# Compressible and Incompressible two phase problem including the phase transition problem

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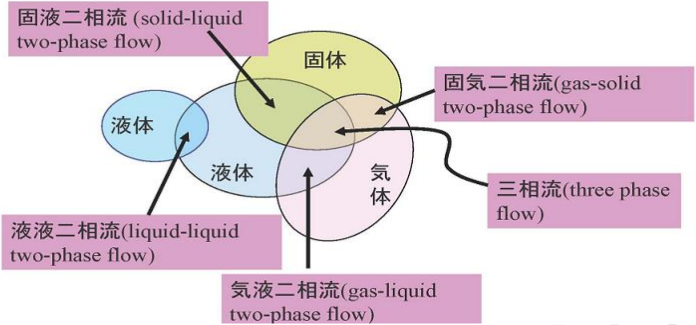
# Phases

The fluid has three phases.

**混相流**
Multiphase Flows

- **気体** (gas)
- **液体** (liquid)
- **固体** (solid)



# Navier-Stokes-Fourier equations

- Balance of Mass
  $\partial_t\rho + \mathrm{div}\,(\rho\mathbf{u}) = 0$
- Balance of Momentum
  $\partial_t(\rho\mathbf{u}) + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) - \mathrm{div}\,\mathbf{T} = \rho\mathbf{f}$
- Balance of Energy
  $\partial_t(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e) + \mathrm{div}\,\{(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e)\mathbf{u}\} - \mathrm{div}\,(\mathbf{T}\mathbf{u}) + \mathrm{div}\,\mathbf{q} = \rho\mathbf{f}\cdot\mathbf{u} + \rho r$

$\rho$: mass field, $\mathbf{u} = (u_1, \ldots, u_N)$: velocity field, $\pi$: pressure field,
$\theta$: absolute temperature field, $e$: internal energy, $\eta$: entropy, $\mathbf{T}$: stress tensor,
$\mathbf{q}$: heat flux
the $i$-th component of $\mathbf{u}\otimes\mathbf{u} = \sum_{j=1}^N u_i u_j$,
$\mathbf{f}$: external force, $r$: heat source

# Interface condition

- $\Omega_+$: region occupied by a gas, $\Omega_-$: region occupied by a liquid
- $\partial\Omega_+ = \partial\Omega_- = \Gamma$: its common boundary, $\mathbb{R}^N = \Omega_+ \cup \Omega_- \cup \Gamma$.
- the time evolution $\Gamma_t$, $\Omega_{t\pm}$ are given by

$$\Gamma_t = \{x = \varphi(\xi, t) \mid \xi \in \Gamma\}, \;\; \Omega_{t\pm} = \{x = \varphi(\xi, t) \mid \xi \in \Omega_\pm\}, \;\; t > 0.$$

- (A) $\quad \frac{d}{dt}\int_{\mathbb{R}^N} f(x,t)\,dx = \int_{\mathbb{R}^N} \partial_t f\,dx + \int_{\Gamma_t} [[f]]\mathbf{v}\cdot\mathbf{n}_\Gamma\,d\Gamma.$
- $\mathbf{v} = \partial_t\varphi$,
- $\mathbf{n}_\Gamma$ is the unit outer normal to $\Gamma_t$ pointing from $\Omega_{+t}$ to $\Omega_{-t}$,
- $[[f]] = (f|_{\Omega_{t+}} - f|_{\Omega_{-t}})|_{\Gamma_t}$: the jump quantity of $f$ accross the $\Gamma_t$

# Interface condition

- Following the argument due to Jan Prüss and Yukihito Suzuki
- $\frac{d}{dt}\int_{\mathbb{R}^N} \rho\, dx = 0$ + Balance of Mass + (A) $\Longrightarrow$

$$\frac{d}{dt}\int_{\mathbb{R}^N} \rho\, dx = -\int_{\mathbb{R}^N} \mathrm{div}\,(\rho\mathbf{u})\, dx + \int_{\Gamma_t} [[\rho]]\mathbf{v}\cdot\mathbf{n}_\Gamma\, d\Gamma$$
$$= -\int_{\Gamma_t} [[\rho(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_\Gamma\, d\Gamma \quad \Longrightarrow$$

$$[[\rho(\mathbf{u}-\mathbf{v})]]\cdot\mathbf{n}_\Gamma = 0 \tag{1}$$

$\Longrightarrow j = \rho_+(\mathbf{u}_+ - \mathbf{v})\cdot\mathbf{n}_\Gamma = \rho_-(\mathbf{u}_- - \mathbf{v})\cdot\mathbf{n}_\Gamma$: phase flux,

- $j \neq 0$ and $[[\rho]] \neq 0 \Longrightarrow j = [[\mathbf{u}]]\cdot\mathbf{n}_\Gamma/[[1/\rho]]$ : with phase transition.
- $j = 0$ : without phase transition $\Longrightarrow [[\mathbf{u}]]\cdot\mathbf{n}_\Gamma = 0$.

# Interface condition

- $\frac{d}{dt}\int_{\mathbb{R}^N} \rho\mathbf{u}\,dx = \int_{\mathbb{R}^N} \rho\mathbf{f}\,dx$ + Balance of Momentum + (A) $\Longrightarrow$

$$[[\rho\mathbf{u}\otimes(\mathbf{u}-\mathbf{v}) - \mathbf{T}]]\mathbf{n}_\Gamma = \mathrm{div}_\Gamma \mathbf{T}_\Gamma. \tag{2}$$

- $\mathbf{T}_\Gamma$: stress tensor on $\Gamma$. Assume that $\mathrm{div}_\Gamma \mathbf{T}_\Gamma = \sigma H_\Gamma \mathbf{n}_\Gamma$ (the surface tension only acts on $\Gamma_t$)
- $\sigma$ : positive constant (the coefficient of the surface tension)
- $H_\Gamma$: double mean curvature of $\Gamma_t$.
- (2) + phase flux $j \Longrightarrow j[[\mathbf{u}]] - [[\mathbf{T}\mathbf{n}_\Gamma]] = \sigma H_\Gamma \mathbf{n}_\Gamma$.
- Thus, we have

$$\begin{aligned} &\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \mathrm{Div}\,\mathbf{T} = \rho\mathbf{f} \quad (x \in \mathbb{R}^N \setminus \Gamma_t), \\ &[[\mathbf{u}]]j - [[\mathbf{T}\mathbf{n}_\Gamma]] = \sigma H_\Gamma \mathbf{n}_\Gamma \quad (x \in \Gamma_t) \end{aligned} \tag{3}$$

# Interface condition

- $\frac{d}{dt}(\frac{1}{2}\int_{\mathbb{R}^N}(\rho|\mathbf{u}|^2 + \rho e)\,dx + \sigma|\Gamma_t|) = \int_{\mathbb{R}^N}\rho\mathbf{f} + \rho r\,dx$
  $+$ Balance of Energy $+$ (A) $\Longrightarrow$

$$[[(\frac{\rho}{2}|\mathbf{u}|^2 + \rho e)(\mathbf{u}-\mathbf{v}) - \mathbf{T}\mathbf{u} + \mathbf{q}]]\cdot\mathbf{n}_\Gamma = \sigma H_\Gamma \mathbf{n}_\Gamma\cdot\mathbf{v}.$$

- Using Balance of Mass, (3) and $j$, we have

$$\begin{aligned} &\rho(\partial_t e + \mathbf{u}\cdot\nabla e) + \operatorname{div} q - \mathbf{T}:\nabla\mathbf{u} = \rho r \ \ (x\in\mathbb{R}^N\setminus\Gamma_t),\\ &([[e]] + [[\frac{1}{2}|\mathbf{u}-\mathbf{v}|^2]])j - [[(\mathbf{u}-\mathbf{v})\cdot\mathbf{T}\mathbf{n}_\Gamma]] + [[\mathbf{q}\cdot\mathbf{n}_\Gamma]] = 0 \ \ (x\in\Gamma_t). \end{aligned} \tag{4}$$

# Constitutive Laws in the Phases

- $\mathbf{T} = \mathbf{S} - \pi \mathbf{I}$, $\mathbf{S} = 2\mu\mathbf{D} + (\lambda - \mu)(\mathrm{div}\,\mathbf{v})\mathbf{I}$, $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + {}^T\nabla\mathbf{u})$ (Newton's law) $\mu = \mu(\theta, \rho) > 0$, $\lambda = \lambda(\theta, \rho)$, $\lambda + \frac{2-N}{N}\mu \geq 0$.
- $q =: -d\nabla\theta$ (Fourier's law) $d = d(\theta, \rho) > 0$.
- $de = \theta d\eta + \frac{\pi}{\rho^2} d\rho$ (the first law of the thermodynamics)
- $\psi := e - \theta\eta$: Helmholtz free energy $\Longrightarrow d\psi = \frac{\pi}{\rho^2} d\rho - \eta d\theta \Longrightarrow$

$$\eta = -\frac{\partial\psi}{\partial\theta}.$$

- $\kappa$: specific heat at constant volume

$$\kappa := \frac{\partial e}{\partial\theta} = -\theta\frac{\partial^2\psi}{\partial\theta^2} > 0.$$

# Entropy Production

- the first law of the thermodynamics $\Longrightarrow$
  $\rho(\partial_t e + \mathbf{u}\cdot\nabla e) = \theta[\partial_t(\rho\eta) + \mathrm{div}\,(\rho\eta\mathbf{u})] - \pi\mathrm{div}\,\mathbf{u}$ + (4) $\Longrightarrow$

$$\partial_t(\rho\eta) + \mathrm{div}\,(\rho\eta\mathbf{u}) = \frac{1}{\theta}[\pi\mathrm{div}\,\mathbf{u} - \mathrm{div}\,q + \mathbf{T} : \nabla\mathbf{u} + \rho r].$$

- $\frac{d}{dt}\int_{\mathbb{R}^N}\rho\eta\,dx \geq 0 \Longrightarrow$

$$\frac{d}{dt}\int_{\mathbb{R}^N}\rho\eta\,dx = \int_{\mathbb{R}^N}\Big\{d\frac{|\nabla\theta|^2}{\theta^2} + \frac{2\mu|\mathbf{D}|^2}{\theta} + \frac{(\lambda-\mu)(\mathrm{div}\,\mathbf{u})^2}{\theta}\Big\}\,dx + \int_{\mathbb{R}^N}\rho r\,dx + \int_{\Gamma_t}[[\rho\eta(\mathbf{u}-\mathbf{v}) + \frac{q}{\theta}]]\cdot\mathbf{n}_\Gamma\,d\Gamma \geq 0.$$

- $[[\theta]] = 0$ + $[[\rho\eta(\mathbf{u}-\mathbf{v}) + \frac{q}{\theta}]]\cdot\mathbf{n}_\Gamma = 0 \Longrightarrow$
  $j[[\theta\eta]] - [[d\partial_{\mathbf{n}_\Gamma}\theta]] = 0$ (Stefan Low)
- $j = 0 \Longrightarrow [[d\partial_{\mathbf{n}_\Gamma}\theta]] = 0.$

# generlized Gibbs-Thomson law

- Stefan Law + the second eq of (4) (the energy balance) + $\psi = e - \theta\eta$ + $[[\theta]] = 0$ + $[[\mathbf{u}']] = 0$ ($\mathbf{u}' = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n}_\Gamma)\mathbf{n}_\Gamma$ : the tangent component of $\mathbf{u}$ $\Longrightarrow$

$$j([[\psi]] + j^2[[\frac{1}{2\rho^2}]] - [[\frac{1}{\rho}\mathbf{n}_\Gamma\cdot\mathbf{T}\mathbf{n}_\Gamma]]) = 0.$$

- $j \neq 0 \Longrightarrow [[\psi]] + j^2[[\frac{1}{2\rho^2}]] - [[\frac{1}{\rho}\mathbf{n}_\Gamma\cdot\mathbf{T}\mathbf{n}_\Gamma]] = 0$ (generalized Gibbs-Thomson law).
- $de = \kappa d\theta + \frac{\partial e}{\partial\rho}d\rho$ + Newton's Law + Fourier's Law + the balance of energy $\Longrightarrow$

$$\rho\kappa(\partial_t\theta + \mathbf{u}\cdot\nabla\theta) - \mathrm{div}\,(d\nabla\theta) - [2\mu|\mathbf{D}|^2 + (\lambda-\mu)(\mathrm{div}\,\mathbf{v})^2] + (\pi - \rho^2\frac{\partial e}{\partial\rho})\mathrm{div}\,\mathbf{u} = \rho r.$$

# Equations, $j \neq 0$, with phase transition

$$\begin{cases} \rho_+(\partial_t \mathbf{u}_+ + \mathbf{u}_+ \cdot \nabla \mathbf{u}_+) - \mathrm{Div}\, \mathbf{T}_+ = 0, \;\; \partial_t \rho_+ + \mathrm{div}\,(\rho_+ \mathbf{u}_+) = 0, \\ \rho_+ \kappa_+ (\partial_t \theta_+ + \mathbf{u}_+ \cdot \nabla \theta_+) - \nabla(d_+ \nabla \theta_+) \\ \quad - (2\mu_+ |\mathbf{D}_+|^2 + (\lambda_+ - \mu_+)(\mathrm{div}\, \mathbf{u}_+)^2) + (\pi_+ - \rho_+^2 \dfrac{\partial e_+}{\partial \rho}) \mathrm{div}\, \mathbf{u}_+ = 0 \quad \text{in } \Omega_{t+}, \end{cases}$$

$$\begin{cases} \rho_{*-}(\partial_t \mathbf{u}_- + \mathbf{u}_- \cdot \nabla \mathbf{u}_-) - \mathrm{Div}\, \mathbf{T}_- = 0, \;\; \mathrm{div}\, \mathbf{u}_- = 0, \\ \rho_{*-} \kappa_- (\partial_t \theta_- + \mathbf{u}_- \cdot \nabla \theta_-) - \mathrm{div}\,(d_- \nabla \theta_-) - 2\mu_- |\mathbf{D}_-|^2 = 0 \quad \text{in } \Omega_{t-}, \end{cases}$$

$$\begin{cases} [[\dfrac{1}{\rho}]] j^2 \mathbf{n}_\Gamma - [[\mathbf{T} \mathbf{n}_\Gamma]] = \sigma H_\Gamma \mathbf{n}_\Gamma, \quad [[\mathbf{u}]] = [[\dfrac{1}{\rho}]] j \mathbf{n}_\Gamma, \\ [[\theta \eta]] j - [[d \dfrac{\partial \theta}{\partial \mathbf{n}_\Gamma}]] = 0, \quad [[\theta]] = 0, \\ [[\psi]] + [[\dfrac{1}{2\rho^2}]] j^2 - [[\dfrac{1}{\rho} \mathbf{n}_\Gamma \cdot \mathbf{T} \mathbf{n}_\Gamma]] = 0, \quad V_\Gamma = \mathbf{v} \cdot \mathbf{n}_\Gamma = [[\rho \mathbf{u} \cdot \mathbf{n}_\Gamma]]/[[\rho]] \quad \text{on } \Gamma_t. \end{cases}$$

$(\mathbf{u}_+, \rho_+, \theta_+)|_{t=0} = (\mathbf{u}_{0+}, \rho_{*+} + \rho_{0+}, \theta_{*+} + \theta_{0+})$ in $\mathbb{R}^N_+ = \{x_N > 0\}$
$(\mathbf{u}_-, \theta_-)|_{t=0} = (\mathbf{u}_{0-}, \theta_{*-} + \theta_{0-})$ in $\mathbb{R}^N_- = \{x_N < 0\}$.

# Equations

- $\rho_{*\pm}$, $\theta_{*\pm}$ are reference densities and temperatures, respectively.
- $\mathbf{T}_+ = \mu_+ \mathbf{D}_+ + (\lambda_+ - \mu_+)\mathrm{div}\, \mathbf{u}_+ \mathbf{I} - \pi_+ \mathbf{I}$, $\mathbf{T}_- = \mu_- \mathbf{D}_- - \pi_-$
- $\kappa_+ = \kappa_+(\rho, \theta) > 0$, $\mu_+ = \mu_+(\rho, \theta) > 0$, $\lambda_+ = \lambda_+(\rho, \theta) > 0$, $d_+ = d_+(\rho, \theta) > 0$, $e_+ = e_+(\rho, \theta)$ : $C^\infty$ functions defined on $\rho \in (0, \infty)$ and $\theta \in (0, \infty)$.
- $\pi_+ = P(\rho, \theta)$, $\partial P/\partial \rho > 0$, $C^\infty$ function defined on $\rho \in (0, \infty)$ and $\theta \in (0, \infty)$.
- $\kappa_- = \kappa_-(\theta) > 0$, $d_- = d_-(\theta) > 0$: $C^\infty$ functions defined on $\theta \in (0, \infty)$.
- $\eta_+ = \eta(\rho, \theta)$, $\psi = \psi_+(\rho, \theta)$: $C^\infty$ functions defined on $\rho \in (0, \infty)$ and $\theta \in (0, \infty)$.
- $\eta_- = \eta_-(\theta)$, $\psi_- = \psi_-(\theta)$: $C^\infty$ function defined on $\theta \in (0, \infty)$.

# Equations, $j = 0$, without phase transition

$$\begin{cases} \rho_+(\partial_t \mathbf{u}_+ + \mathbf{u}_+ \cdot \nabla \mathbf{u}_+) - \mathrm{Div}\, \mathbf{T}_+ = 0, \ \ \partial_t \rho_+ + \mathrm{div}\, (\rho_+ \mathbf{u}_+) = 0, \\ \rho_+ \kappa_+ (\partial_t \theta_+ + \mathbf{u}_+ \cdot \nabla \theta_+) - \nabla (d_+ \nabla \theta_+) \\ \qquad - (2\mu_+ |\mathbf{D}_+|^2 + (\lambda_+ - \mu_+)(\mathrm{div}\, \mathbf{u}_+)^2) + (\pi_+ - \rho_+^2 \dfrac{\partial e_+}{\partial \rho}) \mathrm{div}\, \mathbf{u}_+ = 0 \quad \text{in } \Omega_{t+}, \end{cases}$$

$$\begin{cases} \rho_{*-}(\partial_t \mathbf{u}_- + \mathbf{u}_- \cdot \nabla \mathbf{u}_-) - \mathrm{Div}\, \mathbf{T}_- = 0, \ \ \mathrm{div}\, \mathbf{u}_- = 0, \\ \rho_{*-} \kappa_- (\partial_t \theta_- + \mathbf{u}_- \cdot \nabla \theta_-) - \mathrm{div}\, (d_- \nabla \theta_-) - 2\mu_- |\mathbf{D}_-|^2 = 0 \quad \text{in } \Omega_{t-}, \end{cases}$$

$$\begin{cases} [[\mathbf{T} \mathbf{n}_\Gamma]] = -\sigma H_\Gamma \mathbf{n}_\Gamma, \quad [[\mathbf{u}]] = 0, \\ [[d \dfrac{\partial \theta}{\partial \mathbf{n}_\Gamma}]] = 0, \quad [[\theta]] = 0, \\ V_\Gamma = \mathbf{u} \cdot \mathbf{n}_\Gamma \quad \text{on } \Gamma_t. \end{cases}$$

$(\mathbf{u}_+, \rho_+, \theta_+)|_{t=0} = (\mathbf{u}_{0+}, \rho_{*+} + \rho_{0+}, \theta_{*+} + \theta_{0+})$ in $\mathbb{R}^N_+ = \{x_N > 0\}$
$(\mathbf{u}_-, \theta_-)|_{t=0} = (\mathbf{u}_{0-}, \theta_{*-} + \theta_{0-})$ in $\mathbb{R}^N_- = \{x_N < 0\}$.

# reduction of the problem

- Assume that the reference temperatures $\theta_{*\pm}$ satisfy the equation:

$$\psi_+(\theta_{*+}) - \psi_-(\theta_{*-}) - \Big(\frac{P(\rho_{*+}, \theta_{*+})}{\rho_{*+}} - \frac{\pi_{**-}}{\rho_{*-}}\Big) = 0$$

  with some constant $\pi_{**-}$.

- We consider the problem nearly flat interface represented by a graph over $\mathbb{R}^{N-1}$ :

$$\Gamma_t = \{x \in \mathbb{R}^N \mid x_N = h(x', t) \ \text{for}\ x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \ \text{and}\ t \geq 0\},$$
$$\Omega_{t\pm} = \{x \in \mathbb{R}^N \mid \pm(x_N - h(x', t)) > 0 \ \text{for}\ x' \in \mathbb{R}^{N-1} \ \text{and}\ t \geq 0\}.$$

- $H(x, t)$ sol. to $(1 - \Delta)H = 0$ in $\mathbb{R}^N$, $H|_{x_N=0} = h(x', t)$.
- $H_\epsilon(x, t) = H(x', \epsilon x_N, t)$, $1 + \frac{\partial}{\partial x_N} H_\epsilon \geq 1/2$ with some small $\epsilon > 0$.
- $x \mapsto \varphi(x, t) = (x', x_N + H_\epsilon(x, t)) \Longrightarrow$
  $\Gamma_t = \{y = \varphi(x', 0, t) \mid x' \in \mathbb{R}^{N-1}\}$, $\Omega_{t\pm} = \{y = \varphi(x, t) \mid \pm x_N > 0\}$.

# reduction of the problem

- $\mathbf{n}_\Gamma = (\nabla' h, -1)/\sqrt{|\nabla' h|^2+1}$, $\nabla' h = (\partial_1 h, \ldots, \partial_{N-1}h)$
- $V_\Gamma = \partial_t\varphi \cdot \mathbf{n}_\Gamma = -\partial_t H//\sqrt{|\nabla' H|^2+1}$
- $\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \frac{H_0}{1+H_N}\frac{\partial}{\partial x_N}$, $\frac{\partial}{\partial y_j} = \frac{\partial}{\partial x_j} - \frac{H_j}{1+H_N}\frac{\partial}{\partial x_N}$, $H_0 = \partial_t H_\epsilon$, $H_j = \partial_j H_\epsilon$.
- $\mathrm{div}_{\,y}\, a = 0 \Longleftrightarrow$
  $\mathrm{div}\, a + \sum_{\ell=1}^N\{\frac{\partial}{\partial x_\ell}(H_N a_\ell) - \frac{\partial}{\partial x_N}(H_\ell a_\ell)\} = \mathrm{div}\, a + \sum_{\ell=1}^{N-1}(H_N - H_\ell)\frac{\partial a_\ell}{\partial x_\ell} = 0.$
- $\partial_t\rho_+ + \mathrm{div}\,(\rho_+\mathbf{u}_+) = 0 \Longrightarrow \partial_t\rho_+ + \mathrm{div}\,(\rho_+\tilde{\mathbf{u}}_+) + \rho_+ G = 0$ in $\mathbb{R}^N$
- $\tilde{\mathbf{u}}_+ = (\tilde{u}_{+1}, \ldots, \tilde{u}_{+N})$: Lions extension of $\mathbf{u}_+(x,t)$ to $\mathbb{R}^N$ for each $t \geq 0$.
- $G = \frac{1}{1+H_N}\{\partial_N(\sum_{j=1}^{N-1} H_j\tilde{u}_{+j} + H_0) + (\partial_N H_N)\tilde{u}_{+N} - \sum_{j=1}^{N-1} H_j\partial_j\tilde{u}_{+j}\}$.
- $\int_0^T \|\nabla\tilde{\mathbf{u}}_+(\cdot,s)\|_{L_\infty(\mathbb{R}^N)}ds \leq \epsilon << 1 \Longrightarrow \xi \mapsto x(\xi,t)$ is diffeomorphism on $\mathbb{R}^N$, where $x$ is the solution to the Cauchy problem:

$$\frac{d}{dt}x(\xi,t) = \tilde{\mathbf{u}}_+(x,t), \quad x(\xi,0) = \xi \in \mathbb{R}^N.$$

- $J(\xi, t) := \det \frac{\partial x}{\partial \xi}, \quad \partial_t J(\xi, t) = (\mathrm{div}\, \tilde{\mathbf{u}}_+)(x(\xi, t), t) J(\xi, t).$
- $\partial_t(\rho_+(x(\xi, t), t)J(\xi, t)) = (\partial_t \rho_+ + \mathrm{div}\,(\rho_+ \tilde{\mathbf{u}}_+)(x(\xi, t), t)J(\xi, t) = -(\rho_+(x(\xi, t), t)J(\xi, t)G(x(\xi, t), t)$
- $\rho(x(\xi, t), t) = (\rho_{*+} + \rho(\xi))J(\xi, t)^{-1} e^{\int_0^t G(x(\xi,s).s)\, ds}.$
- $\partial_t J(\xi, t) = (\mathrm{div}\, \tilde{\mathbf{u}}_+)(x(\xi, t), t)J(\xi, t) \Longrightarrow J(\xi, t) = e^{\int_0^t (\mathrm{div}\, \tilde{\mathbf{u}}_+)(x(\xi,s),s)\, ds}.$
- $x(\xi, t) = \xi + \int_0^t \tilde{\mathbf{u}}_+(x(\xi, s), s)\, ds \Longrightarrow \xi = \xi(x, t)$ : the inverse map of $x = x(\xi, t)$
- $\rho_+(x, t) = (\rho_{*+} + \rho(\xi))J(\xi, t)^{-1} e^{\int_0^t (G(x(\xi,s).s) - (\mathrm{div}\, \tilde{\mathbf{u}}_+)(x(\xi,s),s))\, ds}|_{\xi = \xi(x,t)}$
- Insert this formula into the equations.

# Final form of the equations, $j \neq 0$, with phase transition

$$\begin{cases} \rho_+\partial_t\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_+ = \mathbf{f}_+ & \text{in } \mathbb{R}^N_+ \times (0, T), \\ \rho_-\partial_t\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_- = \mathbf{f}_-, \quad \mathrm{div}\,\mathbf{u}_- = f_{0-} = \mathrm{div}\,\mathbf{f}_{0-} & \text{in } \mathbb{R}^N_- \times (0, T), \\ [[\mathbf{S}\mathbf{n}]] + \sigma(\Delta' h)\mathbf{n} = \mathbf{g}, \quad [[\mathbf{u}']] = 0, \quad [[\rho^{-1}\mathbf{n}\cdot\mathbf{S}\mathbf{n}]] = g_0, & \text{on } \mathbb{R}^N_0 \times (0, T), \\ \partial_t h - \left(\dfrac{\rho_+}{\rho_+ - \rho_-}u_{+N} - \dfrac{\rho_-}{\rho_+ - \rho_-}u_{-N}\right) = g_{N+1} & \text{on } \mathbb{R}^N_0 \times (0, T), \end{cases} \tag{5}$$

$$\begin{cases} \kappa_+\partial_t\theta_+ - d_+\Delta\theta_+ = f_{\tau+} & \text{in } \mathbb{R}^N_+ \times (0, T), \\ \kappa_-\partial_t\theta_- - d_-\Delta\theta_- = f_{\tau-} & \text{in } \mathbb{R}^N_+ \times (0, T), \\ [[\theta]] = 0, \quad [[d\partial_N\theta]] = g_\tau & \text{on } \mathbb{R}^N_0 \times (0, T), \end{cases} \tag{6}$$

$$(\mathbf{u}_\pm, \theta_\pm)|_{t=0} = (\mathbf{u}_{0\pm}, \theta_{0\pm}) \quad \text{in } \mathbb{R}^N_\pm, \quad h|_{t=0} = h_0 \quad \text{in } \mathbb{R}^N_0, \tag{7}$$

when $j \neq 0$.

- $\mathbb{R}^N_\pm = \{x \in \mathbb{R}^N \mid \pm x_N > 0\}$, $\mathbb{R}^N_0 = \{x \in \mathbb{R}^N \mid x_N = 0\}$,

# Final form of the equations, $j = 0$, without phase transition

$$\begin{cases} \rho_+\partial_t\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_+ = \mathbf{f}_+ & \text{in } \mathbb{R}^N_+ \times (0, T), \\ \rho_-\partial_t\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_- = \mathbf{f}_-, \quad \mathrm{div}\,\mathbf{u}_- = f_{0-} = \mathrm{div}\,\mathbf{f}_{0-} & \text{in } \mathbb{R}^N_- \times (0, T), \\ [[\mathbf{S}\mathbf{n}]] + \sigma(\Delta' h)\mathbf{n} = \mathbf{g}, \quad [[\mathbf{u}]] = 0, & \text{on } \mathbb{R}^N_0 \times (0, T), \\ \partial_t h - u_{+N} = g_{N+1} & \text{on } \mathbb{R}^N_0 \times (0, T), \end{cases} \tag{8}$$

$$\begin{cases} \kappa_+\partial_t\theta_+ - d_+\Delta\theta_+ = f_{\tau+} & \text{in } \mathbb{R}^N_+ \times (0, T), \\ \kappa_-\partial_t\theta_- - d_-\Delta\theta_- = f_{\tau-} & \text{in } \mathbb{R}^N_+ \times (0, T), \\ [[\theta]] = 0, \quad [[d\partial_N\theta]] = 0 & \text{on } \mathbb{R}^N_0 \times (0, T), \end{cases} \tag{9}$$

$$(\mathbf{u}_\pm, \theta_\pm)|_{t=0} = (\mathbf{u}_{0\pm}, \theta_{0\pm}) \quad \text{in } \mathbb{R}^N_\pm, \quad h|_{t=0} = h_0 \quad \text{in } \mathbb{R}^N_0, \tag{10}$$

when $j = 0$.

# Local Wellposedness

- $\rho_\pm = \rho_{*\pm}$, $d_+ = d_+(\theta_{*+}, \rho_{*+}) > 0$, $d_- = d_-(\theta_{*-}) > 0$.
- $\mathbf{S}_+ = \lambda_+ \mathbf{D}(\mathbf{u}_+) + (\lambda_+ - \mu_+)(\mathrm{div}\, \mathbf{u}_+)\mathbf{I}$, $\mathbf{S}_- = \mu_- \mathbf{D}(\mathbf{u}_-) - \pi_- \mathbf{I}$
- $\mu_+ = \mu_+(\theta_{*+}, \rho_{*+}) > 0$, $\lambda_+ = \lambda_+(\theta_{*+}, \rho_{*+}) > 0$, $\mu_- = \mu_-(\theta_{*-}) > 0$.
- $\kappa_+ = \rho_{*+}\kappa_+(\theta_{*+}, \rho_{*+})$, $\kappa_- = \rho_{*-}\kappa_-(\theta_{*-})$.
- The maximal $L_p$-$L_q$ regularity result for the linearized equations yields that for given time $T$ there exists a small number $\epsilon$ such that initial data satisfy the compatibility condition and smallness condition:

$$\|\theta_{0\pm}\|_{B_q^{2(1-1/p)}(\mathbb{R}^N_\pm)} + \|\mathbf{u}_{0\pm}\|_{B_p^{2(1-1/p)}(\mathbb{R}^N_\pm)} + \|\rho_{0+}\|_{W_q^1(\mathbb{R}^N_+)} + \|h_0\|_{B_{q,p}^{3-1/p}(\mathbb{R}^{N-1})} \le \epsilon$$

then, the local wellposedness holds in the following solution classes:

$$\mathbf{u}_\pm, \theta_\pm \in L_p((0,T), W_q^2(\mathbb{R}^N_\pm)) \cap W_p^1((0,T), L_q(\mathbb{R}^N_\pm)),$$
$$\rho_+ \in W_p^1((0,T), L_q(\mathbb{R}^N_+)) \cap L_p((0,T), W_q^1(\mathbb{R}^N_+)).$$

# Introduction for the linear theory

Quasilinear Parabolic Equations:

$$u_t - A(u)u = 0, \quad B(u)u|_\Gamma = 0, \quad u|_{t=0} = u_0$$

Maximal Regularity for the corresponding linearized equations:

$$u_t - A(0)u = f, \quad B(0)u = g, \quad u|_{t=0} = u_0$$

with $f = (A(u) - A(0))u$ and $g = -(B(u) - B(0))u$.

**Maximal regularity**

$$\|u_t\|_0 + \|A(0)u\|_0 \leq C(\|u_0\| + \|f\|_0 + \|g\|_1).$$

- Solonnikov: Potential estimate
- Mucha - Zajaczkowski: Fourier multiplier theorem
- Pruess..: $H^\infty$ calculus
- Shibata: $R$ bounded solution operators

# Maximal Regularity for Model Prob. $j \neq 0$

$$
\begin{aligned}
&\rho_+\partial_t\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_+ = 0 && \text{in } \mathbb{R}^N_+ \times (0, T),\\
&\rho_-\partial_t\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_- = 0, \ \ \mathrm{div}\,\mathbf{u}_- = 0 && \text{in } \mathbb{R}^N_- \times (0, T),\\
&[[\mathbf{Sn}]] + \sigma(\Delta' h)\mathbf{n} = \mathbf{g}, \ \ [[\mathbf{u}']] = 0, \ \ [[\frac{1}{\rho}\mathbf{n}\cdot\mathbf{Sn}]] = g_0 && \text{on } \mathbb{R}^N_0 \times (0, T),\\
&\partial_t h - \Big(\frac{\rho_+}{\rho_+ - \rho_-}u_{+N} - \frac{\rho_-}{\rho_+ - \rho_-}u_{-N}\Big) = g_{N+1} && \text{on } \mathbb{R}^N_0 \times (0, T)\\
&(\mathbf{u}_\pm, \theta_\pm)|_{t=0} = (0, 0), && (11)
\end{aligned}
$$

where $\mathbf{n} = (0, \ldots, 0, 1)$.

# Generalized Resolvent Problem $j \neq 0$

- Laplace transform with respect to $t$
- $$\begin{aligned}
&\rho_+ \lambda \mathbf{u}_+ - \mathrm{Div}\, \mathbf{S}_+ = 0 && \text{in } \mathbb{R}^N_+, \\
&\rho_- \lambda \mathbf{u}_- - \mathrm{Div}\, \mathbf{S}_- = 0, \ \ \mathrm{div}\, \mathbf{u}_- = 0 && \text{in } \mathbb{R}^N_-, \\
&[[\mathbf{S}\mathbf{n}]] + \sigma(\Delta' h)\mathbf{n} = \mathbf{g}, \ \ [[\mathbf{u}']] = 0, \ \ [[\frac{1}{\rho}\mathbf{n} \cdot \mathbf{S}\mathbf{n}]] = g_0 && \text{on } \mathbb{R}^N_0, \\
&\lambda h - \Big(\frac{\rho_+}{\rho_+ - \rho_-} u_{+N} - \frac{\rho_-}{\rho_+ - \rho_-} u_{-N}\Big) = g_{N+1} && \text{on } \mathbb{R}^N_0
\end{aligned} \tag{12}$$

  where $\mathbf{n} = (0, \ldots, 0, 1)$.
- Laplace inverse transform + Weis Operator Valued Fourier Multiplier Theorem

## Theorem

Let $1 < p, q < \infty$ and $0 < T < \infty$. Let $g_0$, $\mathbf{g} = (g_1, \ldots, g_N) \in L_p((0, T), W^1_q(\mathbb{R}^N)) \cap W^1_p((0, T), W^{-1}_q(\mathbb{R}^N))$, $g_{N+1} \in L_p((0, T), W^2_q(\mathbb{R}^N)) \cap W^1_p((0, T), L_q(\mathbb{R}^N))$. Assume that $g_i|_{t=0} = 0$ for $i = 0, \ldots, N+1$. Then, problem (11) admits a unique solutions $\mathbf{u}_\pm$, $\pi_-$ and $h$ with

$$\mathbf{u}_\pm \in L_p((0, T), W^2_q(\mathbb{R}^N_\pm)) \cap W^1_p((0, T), L_p(\mathbb{R}^N_\pm)), \quad \pi \in L_p((0, T), \hat W^1_q(\mathbb{R}^N_-)),$$
$$h \in L_p((0, T), W^3_q(\mathbb{R}^N)) \cap W^1_p((0, T), W^2_q(\mathbb{R}^N))$$

possessing the estimate:

$$\|\partial_t h\|_{L_p((0,t),W^2_q(\mathbb{R}^N))} + \|\nabla h\|_{L_p((0,t),W^2_q(\mathbb{R}^N))} + \|\mathbf{u}_\pm\|_{L_p((0,t),L_q(\mathbb{R}^N_\pm))}$$
$$+ \|\partial_t \mathbf{u}_\pm\|_{L_p((0,t),W^2_q(\mathbb{R}^N))} \leq Ce^{\gamma t}(\|(g_0, \mathbf{g})\|_{L_p((0,t),W^1_q(\mathbb{R}^N))}$$
$$+ \|\partial_t(g_0, \mathbf{g})\|_{L_p((0,t),W^{-1}_q(\mathbb{R}^N))} + \|g_{N+1}\|_{L_p((0,t),W^2_q(\mathbb{R}^N))} + \|\partial_t g_{N+1}\|_{L_p((0,t),L_q(\mathbb{R}^N))})$$

for any $t \in (0, T)$ with some positive constants $C$ and $\gamma$.

# Generalized Resolvent Problem, $j \neq 0$

We consider the following two problems:

$$\begin{aligned}
&\rho_+\lambda\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_+ = 0 && \text{in } \mathbb{R}^N_+,\\
&\rho_-\lambda\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_- = 0, \ \ \mathrm{div}\,\mathbf{u}_- = 0 && \text{in } \mathbb{R}^N_-,\\
&[[\mathbf{Sn}]] = \mathbf{g}, \ \ [[\mathbf{u}']] = 0, \ \ [[\frac{1}{\rho}\mathbf{n}\cdot\mathbf{Sn}]] = g_0 && \text{on } \mathbb{R}^N_0,
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\rho_+\lambda\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_+ = 0 && \text{in } \mathbb{R}^N_+,\\
&\rho_-\lambda\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_- = 0, \ \ \mathrm{div}\,\mathbf{u}_- = 0 && \text{in } \mathbb{R}^N_-,\\
&[[\mathbf{Sn}]] = -\sigma\Delta' h, \ \ [[\mathbf{u}']] = 0, \ \ [[\frac{1}{\rho}\mathbf{n}\cdot\mathbf{Sn}]] = 0 && \text{on } \mathbb{R}^N_0,\\
&\lambda h - \Big(\frac{\rho_+}{\rho_+ - \rho_-}u_{+N} - \frac{\rho_-}{\rho_+ - \rho_-}u_{-N}\Big) = g_0 && \text{on } \mathbb{R}^N_0
\end{aligned} \tag{14}$$

# Partial Fourier Transform of (13)

$$
\begin{aligned}
&\rho_+\lambda v_{+j} + \mu_+|\xi'|^2 v_{+j} - \mu_+ D_N^2 v_{+j} - \nu_+ i\xi_j(i\xi'\cdot v'_+ + D_N v_{+N}) = 0,\\
&\rho_+\lambda v_{+N} + \mu_+|\xi'|^2 v_{+N} - \mu_+ D_N^2 v_{+N} - \nu_+ D_N(i\xi'\cdot v'_+ + D_N v_{+N}) = 0,\\
&\rho_-\lambda v_{-j} + \mu_-|\xi'|^2 v_{-j} - \mu_+ D_N^2 v_{-j} + i\xi_j\pi_- = 0,\\
&\rho_-\lambda v_{-N} + \mu_+|\xi'|^2 v_{-N} - \mu_+ D_N^2 v_{-N} - D_N\pi_- = 0,\\
&i\xi'\cdot v'_- + D_N v_{-N} = 0 \qquad (15)\\
&\mu_+(D_N v_{+j} + i\xi_j v_{+N})|_{x_N=0+} - \mu_-(D_N v_{-j} + i\xi_j v_{-N})|_{x_N=0-} = f_j,\\
&(2\mu_+ D_N v_{+N} + (\nu_+ - \mu_+)(i\xi'\cdot v'_+ + D_N v_{+N}))|_{x_N=0+}\\
&\quad - (2\mu_- D_N v_{-N} - \pi_-)|_{x_N=0-} = f_N,\\
&v_{+j}|_{x_N=0+} - v_{-j}|_{x_N=0-} = 0\\
&\frac{2\mu_+ D_N v_{+N} + (\nu_+ - \mu_+)(i\xi'\cdot v'_+ + D_N v_{+N})}{\rho_+}\Big|_{x_N=0+}\\
&\quad - \frac{2\mu_- D_N v_{-N} - \pi_-}{\rho_-}\Big|_{x_N=0-} = f_0 \qquad (16)
\end{aligned}
$$

# Solution formula

- $v_{+J} = \alpha_{+J}(e^{-B_+x_N} - e^{-A_+x_N}) + \beta_{+J}e^{-B_+x_N}$,
- $v_{-J} = \alpha_{-J}(e^{B_-x_N} - e^{Ax_N}) + \beta_{-J}e^{-B_-x_N}$, $\pi_- = \gamma_- e^{Ax_N}$
- $A_+ = \sqrt{\rho_+(\mu_+ + \nu_+)^{-1}\lambda + A^2}$, $B_\pm = \sqrt{\rho_\pm\mu_\pm^{-1} + A^2}$, $A = |\xi'|$.

Insert these formulas into the equations and we have

- $$\begin{aligned}
&\mu_+(A_+^2 - B_+^2)\alpha_{+j} + (\nu_+ + \delta)i\xi_j(i\xi' \cdot \alpha'_+ - A_+\alpha_{+N}) = 0,\\
&\mu_+(A_+^2 - B_+^2)\alpha_{+N} - (\nu_+ + \delta)A_+(i\xi' \cdot \alpha'_+ - A_+\alpha_{+N}) = 0,\\
&i\xi' \cdot \alpha'_+ - \alpha_{+N}B_+ + i\xi' \cdot \beta'_+ - \beta_{+N}B_+ = 0,\\
&\mu_-(A^2 - B_-^2)\alpha_{-j} + i\xi_j\gamma_- = 0, \quad \mu_-(A^2 - B_-^2)\alpha_{-N} + A\gamma_- = 0,\\
&i\xi' \cdot \alpha'_- + \alpha_{-N}B_- + i\xi' \cdot \beta'_- + \beta_{-N}B_- = 0, \quad i\xi' \cdot \alpha'_- + A\alpha_{-N} = 0.
\end{aligned}$$

- $i\xi' \cdot \alpha'_+ = \frac{A^2}{A_+B_+-A^2}(i\xi' \cdot \beta'_+ - B_+\beta_{+N})$, $\alpha_{+N} = \frac{A_+}{A_+B_+-A^2}(i\xi' \cdot \beta'_+ - B_+\beta_+)$,
- $i\xi' \cdot \alpha'_- = \frac{A}{B_- - A}(i\xi' \cdot \beta'_- + B_-\beta_{-N})$, $\alpha_{-N} = -\frac{1}{B_- - A}(i\xi' \cdot \beta'_- + B_-\beta_{-N})$,
- $\gamma_- = -\frac{\mu_-(A+B_-)}{A}(i\xi' \cdot \beta'_- + B_-\beta_{-N})$, $\quad \beta_{+j} = \beta_{-j}$
- $-f_j =$ $\mu_+((B_+ - A_+)\alpha_{+j} + B_+\beta_{+j} - i\xi_j\beta_{+N}) - \mu_-((B_- - A)\alpha_{-j} + B_-\beta_{-j} + i\xi_j\beta_{-N})$
- $-f_N = 2\mu_+((B_+ - A_+)\alpha_{+N} + B_+\beta_{+N}) + (\nu_+ - \mu_+)(-i\xi' \cdot \beta'_+ + (B_+ - A_+)\alpha_{+N} + B_+\beta_{+N}) + 2\mu_-((B_- - A)\alpha_{-N} + B_-\beta_{-N}) - \gamma_-$
- $-f_0 = \frac{2\mu_+}{\rho_+}((B_+ - A_+)\alpha_{+N} + B_+\beta_{+N}) + \frac{\nu_+ - \mu_+}{\rho_+}(-i\xi' \cdot \beta'_+ + (B_+ - A_+)\alpha_{+N} + B_+\beta_{+N}) + \frac{2\mu_-}{\rho_-}((B_- - A)\alpha_{-N} + B_-\beta_{-N}) - \frac{\gamma_-}{\rho_-}$

# Lopatinski matrix

- $L = \begin{pmatrix} L_{11}^+ + L_{11}^- & L_{12}^+ A & L_{12}^- A \\ L_{21}^+ + L_{21}^- & L_{22}^+ A & L_{22}^- \\ \frac{L_{21}^+}{\rho_+} + \frac{L_{21}^-}{\rho_-} & \frac{L_{22}^+ A}{\rho_+} & \frac{L_{22}^-}{\rho_-} \end{pmatrix}$
- $L_{11}^+ = \mu_+ \frac{A_+(B_+^2 - A^2)}{A_+B_+ - A^2}$, $L_{11}^- = \mu_-(A + B_-)$, $L_{12}^+ = \mu_+ \frac{A(2A_+B_+ - A^2 - B_+^2)}{A_+B_+ - A^2}$,
  $L_{12}^- = \mu_-(B_- - A)$, $L_{21}^+ = A\{2\mu_+ \frac{A_+(B_+ - A_+)}{A_+B_+ - A^2} - (\nu_+ - \mu_+)\frac{A_+^2 - A^2}{A_+B_+ - A^2}\}$
  $L_{21}^- = \mu_-(B_- - A)$, $L_{22}^+ = (\mu_+ + \nu_+)\frac{B_+(A_+^2 - A^2)}{A_+B_+ - A^2}$, $L_{22}^- = \mu_-(A + B_-)B_-$.
- $L \begin{pmatrix} i\xi' \cdot \beta'_- \\ \beta_{+N} \\ \beta_{-N} \end{pmatrix} = \begin{pmatrix} -i\xi' \cdot f' \\ -Af_N \\ -Af_0 \end{pmatrix}$
- $i\xi' \cdot \beta'_- = \sum_{j=1}^{N-1} i\xi_j \beta_{-j}$, $\quad i\xi' \cdot f'_- = \sum_{j=1}^{N-1} i\xi_j f_j$

# Estimate of Lopatinski determinant

- We prove

$$|\det L| \geq c(|\lambda|^{1/2} + A)^5 \quad \text{for any } \lambda \in \Sigma_\epsilon \text{ and } \xi' \in \mathbb{R}^{N-1} \setminus \{0\},$$

  where $\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \epsilon\}$ $(0 < \epsilon < \pi/2)$.

- We analyze the following three cases:
  Case 1: $R_1|\lambda|^{1/2} \leq |\xi'|$ with some large $R_1 > 0$, Case 2: $R_2|\xi'| \leq |\lambda|^{1/2}$,
  Case 3: $R_-^{-1}|\xi'| \leq |\lambda|^{1/2} \leq R_2|\xi'|$.

- In case 3, we set $\tilde\lambda = \lambda/(|\lambda|^{1/2} + |\xi'|)^2$ and $\tilde\xi_j = \xi_j/(|\lambda|^{1/2} + |\xi'|)$. Then, we have $\det L = (|\lambda|^{1/2} + |\xi'|)^5 \det \tilde L$.

- In case 3, we use the uniqueness of solutions to the ordinary differential equations to prove that $\det \tilde L \neq 0$.

# Partial Fourier Transform of (14)

$$\rho_+\lambda v_{+j} + \mu_+|\xi'|^2 v_{+j} - \mu_+ D_N^2 v_{+j} - \nu_+ i\xi_j(i\xi'\cdot v'_+ + D_N v_{+N}) = 0,$$
$$\rho_+\lambda v_{+N} + \mu_+|\xi'|^2 v_{+N} - \mu_+ D_N^2 v_{+N} - \nu_+ D_N(i\xi'\cdot v'_+ + D_N v_{+N}) = 0,$$
$$\rho_-\lambda v_{-j} + \mu_-|\xi'|^2 v_{-j} - \mu_+ D_N^2 v_{-j} + i\xi_j\pi_- = 0,$$
$$\rho_-\lambda v_{-N} + \mu_+|\xi'|^2 v_{-N} - \mu_+ D_N^2 v_{-N} - D_N\pi_- = 0,$$
$$i\xi'\cdot v'_- + D_N v_{-N} = 0$$
$$\mu_+(D_N v_{+j} + i\xi_j v_{+N})|_{x_N=0+} - \mu_-(D_N v_{-j} + i\xi_j v_{-N})|_{x_N=0-} = 0,$$
$$(2\mu_+ D_N v_{+N} + (\nu_+ - \mu_+)(i\xi'\cdot v'_+ + D_N v_{+N}))|_{x_N=0+}$$
$$\quad - (2\mu_- D_N v_{-N} - \pi_-)|_{x_N=0-} = -\sigma A^2\hat{h} \quad (g_N = -\sigma\Delta' h \Longrightarrow f_N = \sigma A^2 \hat{h}),$$
$$v_{+j}|_{x_N=0+} - v_{-j}|_{x_N=0-} = 0$$
$$\frac{2\mu_+ D_N v_{+N} + (\nu_+ - \mu_+)(i\xi'\cdot v'_+ + D_N v_{+N})}{\rho_+}\Big|_{x_N=0+} - \frac{2\mu_- D_N v_{-N} - \pi_-}{\rho_-}\Big|_{x_N=0-} = 0,$$
$$\lambda\hat{h} - \Big(\frac{\rho_+}{\rho_+ - \rho_-} v_{+N}|_{x_N=0+} - \frac{\rho_-}{\rho_+ - \rho_-} v_{-N}|_{x_N=0-}\Big) = f_{N+1} \qquad (17)$$

- $L\begin{pmatrix} i\xi'\cdot\beta'_- \\ \beta_{+N} \\ \beta_{-N}\end{pmatrix} = \begin{pmatrix} 0 \\ -\sigma A^3\hat h \\ 0\end{pmatrix}$. $L^{-1} = \frac{1}{\det L}\begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33}\end{pmatrix}$.
- $v_{+N}|_{x_N=0+} = \beta_+ = -\sigma\frac{\mathcal{L}_{22}}{\det L}A^3\hat h$, $v_{-N}|_{x_N=0-} = \beta_- = -\sigma\frac{\mathcal{L}_{32}}{\det L}A^3\hat h$
- $\lambda\hat h + \sigma K\hat h = f_{N+1}$ with $K = \frac{(\rho_+\mathcal{L}_{22}-\rho_-\mathcal{L}_{32})A^3}{(\rho_+-\rho_-)\det L} \Longrightarrow \hat h = \frac{1}{\lambda+\sigma K}f_{N+1}$
- We prove that there exist positive constants $c_0$ and $\lambda_0$ such that

$$|\lambda+\sigma K| \geq c_0(|\lambda|+A) \quad \text{for any } \lambda\in\Sigma_{\epsilon,\lambda_0} \text{ and } \xi'\in\mathbb{R}^{N-1}\setminus\{0\},$$

  where $\Sigma_{\epsilon,\lambda_0} = \{\lambda\in\mathbb{C} \mid |\arg\lambda| < \pi-\epsilon \ \ |\lambda|\geq\lambda_0\}$ $(0<\epsilon<\pi/2)$.
- We analyze the following three cases:
  Case 1: $R_1|\lambda|^{1/2}\leq|\xi'|$ with some large $R_1>0$,
  Case 2: $R_2|\xi'|\leq|\lambda|^{1/2}$ for $\lambda\in\Sigma_{\epsilon,\lambda_0}$ with large $R_2$ and $\lambda_0>0$.
  Case 3: $R_1^{-1}|\xi'|\leq|\lambda|^{1/2}\leq R_2|\xi'|$ and $\lambda\in\Sigma_{\epsilon,\lambda_0}$ with large $\lambda_0>0$.
  $|\lambda+\sigma K|\geq((|\lambda|^{1/2}+A)|\tilde\lambda|-M)(|\lambda|^{1/2}+A)\geq c_0|\lambda|^{1/2}(|\lambda|^{1/2}+A)\geq c_1(|\lambda|+A)$ when $|\lambda|$ is large enough.

# Generalized Resolvent Problem, $j = 0$

We consider the following two problems:

$$\begin{aligned}
&\rho_+\lambda\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_+ = 0 && \text{in } \mathbb{R}^N_+,\\
&\rho_-\lambda\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_- = 0, \quad \mathrm{div}\,\mathbf{u}_- = 0 && \text{in } \mathbb{R}^N_-,\\
&[[\mathbf{Sn}]] = \mathbf{g}, \quad [[\mathbf{u}]] = 0, && \text{on } \mathbb{R}^N_0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\rho_+\lambda\mathbf{u}_+ - \mathrm{Div}\,\mathbf{S}_+ = 0 && \text{in } \mathbb{R}^N_+,\\
&\rho_-\lambda\mathbf{u}_- - \mathrm{Div}\,\mathbf{S}_- = 0, \quad \mathrm{div}\,\mathbf{u}_- = 0 && \text{in } \mathbb{R}^N_-,\\
&[[\mathbf{Sn}]] = -\sigma\Delta' h, \quad [[\mathbf{u}]] = 0, && \text{on } \mathbb{R}^N_0,\\
&\lambda h - u_{+N} = g_0 && \text{on } \mathbb{R}^N_0
\end{aligned} \tag{19}$$

# Partial Fourier Transform of (18)

$$
\begin{aligned}
&\rho_+\lambda v_{+j} + \mu_+|\xi'|^2 v_{+j} - \mu_+ D_N^2 v_{+j} - \nu_+ i\xi_j(i\xi'\cdot v'_+ + D_N v_{+N}) = 0,\\
&\rho_+\lambda v_{+N} + \mu_+|\xi'|^2 v_{+N} - \mu_+ D_N^2 v_{+N} - \nu_+ D_N(i\xi'\cdot v'_+ + D_N v_{+N}) = 0,\\
&\rho_-\lambda v_{-j} + \mu_-|\xi'|^2 v_{-j} - \mu_+ D_N^2 v_{-j} + i\xi_j\pi_- = 0,\\
&\rho_-\lambda v_{-N} + \mu_+|\xi'|^2 v_{-N} - \mu_+ D_N^2 v_{-N} - D_N\pi_- = 0,\\
&i\xi'\cdot v'_- + D_N v_{-N} = 0 && (20)\\
&\mu_+(D_N v_{+j} + i\xi_j v_{+N})|_{x_N=0+} - \mu_-(D_N v_{-j} + i\xi_j v_{-N})|_{x_N=0-} = f_j,\\
&(2\mu_+ D_N v_{+N} + (\nu_+ - \mu_+)(i\xi'\cdot v'_+ + D_N v_{+N}))|_{x_N=0+}\\
&\quad - (2\mu_- D_N v_{-N} - \pi_-)|_{x_N=0-} = f_N,\\
&v_{+j}|_{x_N=0+} - v_{-j}|_{x_N=0-} = 0\\
&v_{+N}|_{x_N=0+} - v_{-N}|_{x_N=0-} - 0 && (21)
\end{aligned}
$$

# Solution formula

- $v_{+J} = \alpha_{+J}(e^{-B_+x_N} - e^{-A_+x_N}) + \beta_{+J}e^{-B_+x_N}$,
- $v_{-J} = \alpha_{-J}(e^{B_-x_N} - e^{Ax_N}) + \beta_{-J}e^{-B_-x_N}$, $\pi_- = \gamma_- e^{Ax_N}$
- $A_+ = \sqrt{\rho_+(\mu_+ + \nu_+)^{-1}\lambda + A^2}$, $B_\pm = \sqrt{\rho_\pm \mu_\pm^{-1} + A^2}$, $A = |\xi'|$.

Insert these formulas into the equations and we have

- $$\begin{aligned}
&\mu_+(A_+^2 - B_+^2)\alpha_{+j} + (\nu_+ + \delta)i\xi_j(i\xi' \cdot \alpha'_+ - A_+\alpha_{+N}) = 0, \\
&\mu_+(A_+^2 - B_+^2)\alpha_{+N} - (\nu_+ + \delta)A_+(i\xi' \cdot \alpha'_+ - A_+\alpha_{+N}) = 0, \\
&i\xi' \cdot \alpha'_+ - \alpha_{+N}B_+ + i\xi' \cdot \beta'_+ - \beta_{+N}B_+ = 0, \\
&\mu_-(A^2 - B_-^2)\alpha_{-j} + i\xi_j\gamma_- = 0, \quad \mu_-(A^2 - B_-^2)\alpha_{-N} + A\gamma_- = 0, \\
&i\xi' \cdot \alpha'_- + \alpha_{-N}B_- + i\xi' \cdot \beta'_- + \beta_{-N}B_- = 0, \quad i\xi' \cdot \alpha'_- + A\alpha_{-N} = 0.
\end{aligned}$$

- $i\xi'\cdot\alpha'_+ - \frac{A^2}{A_+B_+-A^2}(i\xi'\cdot\beta'_+ - B_+\beta_{+N})$, $\alpha_{+N} = \frac{A_+}{A_+B_+-A^2}(i\xi'\cdot\beta'_+ - B_+\beta_+)$,
- $i\xi'\cdot\alpha'_- = \frac{A}{B_- -A}(i\xi'\cdot\beta'_- + B_-\beta_{-N})$, $\alpha_{-N} = -\frac{1}{B_- -A}(i\xi'\cdot\beta'_- + B_-\beta_{-N})$,
- $\gamma_- = -\frac{\mu_-(A+B_-)}{A}(i\xi'\cdot\beta'_- + B_-\beta_{-N})$, $\beta_{+j} = \beta_{-j}$, $\beta_{+N} = \beta_{-N}$,
- $-f_j = \mu_+((B_+ - A_+)\alpha_{+j} + B_+\beta_{+j} - i\xi_j\beta_{+N}) - \mu_-((B_- - A)\alpha_{-j} + B_-\beta_{-j} + i\xi_j\beta_{-N})$
- $-f_N = 2\mu_+((B_+ - A_+)\alpha_{+N} + B_+\beta_{+N}) + (\nu_+ - \mu_+)(-i\xi'\cdot\beta'_+ + (B_+ - A_+)\alpha_{+N} + B_+\beta_{+N}) + 2\mu_-((B_- - A)\alpha_{-N} + B_-\beta_{-N}) - \gamma_-$,

# Lopatinski matrix

- $L = \begin{pmatrix} L^+_{11} + L^-_{11} & (L^+_{12} + L^-_{12})A \\ L^+_{21} + L^-_{21} & L^+_{22}A + L^-_{22} \end{pmatrix}$
- $L^+_{11} = \mu_+ \frac{A_+(B_+^2 - A^2)}{A_+B_+ - A^2}$, $L^-_{11} = \mu_-(A + B_-)$, $L^+_{12} = \mu_+ \frac{A(2A_+B_+ - A^2 - B_+^2)}{A_+B_+ - A^2}$,
  $L^-_{12} = \mu_-(B_- - A)$, $L^+_{21} = A\{2\mu_+ \frac{A_+(B_+ - A_+)}{A_+B_+ - A^2} - (\nu_+ - \mu_+)\frac{A_+^2 - A^2}{A_+B_+ - A^2}\}$
  $L^-_{21} = \mu_-(B_- - A)$, $L^+_{22} = (\mu_+ + \nu_+)\frac{B_+(A_+^2 - A^2)}{A_+B_+ - A^2}$, $L^-_{22} = \mu_-(A + B_-)B_-$.
- $L \begin{pmatrix} i\xi' \cdot \beta'_- \\ \beta_{-N} \end{pmatrix} = \begin{pmatrix} -i\xi' \cdot f' \\ -Af_N \end{pmatrix}$
- $i\xi' \cdot \beta'_- = \sum_{j=1}^{N-1} i\xi_j \beta_{-j}$, $\quad i\xi' \cdot f'_- = \sum_{j=1}^{N-1} i\xi_j f_j$

# Estimate of Lopatinski determinant

- We prove

$$|\det L| \geq c(|\lambda|^{1/2} + A)^3 \quad \text{for any } \lambda \in \Sigma_\epsilon \text{ and } \xi' \in \mathbb{R}^{N-1} \setminus \{0\},$$

where $\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| < \pi - \epsilon\}$ $(0 < \epsilon < \pi/2)$.

- We analyze the following three cases:
Case 1: $R_1|\lambda|^{1/2} \leq |\xi'|$ with some large $R_1 > 0$, Case 2: $R_2|\xi'| \leq |\lambda|^{1/2}$,
Case 3: $R_-^{-1}|\xi'| \leq |\lambda|^{1/2} \leq R_2|\xi'|$.

- In case 3, we set $\tilde\lambda = \lambda/(|\lambda|^{1/2} + |\xi'|)^2$ and $\tilde\xi_j = \xi_j/(|\lambda|^{1/2} + |\xi'|)$. Then, we have $\det L = (|\lambda|^{1/2} + |\xi'|)^3 \det \tilde L$.

- In case 3, we use the uniqueness of solutions to the ordinary differential equations to prove that $\det \tilde L \neq 0$.

# Partial Fourier Transform of (19)

$$
\begin{aligned}
&\rho_+\lambda v_{+j} + \mu_+|\xi'|^2 v_{+j} - \mu_+ D_N^2 v_{+j} - \nu_+ i\xi_j(i\xi'\cdot v'_+ + D_N v_{+N}) = 0,\\
&\rho_+\lambda v_{+N} + \mu_+|\xi'|^2 v_{+N} - \mu_+ D_N^2 v_{+N} - \nu_+ D_N(i\xi'\cdot v'_+ + D_N v_{+N}) = 0,\\
&\rho_-\lambda v_{-j} + \mu_-|\xi'|^2 v_{-j} - \mu_+ D_N^2 v_{-j} + i\xi_j \pi_- = 0,\\
&\rho_-\lambda v_{-N} + \mu_+|\xi'|^2 v_{-N} - \mu_+ D_N^2 v_{-N} - D_N \pi_- = 0,\\
&i\xi'\cdot v'_- + D_N v_{-N} = 0\\
&\mu_+(D_N v_{+j} + i\xi_j v_{+N})|_{x_N=0+} - \mu_-(D_N v_{-j} + i\xi_j v_{-N})|_{x_N=0-} = 0,\\
&(2\mu_+ D_N v_{+N} + (\nu_+ - \mu_+)(i\xi'\cdot v'_+ + D_N v_{+N}))|_{x_N=0+}\\
&\quad - (2\mu_- D_N v_{-N} - \pi_-)|_{x_N=0-} = -\sigma A^2 \hat h \quad (g_N = -\sigma\Delta' h \Longrightarrow f_N = \sigma A^2 \hat h),\\
&v_{+j}|_{x_N=0+} - v_{-j}|_{x_N=0-} = 0\\
&v_{+N}|_{x_N=0+} - v_{-N}|_{x_N=0-} = 0\\
&\lambda\hat h - \Big(\frac{\rho_+}{\rho_+ - \rho_-} v_{+N}|_{x_N=0+} - \frac{\rho_-}{\rho_+ - \rho_-} v_{-N}|_{x_N=0-}\Big) = f_{N+1} \qquad (22)
\end{aligned}
$$

- $L\begin{pmatrix} i\xi'\cdot\beta'_- \\ \beta_{-N}\end{pmatrix} = \begin{pmatrix} 0 \\ -\sigma A^3\hat h\end{pmatrix}$. $\quad L^{-1} = \frac{1}{\det L}\begin{pmatrix}\mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22}\end{pmatrix}$.
- $v_{+N}|_{x_N=0+} = \beta_+ = \beta_{-N} = -\sigma\frac{\mathcal{L}_{22}}{\det L}A^3\hat h$
- $\lambda\hat h + \sigma K\hat h = f_{N+1}$ with $K = \sigma\frac{\mathcal{L}_{22}A^3}{\det L} \Longrightarrow \hat h = \frac{1}{\lambda+\sigma K}f_{N+1}$
- We prove that there exist positive constants $c_0$ and $\lambda_0$ such that

  $$|\lambda + \sigma K| \geq c_0(|\lambda| + A) \quad \text{for any } \lambda \in \Sigma_{\epsilon,\lambda_0} \text{ and } \xi' \in \mathbb{R}^{N-1}\setminus\{0\},$$

  where $\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| < \pi - \epsilon \ \ |\lambda| \geq \lambda_0\}$ $(0 < \epsilon < \pi/2)$.
- We analyze the following three cases:
  Case 1: $R_1|\lambda|^{1/2} \leq |\xi'|$ with some large $R_1 > 0$,
  Case 2: $R_2|\xi'| \leq |\lambda|^{1/2}$ for $\lambda \in \Sigma_{\epsilon,\lambda_0}$ with large $\lambda_0 > 0$.
  Case 3: $R_-^{-1}|\xi'| \leq |\lambda|^{1/2} \leq R_2|\xi'|$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$ with large $\lambda_0 > 0$.
  $|\lambda + \sigma K| \geq ((|\lambda|^{1/2} + A)|\tilde\lambda| - M)(|\lambda|^{1/2} + A) \geq c_0|\lambda|^{1/2}(|\lambda|^{1/2} + A) \geq c_1(|\lambda| + A)$ when $|\lambda|$ is large enough.