On a systematic understanding of smoothing estimates for water wave equations

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Equations of water waves

- Korteweg-de Vries (shallow water wave)
  \[ \partial_t u + \partial_x^3 u + u \partial_x u = 0 \]

- Benjamin-Ono (deep water wave)
  \[ \partial_t u - \partial_x |D_x| u + u \partial_x u = 0 \]
• Davey-Stewartson (shallow water wave of 2D)

\[ i \partial_t u - \partial_x^2 u \pm \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x^2 \Delta^{-1} |u|^2 \]

\[ i \partial_t u - \partial_x^2 u \pm \partial_y^2 u = c_1 |u|^2 u + c_2 u \partial_x^2 \Box^{-1} |u|^2 \]

*By further perturbation analysis....*
Dysthe (deep water wave of 2D)

\[ 2i \left( \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u}{\partial x} \right) + \frac{1}{2} \frac{\partial^2 u}{\partial y^2} - \frac{1}{4} \frac{\partial^2 u}{\partial x^2} - u|u|^2 \]

\[ = \frac{i}{8} \left( \frac{\partial^3 u}{\partial x^3} - 6 \frac{\partial^3 u}{\partial x \partial y^2} \right) + \frac{i}{2} u \left( \frac{\partial u}{\partial x} - \overline{u} \frac{\partial \overline{u}}{\partial x} \right) \]

\[ - \frac{5i}{2} |u|^2 \frac{\partial u}{\partial x} - u|D|^{-1} \frac{\partial^2 |u|^2}{\partial x^2} \]
Hogan (deep water wave of 2D)

\[ 2i \left( \frac{\partial u}{\partial t} + c_g \frac{\partial u}{\partial x} \right) + p \frac{\partial^2 u}{\partial y^2} + q \frac{\partial^2 u}{\partial x^2} - \gamma u |u|^2 \]

\[ = -ir \frac{\partial^3 u}{\partial x^3} - is \frac{\partial^3 u}{\partial x \partial y^2} - i\mu |u|^2 \frac{\partial \bar{u}}{\partial x} \]

\[ + iv |u|^2 \frac{\partial u}{\partial x} - u |D|^{-1} \frac{\partial^2 |u|^2}{\partial x^2} \]

\[(c, p, q, \gamma, r, s, \mu, \nu : \text{real parameters})\]
• Shrira (3D packet of internal gravity wave)

\[ i \frac{\partial u}{\partial t} + \frac{\omega_{kk}}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\omega_{ll}}{2} \frac{\partial^2 u}{\partial y^2} + \omega_{kl} \frac{\partial^2 u}{\partial x \partial y} \]

\[ -i \left[ \frac{\omega_{kkk}}{6} \frac{\partial^3 u}{\partial x^3} + \frac{\omega_{kkl}}{2} \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\omega_{kll}}{2} \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\omega_{lll}}{6} \frac{\partial^3 u}{\partial y^3} \right] \]

\[ + i \gamma u \left( u \frac{\partial u}{\partial s} - \bar{u} \frac{\partial \bar{u}}{\partial s} \right) = 0 \]

\( \frac{\partial}{\partial s} \): derivative along a line, \( \gamma, \omega \ldots \): real parameters}
The equation we will consider

\[
\begin{cases}
  i\partial_t u + a(D_1, D_2)u = F(b(D_1, D_2)u), \\
  u(0, x) = \phi(x),
\end{cases}
\]

where

- \(a(\xi_1, \xi_2)\) is a polynomial of order 3,
- \(b(\xi_1, \xi_2)\) is a function of growth order 1.

Dysthe, Hogan, Shrira equations are of this form.
Two important tools to show the well-posedness

- Strichartz estimate

\[ \| e^{ita(D_1,D_2)} \varphi(x) \|_{L_t^q L_x^r} \lesssim \| \varphi \|_{L^2} \]

- Smoothing estimate*

\[ \| \langle x \rangle^{-s} |D| e^{ita(D_1,D_2)} \varphi(x) \|_{L_{t,x}^2} \lesssim \| \varphi \|_{L^2} \]
Historically, smoothing estimate was first shown to the equation

\[
\begin{cases}
\partial_t u + \partial_x^3 u + u \partial_x u = 0, \\
u(0, x) = \varphi(x) \in L^2(\mathbb{R}).
\end{cases}
\]

The solution \( u = u(t, x) \ (t, x \in \mathbb{R}) \) satisfies

\[
\int_{-T}^{T} \int_{-R}^{R} |\partial_x u(x, t)|^2 \, dx \, dt \leq c(T, R, \|\varphi\|_{L^2})
\]

(Kato 1983).
Normal forms

By linear change of variables, polynomials $a(\xi_1, \xi_2)$ of order 3 are reduced to one of the following normal forms:

\[
\begin{align*}
\xi_1^3, & \quad \xi_1 \xi_2^2, \quad \xi_1^3 + \xi_2^3, \quad \xi_1^3 - \xi_1 \xi_2^2, \\
\xi_1^3 + \xi_2^2, & \quad \xi_1^3 + \xi_1 \xi_2^2, \quad \xi_1 \xi_2^2 + \xi_1^2, \\
\xi_1^3 + \xi_2^2 + \xi_1 \xi_2, & \quad \xi_1^3 - \xi_1 \xi_2^2 + \xi_1^2 + \xi_2^2
\end{align*}
\]

(modulo polynomials of order 1)

Strichartz estimates are given for them except for the case $a(\xi_1, \xi_2) = \xi_1^3, \xi_1 \xi_2^2$ (Ben-Artzi, Koch, Saut 2003).
What are known for smoothing estimates?

We consider smoothing estimates for solutions

$$u(t, x) = e^{ita(Dx)} \varphi(x)$$

to general equations

$$\begin{cases} 
(i\partial_t + a(D_x)) u(t, x) = 0 \\
 u(0, x) = \varphi(x) \in L^2(\mathbb{R}^n)
\end{cases}$$

where $a(\xi)$ are real-valued and \textbf{dispersive} in the following senses:
Principal term only

(H) \( a(\xi) = a_m(\xi), \quad \nabla a_m(\xi) \neq 0 \quad (\xi \neq 0), \)

where principal term \( a_m(\xi) \) satisfies

- \( a_m(\xi) \in C^\infty(\mathbb{R}^n \setminus 0), \)

- \( a_m(\lambda \xi) = \lambda^m a_m(\xi) \quad (\lambda > 0, \xi \neq 0) \)

**Example:** \( a(\xi_1, \xi_2) = \xi_1^3 + \xi_2^3, \quad \xi_1^3 - \xi_1 \xi_2^2 \) satisfy (H).
Principal term + Lower oredr terms:

\[ (L) \quad \bullet \quad a(\xi) \in C^\infty(\mathbb{R}^n), \]
\[ \nabla a_m(\xi) \neq 0 \ (\xi \neq 0), \nabla a(\xi) \neq 0 \ (\xi \in \mathbb{R}^n) \]
\[ \bullet \quad |\partial^\alpha (a(\xi) - a_m(\xi))| \leq C\langle \xi \rangle^{m-1-|\alpha|} \quad (|\xi| \geq 1) \]

\[ \iff \]

\[ (L) \quad \bullet \quad a(\xi) \in C^\infty(\mathbb{R}^n), \quad |\nabla a(\xi)| \geq C\langle \xi \rangle^{m-1}, \]
\[ \bullet \quad |\partial^\alpha (a(\xi) - a_m(\xi))| \leq C\langle \xi \rangle^{m-1-|\alpha|} \quad (|\xi| \geq 1) \]

Example: \[ a(\xi) = \xi_1^3 + \xi_2^3 + \xi_1 \] satisfies (L).
**Theorem 1.** Assume (H) or (L). Let $m > 0$ and let $s > 1/2$. Then we have

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \| \varphi \|_{L^2(\mathbb{R}^n)}$$

(Ruzhansky and S. 2012).

**Remark.** Any polynomial $a(\xi)$ which satisfies the estimate in Theorem 1 has to be dispersive, that is

$$\nabla a_m(\xi) \neq 0 \quad (\xi \neq 0).$$

(Hoshiro 2003)
Non-dispersive case

What happens if

\[
\begin{cases}
(i\partial_t + a(D_x))u(t, x) = 0 \\
u(0, x) = \varphi(x) \in L^2(\mathbb{R}^n)
\end{cases}
\]

does not satisfy

\[
\nabla a(\xi) \neq 0 \quad (\xi \in \mathbb{R}^n) ?
\]

We cannot have smoothing estimates (Hoshiro 2003).
But such case appears naturally in equation of water waves:

If fact, normal forms:

\[ \xi_1^3, \quad \xi_1 \xi_2, \quad \xi_1^3 + \xi_2^3, \quad \xi_1^3 - \xi_1 \xi_2^2, \]
\[ \xi_1^3 + \xi_2^2, \quad \xi_1^3 + \xi_1 \xi_2, \quad \xi_1 \xi_2^2 + \xi_1^2, \]
\[ \xi_1^3 + \xi_2^3 + \xi_1 \xi_2, \quad \xi_1^3 - \xi_1 \xi_2^2 + \xi_1^2 + \xi_2^2 \]

does not satisfies (H) nor (L).
Invariant estimate

We suggest an estimate which we expect to have for non-dispersive equations:

\[
\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{i\tau a(D_x)} \varphi(x) \right\|_{L^2(R_t \times R^n_x)} \leq C \| \varphi \|_{L^2(R^n_x)} \quad (s > 1/2)
\]

and let us call it invariant estimate.

This estimate has a number of advantages:
• in the dispersive case $\nabla a(\xi) \neq 0$, it is equivalent to the usual estimate (Theorem 1);

• it is invariant under canonical transformations for the operator $a(D_x)$;

• it does continue to hold for a variety of non-dispersive operators $a(D_x)$, where $\nabla a(\xi)$ may become zero on some set and when the usual estimate fails;
Methods of approach

1. **Comparison principle** ⋯ comparison of the symbol implies the comparison of estimate. (New idea)

2. **Canonical Transformation** ⋯ shift an equation to another simple one. (Egorov’s theorem)

These are new method even for dispersive equations!
Comparison Principle

**Theorem 2** (1D case). Let \( f, g \in C^1(\mathbb{R}) \) be real-valued and strictly monotone. If \( \sigma, \tau \in C^0(\mathbb{R}) \) satisfy

\[
\frac{|\sigma(\xi)|}{|f'(\xi)|^{1/2}} \leq \frac{|\tau(\xi)|}{|g'(\xi)|^{1/2}}
\]

then we have

\[
\| \sigma(D_x)e^{itf(D_x)}\varphi(x) \|_{L^2(\mathbb{R}_t)} \leq \| \tau(D_x)e^{itg(D_x)}\varphi(x) \|_{L^2(\mathbb{R}_t)}
\]

for all \( x \in \mathbb{R} \).
Theorem 3 (2D case). Let $f(\xi, \eta), g(\xi, \eta) \in C^1(\mathbb{R}^2)$ be real-valued and strictly monotone in $\xi \in \mathbb{R}$ for each fixed $\eta \in \mathbb{R}$. If $\sigma, \tau \in C^0(\mathbb{R}^2)$ satisfy

$$\frac{|\sigma(\xi, \eta)|}{|f_\xi(\xi, \eta)|^{1/2}} \leq \frac{|\tau(\xi, \eta)|}{|g_\xi(\xi, \eta)|^{1/2}}$$

then we have

$$\|\sigma(D_x, D_y)e^{itf(D_x, D_y)}\varphi(x, y)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \leq \|\tau(D_x, D_y)e^{itg(D_x, D_y)}\varphi(x, y)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)}$$

for all $x \in \mathbb{R}$. 
Theorem 4 (Radially Symmetric case). Let \( f, g \in C^1(\mathbb{R}_+) \) be real-valued and strictly monotone. If \( \sigma, \tau \in C^0(\mathbb{R}_+) \) satisfy

\[
\frac{|\sigma(\rho)|}{|f'(\rho)|^{1/2}} \leq \frac{|\tau(\rho)|}{|g'(\rho)|^{1/2}}
\]

then we have

\[
\| \sigma(|D_x|)e^{itf(|D_x|)}\varphi(x) \|_{L^2(\mathbb{R}_t)} \leq \| \tau(|D_x|)e^{itg(|D_x|)}\varphi(x) \|_{L^2(\mathbb{R}_t)}
\]

for all \( x \in \mathbb{R}^n \). 

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Low dimensional model estimates

By the comparison principal, we can show the equivalence of low dimensional estimates of various type:

In the 1D case, we have \((l, m > 0)\).

\[
\sqrt{m} \left\| D_x^{(m-1)/2} e^{it|D_x|^m} \phi(x) \right\|_{L^2(\mathbb{R}_t)} = \sqrt{l} \left\| D_x^{(l-1)/2} e^{it|D_x|^l} \phi(x) \right\|_{L^2(\mathbb{R}_t)} \tag{1}
\]

for all \(x \in \mathbb{R}\). Here \(\text{supp} \hat{\phi} \subset [0, +\infty)\) or \((-\infty, 0]\).
In the 2D case, we have \((l, m > 0)\)

\[
\left\| |D_y|^{(m-1)/2} e^{itD_x} |D_y|^{m-1} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} = \left\| |D_y|^{(l-1)/2} e^{itD_x} |D_y|^{l-1} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)}
\]

for all \(x \in \mathbb{R}\).
On the other hand, in 1D case, we have

\[ e^{itD_x} \varphi(x) = \varphi(x + t) \]

hence we have trivially

\[ \| e^{itD_x} \varphi(x) \|_{L^2(\mathbb{R}_t)} = \| \varphi \|_{L^2(\mathbb{R}_x)} \]  \hspace{1cm} (3)

for all \( x \in \mathbb{R} \).

Using the equality (3), the right hand sides of (1) and (2) with \( l = 1 \) can be estimated, and we have:
• 1D Case
\[ \left\| D_x \right\|^{(m-1)/2} e^{it \left| D_x \right|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t)} \leq C \left\| \varphi \right\|_{L^2(\mathbb{R}_x)} \]

• 2D Case
\[ \left\| D_y \right\|^{(m-1)/2} e^{it D_x D_y} \varphi(x, y) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_y)} \leq C \left\| \varphi \right\|_{L^2(\mathbb{R}_{x,y})} \]
for all \( x \in \mathbb{R} \).

The following is straightforward from these estimates:
Proposition 1. Suppose $m > 0$ and $s > 1/2$. Then for $n \geq 1$ we have

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{it |D_n|^m} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)}$$

and for $n \geq 2$ we have

$$\left\| \langle x \rangle^{-s} |D_n|^{(m-1)/2} e^{it D_1 |D_n|^{m-1}} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)},$$

where $D_x = (D_1, \ldots, D_n)$.

The first one gives the invariant estimates for the normal form $a(\xi_1, \xi_2) = \xi_1^3$.
Canonical Transformation

Smoothing estimate for dispersive case (Theorem 1) can be reduced to low dimensional model estimates (Proposition 1) by the Canonical transformation:

For the change of variable \( \psi : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0 \) satisfying \( \psi(\lambda \xi) = \lambda \psi(\xi) \) for all \( \lambda > 0 \) and \( \xi \in \mathbb{R}^n \setminus 0 \), we set

\[
Iu(x) = F^{-1}[(Fu)(\psi(\xi))](x).
\]
Then we have the relation

$$a(D_x) \cdot I = I \cdot \sigma(D_x), \quad a(\xi) = (\sigma \circ \psi)(\xi).$$

In dispersive case, we may replace $a(D)$ by

$$\sigma(D) = |D_n|^m \cdots \text{if } a(\xi) \text{ is elliptic}$$
$$\sigma(D) = D_1|D_n|^{m-1} \cdots \text{if } a(\xi) \text{ is non-elliptic}$$

by canonical transformation!
Summary for dispersive case

- Trivial estimate $\| e^{itD_x} \varphi(x) \|_{L^2(\mathbb{R}_t)} = \| \varphi \|_{L^2(\mathbb{R}_x)}$

  $\downarrow$ (Comparison Principle)

- Low dimensional model estimate

  $\downarrow$ (Canonical Transform)

- Smoothing estimates for dispersive equations
Secondary comparison

By using comparison principle again to the smoothing estimates obtained from the comparison principle, we can have new estimates.

This is a powerful tool to induce the invariant estimates for non-dispersive equations.
Radially symmetric case

From Theorem 1 with $a(\xi) = |\xi|^m$, we obtain

$$\left\| \langle x \rangle^{-s} |D_x|^{(m-1)/2} e^{it|D_x|^m} \varphi \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x^n)}.$$

If we set $g(\rho) = \rho^m$, $\tau(\rho) = \rho^{(m-1)/2}$, then we have

$$|\tau(\rho)|/|g'(\rho)|^{1/2} = 1/\sqrt{m}.$$

Hence by the comparison result for radially symmetric case (Theorem 4), we have
Theorem 5. Let $f \in C^1(\mathbb{R}_+)$ be real-valued and strictly monotone. If $\sigma \in C^0(\mathbb{R}_+)$ satisfy

$$|\sigma(\rho)| \leq |f'(\rho)|^{1/2},$$

then we have

$$\|\langle x \rangle^{-s} \sigma(|D_x|) e^{itf(|D_x|)} \varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^n_x)}$$

for $s > 1/2$
A radial function $a(\xi) = f(|\xi|)$ always satisfies

$$|\nabla a(\xi)| = |f'(|\xi|)|.$$  

From the secondary comparison (Theorem 5), we obtain

**Theorem 6.** Suppose $n \geq 1$ and $s > 1/2$. Let $a(\xi) = f(|\xi|)$ and $f \in C^\omega(\mathbb{R}_+)$ be real-valued. Then we have

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \| \varphi \|_{L^2(\mathbb{R}^n_x)}. $$
Example.

\[ a(\xi) = (|\xi|^2 - 1)^2 \] is non-dispersive because

\[ \nabla a(\xi) = 4(|\xi|^2 - 1)\xi = 0 \]

if \( |\xi| = 0, 1 \).

But we have the invariant estimate by Theorem 6.
Non-radially symmetric case

*Compare again to the low dimensional model estimates

\[ \left\| D_x \right\| (m - 1)/2 e^{it} |D_x|^m \varphi(x) \right\|_{L^2(R_t)} \leq C \left\| \varphi \right\|_{L^2(R_x)} \]

\[ \left\| D_y \right\| (m - 1)/2 e^{it} D_x D_y |m - 1| \varphi(x, y) \right\|_{L^2(R_t \times R_y)} \leq C \left\| \varphi \right\|_{L^2(R_x \times R_y)} \]

then we have:
Theorem 7 (1D secondary comparison). Let $f \in C^1(\mathbb{R})$ be real-valued and strictly monotone. If $\sigma \in C^0(\mathbb{R})$ satisfies

$$|\sigma(\xi)| \leq |f'(\xi)|^{1/2},$$

then we have

$$\|\langle x \rangle^{-s} \sigma(D_x)e^{itf(D_x)}\varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \leq C\|\varphi(x)\|_{L^2(\mathbb{R}_x)}$$

for $s > 1/2$.  

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Theorem 8 (2D secondary comparison). Let $f \in C^1(\mathbb{R}^2)$ be real-valued and $f(\xi, \eta)$ be strictly monotone in $\xi \in \mathbb{R}$ for every fixed $\eta \in \mathbb{R}$. If $\sigma \in C^0(\mathbb{R}^2)$ satisfies

$$|\sigma(\xi, \eta)| \leq |\partial f / \partial \xi(\xi, \eta)|^{1/2},$$

then we have

$$\|\langle x \rangle^{-s} \sigma(D_x, D_y)e^{itf(D_x,D_y)}\varphi(x,y)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_{x,y})} \leq C\|\varphi(x,y)\|_{L^2(\mathbb{R}^2_{x,y})}$$

for $s > 1/2$. 

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Normal forms:

- $a(\xi) = \xi_1^3 + \xi_2^2$

By 1D secondary comparison (Theorem 7), we have

$$\left\| \langle x_1 \rangle^{-s} |D_1| e^{itD_1^3} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \| \varphi \|_{L^2(\mathbb{R}_x^2)}$$

$$\left\| \langle x_2 \rangle^{-s} |D_2|^{1/2} e^{itD_2^2} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^2)} \leq C \| \varphi \|_{L^2(\mathbb{R}_x^2)}$$

for $s > 1/2$. 
Hence by $\langle x \rangle^{-s} \leq \langle x_k \rangle^{-s} (k = 1, 2)$ we have

$$\left\| \langle x \rangle^{-s} \left( |D_1| + |D_2|^{1/2} \right) e^{i\alpha(D_x)} \varphi(x) \right\|_{L^2(R_t \times R^2_x)} \leq C \| \varphi \|_{L^2(R^2_x)}$$

and hence have

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{i\alpha(D_x)} \varphi(x) \right\|_{L^2(R_t \times R^2_x)} \leq C \| \varphi \|_{L^2(R^2_x)}.$$
• $a(\xi) = \xi_1^2 + \xi_1 \xi_2$

By 2D secondary comparison (Theorem 8), we have for $s > 1/2$

$$\|\langle x_1 \rangle^{-s} |2D_1 + D_2^2|^{1/2} e^{i\alpha(D_1, D_2)} \varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^2_x)},$$

$$\|\langle x_2 \rangle^{-s} |D_1 D_2|^{1/2} e^{i\alpha(D_1, D_2)} \varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^2_x)},$$

hence we have similarly

$$\|\langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{i\alpha(D_x)} \varphi(x)\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq C \|\varphi\|_{L^2(\mathbb{R}^2_x)}.$$
• \( a(\xi) = \xi_1 \xi_2^2 \)

By 2D secondary comparison (Theorem 8), we have for \( s > 1/2 \)

\[
\left\| \langle x_1 \rangle^{-s} |D_2| e^{i \alpha (D_1, D_2)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq C \| \varphi \|_{L^2(\mathbb{R}^2_x)} ,
\]
\[
\left\| \langle x_2 \rangle^{-s} |D_1 D_2|^{1/2} e^{i \alpha (D_1, D_2)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq C \| \varphi \|_{L^2(\mathbb{R}^2_x)} ,
\]

hence we have similarly

\[
\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{i \alpha (D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^2_x)} \leq C \| \varphi \|_{L^2(\mathbb{R}^2_x)} .
\]
Non-dispersive case controlled by Hessian

We will show that in the non-dispersive situation the rank of $\nabla^2 a(\xi)$ still has a responsibility for smoothing properties.

First let us consider the case when dispersiveness (L) is true only for large $\xi$:

$$(L') \quad |\nabla a(\xi)| \geq C'\langle \xi \rangle^{m-1} \quad (|\xi| \gg 1),$$

$$|\partial^\alpha (a(\xi) - a_m(\xi))| \leq C'\langle \xi \rangle^{m-1-|\alpha|} \quad (|\xi| \gg 1)$$
**Theorem 9.** Suppose $n \geq 1$, $m \geq 1$, and $s > 1/2$. Let $a \in C^\infty(\mathbb{R}^n)$ be real-valued and assume that it has finitely many critical points. Assume $(L')$ and

$$\nabla a(\xi) = 0 \Rightarrow \det \nabla^2 a(\xi) \neq 0.$$

Then we have

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C \| \varphi \|_{L^2(\mathbb{R}^n_x)}.$$

**Example:** $a(\xi) = \xi_1^3 + \xi_1 \xi_2$, $\xi_1^3 + \xi_2^3 + \xi_1 \xi_2$, $\xi_1^3 - \xi_1 \xi_2 + \xi_2^2 + \xi_1^2$ satisfies assumptions in Theorem 9.
Outline of proof: * $\nabla a(\xi) \neq 0 \Rightarrow$ dispersive.

* $\nabla a(\xi) = 0 \Rightarrow$ by Morse’s lemma

$$a(\xi) = (\sigma \circ \exists \psi)(\xi),$$

$\sigma(\eta) =$ non-degenerate quadratic form,
and $\sigma$ satisfies dispersiveness (H).

Hence the estimate can be reduced to the dispersive case!
Next we consider the case when \( a(\xi) \) is homogeneous (of order \( m \)). Then, by Euler’s identity, we have

\[
\nabla a(\xi) = \frac{1}{m-1} \xi \nabla^2 a(\xi) \quad (\xi \neq 0),
\]

hence

\[
\nabla a(\xi) = 0 \Rightarrow \det \nabla^2 a(\xi) = 0 \quad (\xi \neq 0).
\]

Therefore assumption in Theorem 10 does not make any sense in this case, but we can have the following result if we use the idea of canonical transform wisely:
Theorem 10. Suppose $n \geq 2$ and $s > 1/2$. Let $a \in C^\infty(\mathbb{R}^n \setminus 0)$ be real-valued and satisfy

$$a(\lambda \xi) = \lambda^2 a(\xi) \quad (\lambda > 0, \xi \neq 0).$$

Assume that

$$\nabla a(\xi) = 0 \Rightarrow \text{rank} \, \nabla^2 a(\xi) = n - 1 \quad (\xi \neq 0).$$

Then we have

$$\left\| \langle x \rangle^{-s} |\nabla a(D_x)|^{1/2} e^{ita(D_x)} \varphi(x) \right\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \| \varphi \|_{L^2(\mathbb{R}_x^n)}. $$
Example. \( a(\xi) = \frac{\xi_1^2 \xi_2^2}{\xi_1^2 + \xi_2^2} + \xi_3^2 + \cdots + \xi_n^2 \) satisfies the assumptions in Theorem 11.

In the case \( n = 2 \), this is an illustration of a smoothing estimate for the Cauchy problem for an equation like

\[
i \partial_t u + D_1^2 D_2^2 \Delta^{-1} u = 0
\]

(A mixture of Davey-Stewartson and Benjamin-Ono type equations).
Conclusions

Invariant estimate is true at least for

- radially symmetric \( a(\xi) = f(|\xi|), \ f \in C^\omega(\mathbb{R}_+) \),

- polynomials \( a(\xi_1, \xi_2) \) of order 3.

- non-dispersive \( a(\xi) \) controlled by its Hessian.