho(t)} |\hat{u}(\xi,t)|^2 d\xi + \int_{|\xi|\geq\rho(t)} |\hat{u}(\xi,t)|^2 d\xi \\
&\leq C\int_{|\xi|<\rho(t)} |\xi|^{2\gamma} d\xi + \frac{1}{\rho(t)^2}\int_{|\xi|\geq\rho(t)} |\xi|^2 |\hat{u}(\xi,t)|^2 d\xi \\
&\leq C\rho(t)^{n+2\gamma} + \frac{1}{\rho(t)^2}\|\nabla u(t)\|_{L^2}^2
\end{aligned}
$$

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + \rho(t)^2\|u(t)\|_{L^2}^2 \leq C\rho(t)^{n+2\gamma+2}$$

Use $\rho(t)^2 = \frac{m}{1+t}$ with $m > 0$ large (fixed)

$$\frac{d}{dt}\left[(1+t)^m\|u(t)\|_{L^2}^2\right] \leq C(1+t)^m \left(\frac{m}{1+t}\right)^{\frac{n}{2}+\gamma+1}$$

implies

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2(1+t)^{-m} + C(1+t)^{-\frac{n}{2}-\gamma}$$

L^q-estimate of nonlinearity

$$\langle u(t) - T(t,0)u_0, \psi\rangle = -\int_0^t \langle T(t,\tau) P \mathrm{div}\ (u \otimes u), \psi\rangle \, d\tau$$

Take $q > 1$ close to 1

$$\begin{aligned} & \|u(t) - T(t,0)u_0\|_{L^q} \\ \le & \int_0^t (t-\tau)^{-1/2} \|u\|_{L^2}^{2-\theta} \|\nabla u\|_{L^2}^{\theta} \, d\tau \\ \le & \, C\|u_0\|_{L^2}^{\theta} \left(\int_0^t (t-\tau)^{\frac{-1}{2-\theta}} \|u\|_{L^2}^2 \, d\tau \right)^{1-\theta/2} \end{aligned}$$

Estimate in low frequency region

$$\int_{|\xi|<\rho(t)} |\hat{u}(\xi,t)|^2 d\xi$$

$$\leq \|T(t,0)u_0\|_{L^2}^2 + C\rho(t)^{n(\frac{2}{q}-1)} \|\widehat{w}(t)\|_{L^{q'}}^2$$

taking $\rho(t) = \sqrt{\frac{m}{1+t}}$

$$\leq \|T(t,0)u_0\|_{L^2}^2 + C\left(\frac{m}{1+t}\right)^{n(\frac{1}{q}-\frac{1}{2})} \|w(t)\|_{L^q}^2$$

where $w(t) = u(t) - T(t,0)u_0$

Proposition 3 *Suppose* $\sup_t \|V(t)\|_{L^{n,\infty}}$ *is small enough. Every weak solution with (SEI) enjoys*

$$(1+t)^m \|u(t)\|_{L^2}^2$$

$$\leq \|u_0\|_{L^2}^2 + C \int_0^t (1+\tau)^{m-1} \|T(\tau,0)u_0\|_{L^2}^2 \, d\tau$$

$$+ C \int_0^t (1+\tau)^{m-1-\frac{n}{2}+\theta} \left(\int_0^\tau (\tau - s)^{\frac{-1}{2-\theta}} \|u\|_{L^2}^2 \, ds \right)^{2-\theta} \, d\tau$$