On a systematic understanding of smoothing estimates for water wave equations

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Equations of water waves

- Korteweg-de Vries (shallow water wave)

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0$$

- Benjamin-Ono (deep water wave)

$$\partial_t u - \partial_x |D_x| u + u\partial_x u = 0$$

- Davey-Stewartson (shallow water wave of 2D)

$$i\partial_t u - \partial_x^2 u \pm \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x^2 \Delta^{-1} |u|^2$$

$$i\partial_t u - \partial_x^2 u \pm \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x^2 \Box^{-1} |u|^2$$

*By further perturbation analysis....

- Dysthe (deep water wave of 2D)

$$
\begin{aligned}
&2i\left(\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u}{\partial x}\right)+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}-\frac{1}{4}\frac{\partial^2 u}{\partial x^2}-u|u|^2\\
=&\frac{i}{8}\left(\frac{\partial^3 u}{\partial x^3}-6\frac{\partial^3 u}{\partial x\partial y^2}\right)+\frac{i}{2}u\left(u\frac{\partial \overline{u}}{\partial x}-\overline{u}\frac{\partial u}{\partial x}\right)\\
&-\frac{5i}{2}|u|^2\frac{\partial u}{\partial x}-u|D|^{-1}\frac{\partial^2 |u|^2}{\partial x^2}
\end{aligned}
$$

- Hogan (deep water wave of 2D)

$$2i\left(\frac{\partial u}{\partial t} + c_g \frac{\partial u}{\partial x}\right) + p\frac{\partial^2 u}{\partial y^2} + q\frac{\partial^2 u}{\partial x^2} - \gamma u|u|^2$$
$$= -ir\frac{\partial^3 u}{\partial x^3} - is\frac{\partial^3 u}{\partial x \partial y^2} - i\mu|u|^2\frac{\partial \overline{u}}{\partial x}$$
$$+ i\nu|u|^2\frac{\partial u}{\partial x} - u|D|^{-1}\frac{\partial^2 |u|^2}{\partial x^2}$$

($c, p, q, \gamma, r, s, \mu, \nu$:real parameters)

- Shrira (3D packet of internal gravity wave)

$$
\begin{aligned}
& i\frac{\partial u}{\partial t} + \frac{\omega_{kk}}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\omega_{ll}}{2}\frac{\partial^2 u}{\partial y^2} + \omega_{kl}\frac{\partial^2 u}{\partial x \partial y} \\
& -i\left[\frac{\omega_{kkk}}{6}\frac{\partial^3 u}{\partial x^3} + \frac{\omega_{kkl}}{2}\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\omega_{kll}}{2}\frac{\partial^3 u}{\partial x \partial y^2} + \frac{\omega_{lll}}{6}\frac{\partial^3 u}{\partial y^3}\right] \\
& + i\gamma u\left(u\frac{\partial \overline{u}}{\partial s} - \overline{u}\frac{\partial u}{\partial s}\right) = 0
\end{aligned}
$$

($\frac{\partial}{\partial s}$: derivative along a line, γ, $\omega_{\dots}$: real parameters)

The equation we will consider

$$\begin{cases} i\partial_t u + a(D_1, D_2)u = F(b(D_1, D_2)u), \\ u(0, x) = \varphi(x), \end{cases}$$

where

$a(\xi_1, \xi_2)$ is a polynomial of order 3,

$b(\xi_1, \xi_2)$ is a function of growth order 1.

Dysthe, Hogan, Shrira equations are of this form.

Two important tools to show the well-posedness

- Strichartz estimate

$$\left\| e^{ita(D_1,D_2)}\varphi(x) \right\|_{L_t^q(L_x^r)} \lesssim \|\varphi\|_{L^2}$$

- Smoothing estimate*

$$\left\| \langle x \rangle^{-s} |D| e^{ita(D_1,D_2)}\varphi(x) \right\|_{L^2_{t,x}} \lesssim \|\varphi\|_{L^2}$$

Historically, smoothing estimate was first shown to the equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(0,x) = \varphi(x) \in L^2(\mathbf{R}). \end{cases}$$

The solution $u = u(t,x)$ $(t, x \in \mathbf{R})$ satisfies

$$\int_{-T}^{T} \int_{-R}^{R} |\partial_x u(x,t)|^2 \, dx dt \le c(T, R, \|\varphi\|_{L^2})$$

(Kato 1983).

Normal forms

By linear chage of variables, polynomials $a(\xi_1, \xi_2)$ of order 3 are redueced to one of the following normal forms:

$$\xi_1^3, \quad \xi_1\xi_2^2, \quad \xi_1^3 + \xi_2^3, \quad \xi_1^3 - \xi_1\xi_2^2,$$
$$\xi_1^3 + \xi_2^2, \quad \xi_1^3 + \xi_1\xi_2, \quad \xi_1\xi_2^2 + \xi_1^2,$$
$$\xi_1^3 + \xi_2^3 + \xi_1\xi_2, \quad \xi_1^3 - \xi_1\xi_2^2 + \xi_1^2 + \xi_2^2$$

(modulo polynomials of order 1)

Strichartz estimates are given for them except for the case $a(\xi_1, \xi_2) = \xi_1^3, \xi_1\xi_2^2$ (Ben-Artzi, Koch, Saut 2003).

What are known for smoothing estimates?

We consider smoothing estimates for solutions

$$u(t,x) = e^{ita(D_x)}\varphi(x)$$

to general equations

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0 \\ \qquad\qquad\qquad u(0,x) = \varphi(x) \in L^2(\mathbf{R}^n) \end{cases}$$

where $a(\xi)$ are real-valued and **dispersive** in the following senses:

Principal term only

$$(\mathbf{H}) \quad a(\xi) = a_m(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \neq 0),$$

where principal term $a_m(\xi)$ satisfies

- $a_m(\xi) \in C^\infty(\mathbf{R}^n \setminus 0)$,
- $a_m(\lambda\xi) = \lambda^m a_m(\xi)$ $(\lambda > 0, \xi \neq 0)$

Example: $a(\xi_1, \xi_2) = \xi_1^3 + \xi_2^3$, $\xi_1^3 - \xi_1\xi_2^2$ satisfy (H).

Principal term + Lower oredr terms:

(L)
- $a(\xi) \in C^\infty(\mathbf{R}^n)$,
  $\nabla a_m(\xi) \neq 0$ $(\xi \neq 0)$, $\nabla a(\xi) \neq 0$ $(\xi \in \mathbf{R}^n)$
- $|\partial^\alpha (a(\xi) - a_m(\xi))| \leq C\langle \xi \rangle^{m-1-|\alpha|} \quad (|\xi| \geq 1)$

$\Longleftrightarrow$

(L)
- $a(\xi) \in C^\infty(\mathbf{R}^n), \quad |\nabla a(\xi)| \geq C\langle \xi \rangle^{m-1}$,
- $|\partial^\alpha (a(\xi) - a_m(\xi))| \leq C\langle \xi \rangle^{m-1-|\alpha|} \quad (|\xi| \geq 1)$

Example: $a(\xi) = \xi_1^3 + \xi_2^3 + \xi_1$ satisfies (L).

Theorem 1. *Assume (H) or (L). Let $m > 0$ and let $s > 1/2$. Then we have*

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbf{R}^n)}$$

(Ruzhansky and S. 2012).

Remark. Any polynomial $a(\xi)$ which satisfies the estimate in Theorem 1 has to be dispersive, that is

$$\nabla a_m(\xi) \neq 0 \quad (\xi \neq 0).$$

(Hoshiro 2003)

Non-dispersive case

What happens if

$$\begin{cases} (i\partial_t + a(D_x))\, u(t,x) = 0 \\ \qquad\qquad\qquad u(0,x) = \varphi(x) \in L^2(\mathbf{R}^n) \end{cases}$$

does not satisfy

$$\nabla a(\xi) \neq 0 \qquad (\xi \in \mathbf{R}^n)\,?$$

We cannot have smoothing estimates (Hoshiro 2003).

But such case appears naturally in equation of water waves:

If fact, normal forms:

$$\xi_1^3, \quad \xi_1\xi_2^2, \quad \xi_1^3+\xi_2^3, \quad \xi_1^3-\xi_1\xi_2^2,$$
$$\xi_1^3+\xi_2^2, \quad \xi_1^3+\xi_1\xi_2, \quad \xi_1\xi_2^2+\xi_1^2,$$
$$\xi_1^3+\xi_2^3+\xi_1\xi_2, \quad \xi_1^3-\xi_1\xi_2^2+\xi_1^2+\xi_2^2$$

does not satisfies (H) nor (L).

Invariant estimate

We suggest an estimate which we expect to have for non-dispersive equations:

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbf{R}^n_x)} \quad (s > 1/2)$$

and let us call it **invariant estimate**.

This estimate has a number of advantages:

- in the dispersive case $\nabla a(\xi) \neq 0$, it is equivalent to the usual estimate (Theorem 1);

- it is invariant under canonical transformations for the operator $a(D_x)$;

- it does continue to hold for a variety of non-dispersive operators $a(D_x)$, where $\nabla a(\xi)$ may become zero on some set and when the usual estimate fails;

Methods of approach

1. **Comparison principle** $\cdots$ comparison of the symbol implies the comparison of estimate. (New idea)

2. **Canonical Transformation** $\cdots$ shift an equation to another simple one. (Egorov's theorem)

These are new method even for dispersive equations!

Comparison Principle

Theorem 2 (1D case). *Let* $f, g \in C^1(\mathbf{R})$ *be real-valued and strictly monotone. If* $\sigma, \tau \in C^0(\mathbf{R})$ *satisfy*

$$\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leq \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}$$

then we have

$$\|\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbf{R}_t)} \leq \|\tau(D_x)e^{itg(D_x)}\varphi(x)\|_{L^2(\mathbf{R}_t)}$$

for all $x \in \mathbf{R}$.

Theorem 3 (2D case). *Let* $f(\xi,\eta), g(\xi,\eta) \in C^1(\mathbf{R}^2)$ *be real-valued and strictly monotone in* $\xi \in \mathbf{R}$ *for each fixed* $\eta \in \mathbf{R}$. *If* $\sigma, \tau \in C^0(\mathbf{R}^2)$ *satisfy*

$$\frac{|\sigma(\xi,\eta)|}{\left|f_\xi(\xi,\eta)\right|^{1/2}} \le \frac{|\tau(\xi,\eta)|}{\left|g_\xi(\xi,\eta)\right|^{1/2}}$$

then we have

$$\left\|\sigma(D_x,D_y)e^{itf(D_x,D_y)}\varphi(x,y)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_y)} \le \|\tau(D_x,D_y)e^{itg(D_x,D_y)}\varphi(x,y)\|_{L^2(\mathbf{R}_t\times\mathbf{R}_y)}$$

for all $x \in \mathbf{R}$.

Theorem 4 (Radially Symmetric case)**.** *Let* $f, g \in C^1(\mathbf{R}_+)$ *be real-valued and strictly monotone. If* $\sigma, \tau \in C^0(\mathbf{R}_+)$ *satisfy*

$$\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \leq \frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}$$

then we have

$$\|\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbf{R}_t)} \leq \|\tau(|D_x|)e^{itg(|D_x|)}\varphi(x)\|_{L^2(\mathbf{R}_t)}$$

for all $x \in \mathbf{R}^n$*.*

Low dimensional model estimates

By the comparison principal, we can show the equivalence of low dimensional estimates of various type:

In the 1D case, we have $(l, m > 0)$.

$$\sqrt{m}\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbf{R}_t)} = \sqrt{l}\left\| |D_x|^{(l-1)/2} e^{it|D_x|^l} \varphi(x) \right\|_{L^2(\mathbf{R}_t)} \quad (1)$$

for all $x \in \mathbf{R}$. Here $\operatorname{supp} \widehat{\varphi} \subset [0, +\infty)$ or $(-\infty, 0]$.

In the 2D case, we have $(l, m > 0)$

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x,y) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_y)} = \left\| |D_y|^{(l-1)/2} e^{itD_x|D_y|^{l-1}} \varphi(x,y) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_y)} \quad (2)$$

for all $x \in \mathbf{R}$.

On the other hand, in 1D case, we have

$$e^{itD_x}\varphi(x) = \varphi(x+t)$$

hence we have trivially

$$\left\| e^{itD_x}\varphi(x) \right\|_{L^2(\mathbf{R}_t)} = \|\varphi\|_{L^2(\mathbf{R}_x)} \qquad (3)$$

for all $x \in \mathbf{R}$.

Using the equality (3), the right hand sides of (1) and (2) with $l = 1$ can be estimated, and we have:

- 1D Case

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbf{R}_t)} \le C \|\varphi\|_{L^2(\mathbf{R}_x)}$$

- 2D Case

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x,y) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_y)} \le C \|\varphi\|_{L^2\left(\mathbf{R}^2_{x,y}\right)}$$

for all $x \in \mathbf{R}$.

The following is straightforward from these estimates:

Proposition 1. *Suppose* $m > 0$ *and* $s > 1/2$. *Then for* $n \geq 1$ *we have*

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{it|D_n|^m} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^n)}$$

and for $n \geq 2$ *we have*

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{itD_1|D_n|^{m-1}} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^n)},$$

where $D_x = (D_1, \ldots, D_n)$.

The first one gives the invariant estimates for the normal form $a(\xi_1, \xi_2) = \xi_1^3$

Canonical Transformation

Smoothing estimate for dispersive case (Theorem 1) can be reduced to low dimensional model estimates (Proposition 1) by the **Canonical transformation**:

For the change of variable $\psi : \mathbf{R}^n \setminus 0 \to \mathbf{R}^n \setminus 0$ satisfying $\psi(\lambda\xi) = \lambda\psi(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbf{R}^n \setminus 0$, we set

$$Iu(x) = F^{-1}[(Fu)(\psi(\xi))](x).$$

Then we have the relation

$$a(D_x) \cdot I = I \cdot \sigma(D_x), \quad a(\xi) = (\sigma \circ \psi)(\xi).$$

In dispersive case, we may replace $a(D)$ by

$$\sigma(D) = |D_n|^m \cdots \text{if } a(\xi) \text{ is elliptic}$$
$$\sigma(D) = D_1 |D_n|^{m-1} \cdots \text{if } a(\xi) \text{ is non-elliptic}$$

by canonical transformation!

Summary for dispersive case

- Trivial estimate $\left\| e^{itD_x}\varphi(x) \right\|_{L^2(\mathbf{R}_t)} = \|\varphi\|_{L^2(\mathbf{R}_x)}$

$\Downarrow$ (Comparison Principle)

- Low dimensional model estimate

$\Downarrow$ (Canonical Transform)

- Smoothing estimates for dispersive equations

Secondary comparison

By using comparison principle again to the smoothing estimates obtained from the comparison principle, we can have new estimates.

This is a powerful tool to induce the invariant estimates for non-dispersive equations.

Radially symmetric case

From Theorem 1 with $a(\xi) = |\xi|^m$, we obtain

$$\left\| \langle x\rangle^{-s} |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbf{R}^n_x)}.$$

If we set $g(\rho) = \rho^m$, $\tau(\rho) = \rho^{(m-1)/2}$, then we have

$$|\tau(\rho)|/|g'(\rho)|^{1/2} = 1/\sqrt{m}.$$

Hence by the comparison result for radially symmetric case (Theorem 4), we have

Theorem 5. *Let* $f \in C^1(\mathbf{R}_+)$ *be real-valued and strictly monotone. If* $\sigma \in C^0(\mathbf{R}_+)$ *satisfy*

$$|\sigma(\rho)| \leq |f'(\rho)|^{1/2},$$

then we have

$$\|\langle x\rangle^{-s}\sigma(|D_x|)e^{itf(|D_x|)}\varphi(x)\|_{L^2(\mathbf{R}_t\times\mathbf{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbf{R}^n_x)}$$

for $s > 1/2$

A radial function $a(\xi) = f(|\xi|)$ always satisfies

$$|\nabla a(\xi)| = |f'(|\xi|)|.$$

From the secondary comparison (Theorem 5), we obtain

Theorem 6. *Suppose* $n \geq 1$ *and* $s > 1/2$. *Let* $a(\xi) = f(|\xi|)$ *and* $f \in C^\omega(\mathbf{R}_+)$ *be real-valued. Then we have*

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbf{R}^n_x)}.$$

Example.

$a(\xi) = (|\xi|^2 - 1)^2$ is non-dispersive because

$$\nabla a(\xi) = 4(|\xi|^2 - 1)\xi = 0$$

if $|\xi| = 0, 1$.

But we have the invariant estimate by Theorem 6.

Non-radially symmetric case

*Compare again to the low dimensional model estimates

$$\left\| |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi(x) \right\|_{L^2(\mathbf{R}_t)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x)}$$

$$\left\| |D_y|^{(m-1)/2} e^{itD_x|D_y|^{m-1}} \varphi(x,y) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_y)} \leq C \|\varphi\|_{L^2\left(\mathbf{R}^2_{x,y}\right)}$$

then we have:

Theorem 7 (1D secondary comparison). *Let* $f \in C^1(\mathbf{R})$ *be real-valued and strictly monotone. If* $\sigma \in C^0(\mathbf{R})$ *satisfies*

$$|\sigma(\xi)| \leq |f'(\xi)|^{1/2},$$

then we have

$$\|\langle x\rangle^{-s}\sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x)} \leq C\|\varphi(x)\|_{L^2(\mathbf{R}_x)}$$

for $s > 1/2$.

Theorem 8 (2D secondary comparison). *Let* $f \in C^1(\mathbf{R}^2)$ *be real-valued and* $f(\xi, \eta)$ *be strictly monotone in* $\xi \in \mathbf{R}$ *for every fixed* $\eta \in \mathbf{R}$. *If* $\sigma \in C^0(\mathbf{R}^2)$ *satisfies*

$$|\sigma(\xi, \eta)| \leq |\partial f / \partial \xi(\xi, \eta)|^{1/2},$$

then we have

$$\left\| \langle x \rangle^{-s} \sigma(D_x, D_y) e^{itf(D_x, D_y)} \varphi(x, y) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^2_{x,y})} \leq C \|\varphi(x, y)\|_{L^2(\mathbf{R}^2_{x,y})}$$

for $s > 1/2$.

Normal forms:

- $a(\xi) = \xi_1^3 + \xi_2^2$

By 1D secondary comparison (Theorem 7), we have

$$\left\| \langle x_1 \rangle^{-s} |D_1| e^{itD_1^3} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^2)} \le C \|\varphi\|_{L^2(\mathbf{R}_x^2)}$$

$$\left\| \langle x_2 \rangle^{-s} |D_2|^{1/2} e^{it3D_2^2} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^2)} \le C \|\varphi\|_{L^2(\mathbf{R}_x^2)}$$

for $s > 1/2$.

Hence by $\langle x\rangle^{-s} \le \langle x_k\rangle^{-s}$ $(k = 1, 2)$ we have

$$\left\|\langle x\rangle^{-s}\Big(|D_1| + |D_2|^{1/2}\Big)e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \le C\|\varphi\|_{L^2(\mathbf{R}_x^2)}$$

and hence have

$$\left\|\langle x\rangle^{-s}|\nabla a(D_x)|^{1/2}e^{ita(D_x)}\varphi(x)\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}_x^2)} \le C\|\varphi\|_{L^2(\mathbf{R}_x^2)}.$$

- $a(\xi) = \xi_1^2 + \xi_1 \xi_2^2$

By 2D secondary comparison (Theorem 8), we have for $s > 1/2$

$$\left\| \langle x_1 \rangle^{-s} |2D_1 + D_2^2|^{1/2} e^{ita(D_1, D_2)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^2)},$$

$$\left\| \langle x_2 \rangle^{-s} |D_1 D_2|^{1/2} e^{ita(D_1, D_2)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^2)},$$

hence we have similarly

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^2)}.$$

- $a(\xi) = \xi_1 \xi_2^2$

By 2D secondary comparison (Theorem 8), we have for $s > 1/2$

$$\left\| \langle x_1 \rangle^{-s} |D_2| e^{ita(D_1, D_2)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^2)},$$
$$\left\| \langle x_2 \rangle^{-s} |D_1 D_2|^{1/2} e^{ita(D_1, D_2)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^2)},$$

hence we have similarly

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^2)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^2)}.$$

Non-dispersive case controlled by Hessian

We will show that in the non-dispersive situation the rank of $\nabla^2 a(\xi)$ still has a responsibility for smoothing properties.

First let us consider the case when dispersiveness (L) is true only for large ξ:

$$(\mathbf{L}') \quad |\nabla a(\xi)| \geq C\langle\xi\rangle^{m-1} \quad (|\xi| >> 1),$$
$$|\partial^\alpha (a(\xi) - a_m(\xi))| \leq C\langle\xi\rangle^{m-1-|\alpha|} \quad (|\xi| >> 1)$$

Theorem 9. *Suppose* $n \geq 1$, $m \geq 1$, *and* $s > 1/2$. *Let* $a \in C^\infty(\mathbb{R}^n)$ *be real-valued and assume that it has finitely many critical points. Assume* (L′) *and*

$$\nabla a(\xi) = 0 \Rightarrow \det \nabla^2 a(\xi) \neq 0.$$

Then we have

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbf{R}^n_x)}.$$

Example: $a(\xi) = \xi_1^3 + \xi_1\xi_2$, $\xi_1^3 + \xi_2^3 + \xi_1\xi_2$, $\xi_1^3 - \xi_1\xi_2^2 + \xi_1^2 + \xi_2^2$ satisfies assumptions in Theorem 9.

Outline of proof: * $\nabla a(\xi) \neq 0 \Rightarrow$ dispersive.

* $\nabla a(\xi) = 0 \Rightarrow$ by Morse's lemma

$$a(\xi) = (\sigma \circ {}^{\exists}\psi)(\xi),$$
$$\sigma(\eta) = \text{non-degenerate quadratic form,}$$
$$\text{and } \sigma \text{ satisfies dispersiveness (H).}$$

Hence the estimate can be reduced to the dispersive case!

◎ Next we consider the case when $a(\xi)$ is homogeneous (of oder m). Then, by Euler's identity, we have

$$\nabla a(\xi) = \frac{1}{m-1} \xi \nabla^2 a(\xi) \quad (\xi \neq 0),$$

hence

$$\nabla a(\xi) = 0 \Rightarrow \det \nabla^2 a(\xi) = 0 \quad (\xi \neq 0).$$

Therefore assumption in Theorem 10 does not make any sense in this case, but we can have the following result if we use the idea of canonical transform wisely:

Theorem 10. *Suppose* $n \geq 2$ *and* $s > 1/2$. *Let* $a \in C^\infty(\mathbb{R}^n \setminus 0)$ *be real-valued and satisfy*

$$a(\lambda\xi) = \lambda^2 a(\xi) \quad (\lambda > 0,\ \xi \neq 0).$$

Assume that

$$\nabla a(\xi) = 0 \Rightarrow \operatorname{rank} \nabla^2 a(\xi) = n - 1 \quad (\xi \neq 0).$$

Then we have

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^n)}.$$

Example. $a(\xi) = \dfrac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} + \xi_3^2 + \cdots + \xi_n^2$ satisfies the assumptions in Theorem 11.

In the case $n = 2$, this is an illustration of a smoothing estimate for the Cauchy problem for an equation like

$$i\partial_t u + D_1^2 D_2^2 \Delta^{-1} u = 0$$

(A mixture of Davey-Stewartson and Benjamin-Ono type equations).

Conclusions

Invariant estimate is true at least for

- radially symmetric $a(\xi) = f(|\xi|)$, $f \in C^\omega(\mathbf{R}_+)$,

- polynomials $a(\xi_1, \xi_2)$ of order 3.

- non-dispersive $a(\xi)$ controlled by its Hessian.