FINITE ELEMENT METHODS FOR SADDLEPOINT PROBLEMS WITH APPLICATION TO DARCY AND STOKES FLOW

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Abstract. The goal of this short course is to review the analysis and the systematic discretization of saddlepoint systems that arise in the weak formulation of certain fluid flow problems by Galekrin methods. To demonstrate the applicability of the main arguments in various situations, we consider the slow flow of water through a channel surrounded by a porous aquiver. While the free flow of the fluid may be described by the Stokes system, the flow of the water through the confining porous medium is governed by the Darcy equations. The coupling of these two rather different fluid flow models is accomplished by the interface conditions of Beavers, Joseph, and Saffmann. We establish the well-posedness of this interface problem in the framework of mixed variational problems by verifying the assumptions of Brezzi's splitting theorem. Based on the proper variational setting, we then discuss the Galerkin discretization for the Darcy and Stokes problem. The discretizations are analyzed with the same arguments as the continuous problem, and we will highlight the importance of discrete stability conditions for the construction of reliable schemes. In particular, we will illustrate by explicit examples, that a straight forward discretization will in general not lead to stable numerical schemes. Based on appropriate discretization strategies for the Darcy and Stokes problems, we then propose a combined finite element method which yields quasi-optimal approximations for the coupled Darcy-Stokes flow problem.

1. Introduction

1.1. **The model problem.** We consider the flow of an incompressible viscous fluid through a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, which is separated by an interface Γ_{SD} into two parts Ω_S and Ω_D. In the subdomain Ω_S, the fluid can propagate freely and the steady flow is governed by the Stokes equations

$$-\nu\Delta\mathbf{u}_S + \nabla p_S = \mathbf{f}_S, \qquad \text{in } \Omega_S, \tag{1a}$$

$$\text{div}\mathbf{u}_S = 0, \qquad \text{in } \Omega_S, \tag{1b}$$

$$\mathbf{u}_S = \mathbf{0}, \qquad \text{on } \Gamma_S. \tag{1c}$$

Here $\mathbf{u}_S$ denotes the velocity field, p_S the pressure, and $\mathbf{f}_S$ is the density of external forces. The viscosity ν is assumed to be a positive constant. For ease of presentation, we use homogeneous Dirichlet boundary conditions at the outer boundary $\Gamma_S = \partial\Omega_S \setminus \Gamma_{SD}$, but the incorporation of more general boundary conditions is straight forward.

The second subdomain Ω_D is assumed to be covered by a porous medium and the flow is described by the Darcy equations

$$\rho\mathbf{u}_D + \nabla p_D = \mathbf{0}, \qquad \text{in } \Omega_D, \tag{2a}$$

$$\text{div}\mathbf{u}_D = g_D, \qquad \text{in } \Omega_D, \tag{2b}$$

$$\mathbf{n}_D \cdot \mathbf{u}_D = 0, \qquad \text{on } \Gamma_D. \tag{3b}$$

As before, $\mathbf{u}_D$ and p_D denote the fluid velocity and pressure and $\mathbf{n}_D$ is the normal vector pointing to the outside of Ω_D. The constant $\rho = \kappa^{-1}$ is the inverse of the hydraulic permeability of the porous medium, and the function g_D represents the density of sources and sinks. Homogeneous boundary conditions are again presecribed at the outer part of the boundary $\Gamma_D = \partial\Omega_D \setminus \Gamma_{SD}$ for simplicity.

The coupling of the flow velocity and the pressure field over the interace Γ_{SD} is described by the following set of conditions

$$\mathbf{n}_S \cdot \mathbf{u}_S = \mathbf{n}_S \cdot \mathbf{u}_D, \qquad \text{on } \Gamma_{SD}, \tag{3a}$$

$$\mathbf{n}_S \cdot (-\nu\partial_n \mathbf{u}_S + p_S \mathbf{n}_S) = p_D, \qquad \text{on } \Gamma_{SD}, \tag{3b}$$

$$\mathbf{n}_S \times (-\nu\partial_n \mathbf{u}_S) = \gamma\rho^{1/2}\mathbf{n}_S \times \mathbf{u}_S, \qquad \text{on } \Gamma_{SD}. \tag{3c}$$

Here $\mathbf{n}_S = -\mathbf{n}_D$ denotes the normal vector at the interface Γ_{SD} pointing to the outside of Ω_S. The first condition expresses the mass conservation due to incompressibility of the fluid, and the second models the equilibrium of normal forces at the interface. The momentum balance is completed by the third equation, which relates the tangential forces to the tangential velocity. The interface condition (3c) has been first derived experimentally in a slightly different form by Beavers and Joseph [2], and then simplified and justified theoretically by Saffman [23]. A rigorous mathematical justification for the complete model (1a)–(3c) via homogenization was given by Jäger and Mikelic [18].

1.2. **A variational characterization.** Let $(\mathbf{u}, p)$ be a smooth solution to the coupled Darcy-Stokes problem, i.e., such that the restrictions $\mathbf{u}_S = \mathbf{u}|_{\Omega_S}$, $p_S = p|_{\Omega_S}$ and $\mathbf{u}_D = \mathbf{u}|_{\Omega_D}$, $p_D = p|_{\Omega_D}$ solve (1a)–(1c) and (2a)–(2c), respectively, and also satisfy (3a)–(3c). Multiplying (1a) by a smooth test function $\mathbf{v}_S$, integrating over Ω_S, and employing integration-by-parts, we get

$$\begin{aligned}(\mathbf{f}_S, \mathbf{v}_S)_{\Omega_S} &= (-\nu\Delta\mathbf{u}_S + \nabla p_S, \mathbf{v}_S)_{\Omega_S}\\ &= (\nu\nabla\mathbf{u}_S, \nabla\mathbf{v}_S)_{\Omega_S} - (p_S, \mathrm{div}\mathbf{v}_S)_{\Omega_S} + \langle -\nu\partial_n\mathbf{u}_S + p_S\mathbf{n}_S, \mathbf{v}_S\rangle_{\partial\Omega_S}.\end{aligned}$$

By testing the (2a) with a smooth function $\mathbf{v}_D$, we obtain in a similar manner

$$\begin{aligned}(\mathbf{f}_D, \mathbf{v}_D)_{\Omega_D} &= (\rho\mathbf{u}_D + \nabla p_D, \mathbf{v}_D)_{\Omega_D}\\ &= (\rho\mathbf{u}_D, \mathbf{v}_D)_{\Omega_D} - (p_D, \mathrm{div}\mathbf{v}_D)_{\Omega_D} + \langle p_D\mathbf{n}_D, \mathbf{v}_D\rangle_{\partial\Omega_D}.\end{aligned}$$

The boundary terms can be simplified by utilizing the interface conditions (3b)–(3c) and by imposing appropriate boundary and interface conditions on the test function $\mathbf{v}_S$ and $\mathbf{v}_D$. Let us therefore assume that $\mathbf{v}_S = \mathbf{v}|_{\Omega_S}$ and $\mathbf{v}_D = \mathbf{v}|_{\Omega_D}$ are the restriction a function $\mathbf{v}$ defined on Ω, which is smooth on the two sub-domains, and additionally satisfies

$$\mathbf{v}_S = \mathbf{0} \quad \text{on } \Gamma_S, \qquad \mathbf{n}_D \cdot \mathbf{v}_D = 0 \quad \text{on } \Gamma_D, \quad \text{and} \quad \mathbf{n}_S \cdot \mathbf{v}_S = \mathbf{n}_S \cdot \mathbf{v}_D \quad \text{on } \Gamma_{SD}. \tag{4}$$

Note that the conditions in (4) corresponds to the boundary conditions (1c) and (2c), and to the interface condition (3a). The sum of the two bundary terms then reads

$$\begin{aligned}&\langle -\nu\partial_n\mathbf{u}_S + p_S\mathbf{n}_S, \mathbf{v}_S\rangle_{\partial\Omega_S} + \langle p_D\mathbf{n}_D, \mathbf{v}_D\rangle_{\partial\Omega_D}\\ &\quad = \langle -\nu\partial_n\mathbf{u}_S + p_S\mathbf{n}_S, \mathbf{v}_S\rangle_{\Gamma_{SD}} - \langle p_D\mathbf{n}_S, \mathbf{v}_D\rangle_{\Gamma_{SD}}\\ &\quad = \langle \mathbf{n}_S \cdot (-\nu\partial_n\mathbf{u}_S + p_S\mathbf{n}_S - p_D, \mathbf{n}_S \cdot \mathbf{v}_S\rangle_{\Gamma_{SD}} + \langle \mathbf{n}_S \times (-\nu\partial_n\mathbf{u}_S), \mathbf{n}_S \times \mathbf{v}_S\rangle_{\Gamma_{SD}}\\ &\quad = \langle \gamma\rho^{1/2}\mathbf{n}_S \times \mathbf{u}_S, \mathbf{n}_S \times \mathbf{v}_S\rangle_{\Gamma_{SD}}.\end{aligned}$$

For the last transformation, we used the interface conditions (3b) and (3c), and in the line before, we employed to normal continuity of $\mathbf{v}$ across the interface. We thuse observe that any piecewise smooth solution of the coupled Darcy-Stokes problem (1a)–(3c) satisfies

$$\nu(\nabla\mathbf{u},\nabla\mathbf{v})_{\Omega_S}+\rho(\mathbf{u},\mathbf{v})_{\Omega_D}+\gamma\rho^{1/2}\langle\mathbf{n}_S\times\mathbf{u}_S,\mathbf{n}_S\times\mathbf{v}_S\rangle_{\Gamma_{SD}}-(p,\mathrm{div}\mathbf{v})_\Omega=(\mathbf{f},\mathbf{v})_\Omega, \tag{5a}$$
$$-(\mathrm{div}\mathbf{u},q)_\Omega=-(g,q)_\Omega. \tag{5b}$$

The first equation holds for any piecewise smooth test function $\mathbf{v}$ that satisfies the boundary and interface conditions given in (4), and the second in this variational prinicple, which was added to give a complete characterization, is valid for any sufficiently integrable test function q.

1.3. **Weak formulation.** The equations (5a)–(5b) can be written in compact form as

$$a(\mathbf{u},\mathbf{v})+b(p,\mathbf{v})=l(\mathbf{v}), \tag{6a}$$
$$b(\mathbf{u},q)=j(\mathbf{v}), \tag{6b}$$

with bilinear forms defined by

$$a(\mathbf{u},\mathbf{v})=(\nu\nabla\mathbf{u},\nabla\mathbf{v})_{\Omega_S}+(\rho\mathbf{u},\mathbf{v})_{\Omega_D}+\langle\gamma\rho^{1/2}\mathbf{n}_S\times\mathbf{u}_S,\mathbf{n}_S\times\mathbf{v}_S\rangle_{\Gamma_{SD}}, \tag{6c}$$
$$b(\mathbf{v},q)=-(q,\mathrm{div}\mathbf{v})_\Omega, \tag{6d}$$

and linear forms given by

$$l(\mathbf{v})=(\mathbf{f},\mathbf{v})_\Omega \qquad \text{and} \qquad j(q)=-(g,q)_\Omega. \tag{6e}$$

For simplicity, we extended $\mathbf{f}$ and g by zero to the whole domain. The bilinear and linear forms are well-defined for functions

$$\mathbf{u},\mathbf{v}\in H(\mathrm{div};\Omega) \quad \text{with} \quad \mathbf{u}|_{\Omega_S},\mathbf{v}|_{\Omega_s}\in H^1(\Omega_S)^d \quad \text{and} \quad p,q\in L^2(\Omega).$$

The Hilbert space

$$H(\mathrm{div};\Omega)=\{\mathbf{v}\in L^2(\Omega)^d:\mathrm{div}\mathbf{v}\in L^2(\Omega)\} \tag{7}$$

of vector fields with square integrable divergence will play a major role in our considerations and is equipped with the graph norm $\|\mathbf{u}\|^2_{H(\mathrm{div};\Omega)}=\|\mathbf{u}\|^2_{L^2(\Omega)}+\|\mathrm{div}\mathbf{u}\|^2_{L^2(\Omega)}$. Incorporiating the boundary and interface conditions, and the usual scaling condition $\int_\Omega p=0$ in order to guarantee uniqueness of the pressure, we obtain the following function spaces

$$V=\{\mathbf{v}\in H(\mathrm{div};\Omega):\mathbf{v}|_{\Omega_S}\in H^1(\Omega_S)^d,\ \mathbf{u}=\mathbf{0}\text{ on }\Gamma_S,\ \mathbf{n}\cdot\mathbf{v}_D=0\text{ on }\Gamma_D\}, \tag{8a}$$
$$Q=\{q\in L^2(\Omega):\int_\Omega q\,dx=0\}, \tag{8b}$$

which can be considered to be the natural choice for the problem under investigation. Summarizing we arrive at the following weak formulation

Problem 1. *Let* $\mathbf{f}\in L^2(\Omega)^d$ *and* $g\in L^2(\Omega)$ *with* $\int_\Omega g\,dx=0$. *Find* $\mathbf{u}\in V$ *and* $p\in Q$, *such that (6a)–(6b) holds for all* $\mathbf{v}\in V$ *and* $q\in Q$.

The scaling condition on the function g is required to ensure the existence of solutions. Using the theory of mixed variational problems, we will be able to establish the well-posedness of this weak formulation for the Darcy-Stokes problem.

Theorem 2. *Problem 1 has a unique solution, and*

$$\|\mathbf{u}\|_{H(\mathrm{div},\Omega)}+\|\mathbf{u}|_{\Omega_S}\|_{H^1(\Omega_S)}+\|p\|_{L^2(\Omega)}\le C(\|\mathbf{f}\|_{L^2(\Omega)}+\|g\|_{L^2(\Omega)}).$$

A complete proof of this result will be given at the end of the next section.

1.4. **Plan for the rest of the lecture.** In Section 2, we review the basic existence and unqiqueness results about mixed variational problems of the form (6a)–(6b). In particular, we present an extension of the Lax-Milgram lemma due to Brezzi [5], which is suitable for the analysis of problems of this form. Sufficient conditions on the bilinear forms a and b for the well-posedness of the mixed problems will be discussed, and their necessity for many relevant cases will be highlighted. Based on the abstract framework, we then present short proofs for the existence and uniqueness of weak solutions to the Stokes, the Darcy, and the coupled Darcy-Stokes problem.

In Section 3, we then discuss the systematic approximation of mixed variational problems (6a)–(6b) by Galerkin methods. We recall the abstract stability and error estimates, and highlight the importance of two discrete stability conditions, which are sufficient and in many cases also necessary for a stable discretization. As a preparation for the last section, some basic results about piecewise polynomial approximations are recalled.

Section 4 is then devoted to the stable discretization of the flow problems under consideration by appropriate finite element methods. We first discuss the discretization of the Stokes problem and illustrate that not every approximation, that might seem reasonable at first sight, yields a stable discretization. We then investigate finite element methods for the Darcy problem, and again highlight that seemingly reasonable discretization may fail in practice. In both cases, standard choices of approximation spaces may lead to instability and possible non- or suboptimal convergence. For both problems, stable finite element approximations will be discussed as well. These can finally be combined to a valid and convergence discretization for the coupled Darcy-Stokes problem (1)–(3).

2. Mixed variational problems

Let V and Q be separable Hilbert spaces, $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ be given bilinear forms, and $f : V \to \mathbb{R}$ and $j : Q \to \mathbb{R}$ be some prescribed linear functionals. We consider mixed variational problems of the following abstract form:

Problem 3. *Find $u \in V$ and $p \in Q$ such that*

$$a(u, v) + b(p, v) = l(v), \qquad \text{for all } v \in V, \tag{9a}$$

$$b(u, q) = j(v), \qquad \text{for all } q \in Q. \tag{9b}$$

This problem is equivalent to the system of operator equations

$$Au + B'p = l \qquad \text{in } V', \tag{10a}$$

$$Bu = j \qquad \text{in } Q', \tag{10b}$$

where $A : V \to V'$, $B : V \to Q'$, and $B' : Q \to V'$ are linear operators associated to the bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ by

$$\langle Au, v\rangle_{V'\times V} = a(u, v) \quad \text{and} \quad \langle Bu, q\rangle_{Q'\times Q} = b(v, q) = \langle u, B'q\rangle_{V\times V'}.$$

Here V', Q' denote the dual spaces of V and Q, and $\langle\cdot,\cdot\rangle_{V'\times V}$ is the duality product on $V' \times V$. The operator B' is called the adjoint operator of B. The kernel

$$N(B) = \ker(B) = \{v \in V : Bv = 0\} = \{v \in V : b(v, q) = 0 \ \forall q \in Q\} \tag{11}$$

of the operator B will play an important role in the subsequent analysis.

If the bilinear form a is symmetric and non-negative, then the problem (9a)–(9b) is the first order optimality system for the constrained minimization problem

$$\min_{u\in V} \tfrac{1}{2}a(u, u) - l(u), \qquad \text{such that } Bu = j.$$

In fact, the solution of (9a)–(9b) characterizes the saddlepoint of the Lagrangian

$$\mathcal{L}(u,p) = \tfrac{1}{2}a(u,u) - l(u) + b(u,p) - j(p),$$

which uniquely decribes the solution of the constrained minimization problem, if the variational problem (9a)–(9c) is uniquely solvable. For this reason, problems of the form (9a)–(9b) are usually called saddlepoint problems.

2.1. **Brezzi's theorem.** The well-posedness of the mixed variational problem (9a)–(9b) and of the system (10a)–(10b) is guaranteed by the following result due to Franco Brezzi [5].

Theorem 4 (Brezzi's splitting theorem).
Assume that the bilinear forms a and b are continuous, i.e., that

(a1) $a(u,v) \le C_a \|u\|_V \|v\|_V$ *for all* $u, v \in V$, *(continuity of a)*

(b1) $b(u,q) \le C_b \|u\|_V \|q\|_V$ *for all* $u \in V$ *and* $q \in Q$, *(continuity of b)*

with some $C_a, C_b > 0$, *and that for some* $\alpha, \beta > 0$ *there holds*

(a2) $a(u_0,u_0) \ge \alpha \|u_0\|_V^2$ *for all* $u_0 \in N(B)$, *(kernel ellipticity)*

(b2) $\sup_{u\in V} b(u,q)/\|u\|_V \ge \beta \|q\|_Q$ *for all* $q \in Q$. *(inf-sup stability)*

Then for any data $l \in V'$ *and* $j \in Q'$, *Problem 2 has a unique solution, and*

$$\|u\|_V + \|p\|_Q \le C(\|l\|_{V'} + \|j\|_{Q'}),$$

with C depending only on the constants α, β, C_a *of the conditions (a1)–(b2).*

Before we present the proof of this result, let us collect some remarks on the meaning of the conditions and their necessity for the well-posedness in many situations.

2.2. **Discussion of the conditions.** The continuity conditions (a1) and (b1) ensure that $A : V \to V'$, $B : V \to Q'$, and $B' : Q \to V'$ are bounded linear operators. The stability conditions (a2) and (b2), on the other hand, imply that the combined operator

$$L : V \times Q \to V' \times Q', \qquad L(u,p) = (Au + B'p, Bu)$$

is boundedly invertible. Together the four conditions of Brezzi's theorem thus guarantee that the combined operator L is an isomorphism between the solution and the data space.

Let us now discuss in bit more detail the two stability conditions: The condition (a2) ensures that the operator A is invertible on the kernel $N(B)$, which could also be guaranteed under more general conditions on a. Provided that a is symmetric and non-negative, which is the case for the problems under consideration, the condition (a2) is however necessary to guarantee invertibility of A on $N(B)$. Condition (b2) can also be written as

(b2') $\inf_{q\in Q} \sup_{u\in V} \frac{b(u,q)}{\|u\|_V \|q\|_Q} \ge \beta$,

which explains the name inf-sup condition. For many problem including those under consideration, the conditions of Brezzi's theorem can be shown to be not only sufficient but even necessary for the well-posedness.

The conditions (b1) and (b2) on the bilinearform b imply some properties of the operators B and B', which we summarize for later reference.

Lemma 5. *Let (b1) and (b2) hold. Then*
(i) $B : V \to Q'$ *is surjective, i.e., for any* $j \in Q'$ *there exists a unique solution* $u_1 \in N(B)^\perp$ *such that* $Bu_1 = j$. *Moreover,* $\|u_1\|_V \le \frac{1}{\beta}\|j\|_{Q'}$.
(ii) $B' : Q \to V'$ *is injective with closed range, and for any*

$$\tilde{l} \in N(B)^\circ = \{l \in V' : \langle l, v_0\rangle_{V'\times V} = 0 \ \forall v_0 \in N(B)\} = R(B'),$$

there exists a unique solution p to $B'p = \tilde{l}$ *and* $\|p\|_Q \le \frac{1}{\beta}\|\tilde{l}\|_{V'}$.

Remark on the proof: The inf-sup condition implies that B' is injective and has closed range. The assertions then follow by appropriate restriction of the operators and duality.

2.3. **Proof of Brezzi's theorem.** We are now in the position to present a short proof of Brezzi's theorem. The proof is constructive and follows in three steps:
STEP 1: By Lemma 5(i), there exists $u_1 \in N(B)$ such that $Bu_1 = j$ and $\|u_1\|_V \le \frac{1}{\beta}\|j\|_{Q'}$.
STEP 2: In order to satisfy (9b) respectively (10b), we have to find the solution u in the form $u = u_0 + u_1$ with $u_0 \in N(B)$. We test (9a) with $v = v_0 \in N(B)$ to get

$$l(v_0) = a(u_0 + u_1, v_0) + b(v_0, p) = a(u_0, v_0) + a(u_1, v_0),$$

where we employed that $b(v_0, p) = \langle Bv_0, p\rangle_{Q'\times Q} = 0$ since $v_0 \in N(B)$. By the Lax-Milgram lemma and the conditions (a1) and (a2), the problem

$$a(u_0, v_0) = l(v_0) - a_1(u_1, v_0) =: \bar{l}(v_0) \qquad \text{for all } v_0 \in N(B),$$

has a unique solution $u_0 \in N(B)$ and $\|u_0\|_V \le \frac{1}{\alpha}\|\bar{l}\|_{V'} \le \frac{1}{\alpha}(\|l\|_{V'} + C_a\|u_1\|_V)$. Summarizing, we have obtained $u = u_0 + u_1$ satisfying (9a) and (9b) for all $v = v_0 \in N(B)$. It remains to define p such that (9a) holds for all $v \in V$.
STEP 3: By rearranging (10a), we get the condition

$$B'p = l - Au =: \tilde{l}.$$

By construction of u in Step 1 and 2, we have $\langle \tilde{l}, v_0\rangle_{V'\times V} = l(v_0) - a(u, v_0) = 0$ for all $v_0 \in N(B)$, i.e., $\tilde{l} \in N(B)^\circ$. Hence by Lemma 5(ii), there exists a unique solution $p \in Q$ for this problem and there holds $\|p\|_Q \le \frac{1}{\beta}\|\tilde{l}\|_{V'} \le \frac{1}{\beta}(\|l\|_{V'} + C_a\|u\|_V)$.
This complete the proof of existence; uniqueness and the estimates are left as an exercise.

2.4. **Application to model problems.** We now illustrate the application of Brezzi's theorem by establishing well-posedness for the flow problems considered in Section 1.

2.4.1. *Stokes flow.* Let us consider the Stokes problem

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \qquad \text{in } \Omega, \tag{12a}$$
$$\mathrm{div}\mathbf{u} = 0, \qquad \text{in } \Omega, \tag{12b}$$
$$\mathbf{u} = \mathbf{0}, \qquad \text{on } \partial\Omega, \tag{12c}$$

which arises from the problem of Section 1 as a special case by setting $\Omega_D = \{\}$. The weak formulation leads to a mixed variational problem of the form (9a)–(9b) with spaces $V = H_0^1(\Omega)^d = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v}|_{\partial\Omega} = \mathbf{0}\}$ and $Q = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$, with bilinear forms $a(\mathbf{u}, \mathbf{v}) = \nu(\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega$ and $b(\mathbf{v}, q) = -(\mathrm{div}\mathbf{u}, q)_\Omega$, and with linear functionals $l(\mathbf{v}) = (\mathbf{f}, \mathbf{v})_\Omega$ and $j(q) = 0$. The linear forms can easily be shown to be continuous, and by the Cauchy-Schwarz inequality, we have $a(\mathbf{u}, \mathbf{v}) = \nu(\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega \le \nu\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)}$ and $b(u, q) = -(\mathrm{div}\mathbf{u}, q)_\Omega \le \|\mathrm{div}\mathbf{u}\|_\Omega\|q\|_\Omega \le \sqrt{d}\|\mathbf{u}\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}$. Thus (a1) and (b1) hold with $C_a = \nu$ and $C_b = \sqrt{d}$. Using the Friedrichs inequality, we get $a(\mathbf{u}, \mathbf{u}) = \nu\|\nabla\mathbf{u}\|_\Omega^2 \ge C_F\nu\|\mathbf{u}\|_{H^1(\Omega)}^2$ for all $\mathbf{u} \in H_0^1(\Omega)^d$. Thus kernel ellipticity (a2) holds with $\alpha = \nu C_F$. The remaining inf-sup stability condition (b2) is provided by

Lemma 6 (Necas/Nitsche/Bramble)**.** *There exists a constant $\beta = \beta(\Omega)$ such that*

$$\sup_{\mathbf{u}\in H_0^1(\Omega)^d} (\mathrm{div}\mathbf{u}, q)_\Omega / \|\mathbf{u}\|_{H^1(\Omega)} \ge \beta\|q\|_{L^2(\Omega)} \qquad \textit{for all } q \in L_0^2(\Omega). \tag{LBB}$$

A recent proof of this important result for Lipschitz domains can be found in [3]. By using the previous estimates and applying Brezzi's theorem, we obtain

Theorem 7. *The Stokes problem (12a)–(12c) has a unique solution $\mathbf{u} \in H_0^1(\Omega)^d$, $p \in L_0^2(\Omega)$, and $\|\mathbf{u}\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \le C\|\mathbf{f}\|_{L^2(\Omega)}$ with C only depending on Ω and ν.*

2.4.2. *Darcy flow.* The flow in the porous medium is governed by

$$\rho\mathbf{u} + \nabla p = \mathbf{f}, \qquad \text{in } \Omega, \tag{13a}$$

$$\text{div}\mathbf{u} = g, \qquad \text{in } \Omega, \tag{13b}$$

$$\mathbf{n} \cdot \mathbf{u} = 0, \qquad \text{on } \partial\Omega. \tag{13c}$$

This problem results from the general model discussed in Section 1 when $\Omega_S = \{\}$. The weak formulation again leads to a mixed variational problem of the form (9a)–(9b) with spaces $V = H_0(\text{div};\Omega) = \{\mathbf{v} \in H(\text{div};\Omega) : \mathbf{n}\cdot\mathbf{v} = 0 \quad \text{on } \partial\Omega\}$ and $Q = L^2_0(\Omega)$, with bilinear forms $a(\mathbf{u},\mathbf{v}) = \rho(\mathbf{u},\mathbf{v})_\Omega$ and $b(\mathbf{u},q) = -(\text{div}\mathbf{u},q)_\Omega$, and with linear functionals $l(\mathbf{v}) = 0$ and $j(q) = -(g,q)_\Omega$. Again, continuity of the linear a bilinear forms is easily established and (a1) and (b1) hold with $C_a = \rho$ and $C_b = 1$, respectively. Note that ther kernel of the constraint is given by $N(B) = \{\mathbf{u}_0 \in V : \text{div}\mathbf{u}_0 = 0\}$ and thus,

$$a(\mathbf{u}_0,\mathbf{u}_0) = \rho\|\mathbf{u}_0\|^2_\Omega = \rho(\|\mathbf{u}_0\|^2_\Omega + \|\text{div}\mathbf{u}_0\|^2_\Omega) = \rho\|\mathbf{u}_0\|_{H(\text{div};\Omega)}, \qquad \forall\mathbf{u}_0 \in N(B),$$

which shows kernel ellipticity (a2) with $\alpha = \rho$. Let us emphasize that a is uniformly elliptic only on the kernel $N(B)$ here, but not on the whole space $H_0(\text{div};\Omega)$! Since $H_0(\text{div};\Omega) \supset H^1_0(\Omega)^d$ with $\|\mathbf{u}\|_{H(\text{div};\Omega)} \le \sqrt{d}\|\mathbf{u}\|_{H^1(\Omega)}$, the inf-sup condition (b2) follows directly from Lemma 6. Application of Brezzi's theorem thus yields

Theorem 8. *The problem (13a)–(13c) has a unique solution $\mathbf{u} \in H_0(\text{div};\Omega)$, $p \in L^2_0(\Omega)$, and $\|\mathbf{u}\|_{H(\text{div};\Omega} + \|p\|_{L^2(\Omega)} \le C(\|\mathbf{f}\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)})$ with C depending only on Ω and ρ.*

2.4.3. *Coupled Darcy-Stokes flow.* Let us finally return to the full problem discussed in Section 1. This problem can be cast in the form (9a)–(9b) with inhomogeneous spaces V and Q defined in (8a) and (8b), with bilinear forms $a(\mathbf{u},\mathbf{v})$ and $b(\mathbf{u},q)$ given in (6c)–(6d) and linear forms $l(v)$ and $j(q)$ defined in (6e). It is easy to verify that the linear forms are continuous. Moreover,

$$a(\mathbf{u},\mathbf{v}) \le \nu\|\nabla\mathbf{u}_S\|_{\Omega_S}\|\nabla\mathbf{v}_S\|_{\Omega_S} + \rho\|\mathbf{u}_D\|_{\Omega_D}\|\mathbf{v}_D\|_{\Omega_D} + \gamma\rho^{1/2}\|\mathbf{n}\times\mathbf{u}_S\|_{\Gamma_{SD}}\|\mathbf{n}\times\mathbf{v}_S\|_{\Gamma_{SD}}.$$

The last term can be estimated via a trace inequality $\|\mathbf{u}_S\|_{\Gamma_{SD}} \le C_{tr}\|\mathbf{u}_S\|_{H^1(\Omega_S)}$. This yields the continuity condition (a1) for a with $C_a = C_a(\nu,\rho,\gamma;\Omega)$. The second continuity condition (b1) holds with $C_b = 1$ again. Because of the Friedrichs' inequality and the divergence free condition, we have

$$a(\mathbf{u}_0,\mathbf{u}_0) = \nu\|\nabla\mathbf{u}_0\|_{\Omega^2_S} + \gamma\rho^{1/2}\|\mathbf{n}_s\times(\mathbf{u}_0)_S\|^2_{\Gamma_{SD}} + \rho\|\mathbf{u}_0\|^2_{\Omega_D} \ge \alpha\|\mathbf{u}_0\|^2_V$$

for all $\mathbf{u}_0 \in N(B) = \{\mathbf{u} \in V : \text{div}\mathbf{u} = 0\}$ with some $\alpha > 0$ depending on ρ, ν, and the domain Ω, which yields the kernel ellipticity (a2). The inf-sup condition (b2) finally is guaranteed by Lemma 6. Theorem 1 now directly follows from Brezzi's theorem.

3. Finite element approximation for Stokes

The mixed variational framework (9a)–(9b) allows a systematic construction of numerical approximations. We first introduce the general concept, discuss the conditions required to ensure well-posedness of the numerical approximations, and then turn to the construction of particular approximations for the problems under investigation.

3.1. **Galerkin approximations.** Let $V_h \subset V$ and $Q_h \subset Q$ be finite dimensional subspaces of V and Q, equipped with the norms of V and Q, respectively. As numerical approximation for the problem (9a)–(9b), we consider the discrete variational problem

Problem 9. *Find $u_h \in V_h$ and $p_h \in Q_h$ such that*

$$a(u_h,v_h) + b(v_h,p_h) = l(v_h), \qquad \textit{for all } v_h \in V_h, \tag{14a}$$

$$b(u_h,q_h) = j(q_h), \qquad \textit{for all } q_h \in Q_h. \tag{14b}$$

This problem can again be written equivalently as a discrete operator equation

$$A_h u_h + B_h' p_h = l_h, \qquad \text{in } V_h', \tag{15a}$$

$$B_h u_h = j_h, \qquad \text{in } Q_h', \tag{15b}$$

where $A_h : V_h \to V_h'$, $B_h : V_h \to Q_h'$, and $B_h' : Q_h \to V_h'$ are restrictions of the operators A, B, and B' to the finite dimensional spaces V_h and Q_h, respectively. The kernel

$$N(B_h) = \{v_h \in V_h : B_h v_h = 0\} = \{v_h \in V_h : b(v_h, q_h) = 0, \ \forall q_h \in Q_h\} \tag{16}$$

of the discrete operator B_h will again be of special interest. By application of Brezzi's theorem to the discrete variational problem, we immediately obtain

Theorem 10. *Assume that (a1)–(a2) and the discrete stability conditions*

(a2h) $a(u_h, u_h) \geq \alpha_h \|u_h\|_V^2$ *for all* $u_h \in N(B_h)$,

(b2h) $\sup_{u_h \in V_h} b(u_h, q_h)/\|u_h\|_V \geq \beta_h \|q_h\|_Q$ *for all* $q_h \in Q_h$,

hold. Then Problem 9 admits a unique solution $u_h \in V_h$, $p_h \in Q_h$ *with*

$$\|u_h\|_V + \|p_h\|_Q \leq C_h(\|l\|_{V'} + \|j\|_{Q'})$$

with C_h *depending only on the constants* $C_a, C_b, \alpha_h, \beta_h$ *of (a1)–(b1) and (a2h)–(b2h).*

Note that the two discrete stability conditions (a2h) and (b2h) are not only sufficient, but if a is symmetric and non-negative, which is the case for the problems of Section 1, they are even necessary for the existence of a unique solution for the discretized problem!

3.2. **Transformation to a linear system of equations.** By choosing bases for the finite dimensional spacesV_h and Q_h, the discrete variational problem can be turned into a linear system of equations: Let $\{\phi_i\}$ and $\{\psi_j\}$ be linear independent systems such that $V_h = \text{span}\{\phi_i : i = 1, \ldots, N\}$ and $Q_h = \text{span}\{\psi_j : j = 1, \ldots, M\}$. Then the discrete functions can be expanded as $u_h = \sum_i \mathtt{u}_i \phi_i$ and $p_h = \sum_j \mathtt{p}_j \psi_j$, and the discrete variational problem (14a)–(14b) can be expressed equivalently as a linear system

$$\mathtt{Au} + \mathtt{B}^\top \mathtt{p} = \mathtt{l}, \qquad \text{in } \mathbb{R}^N, \tag{17a}$$

$$\mathtt{Bu} = \mathtt{j}, \qquad \text{in } \mathbb{R}^M, \tag{17b}$$

with $\mathtt{A}_{ij} = a(\phi_j, \phi_i)$, $\mathtt{B}_{ij} = b(\phi_j, \psi_i)$, $\mathtt{l}_i = l(\phi_i)$, and $\mathtt{j}_i = j(\psi_i)$. The condition (b2h) means that $\mathtt{B}$ has full rank, and in particular, that $M \leq N$, i.e., the number of constraints in (17b) is less than the number of degrees of freedom in $\mathtt{u}$. The conditions (a2h) and (b2h) together imply that $[\mathtt{A};\mathtt{B}]$ has full rank and that the system matrix $\mathtt{L} = [\mathtt{A}, \mathtt{B}^\top; \mathtt{B}, \mathtt{0}]$ is regular. This yields the well-posedness of the discrete problem.

3.3. **Error estimates.** Note that the Galerkin method (14a)–(14b) is already completely defined by the choice of the spaces V_h and Q_h. The following results states that the error of the numerical approximation does also depend only on the choice of the spaces.

Theorem 11. *Let (a1)–(b1) and (a2h)–(b2h) hold, and let* (u, p) *and* (u_h, p_h) *denote solutions of the continuous and discrete variational problems, respectively. Then*

$$\|u - u_h\|_V + \|p - p_h\|_Q \leq C_h(\inf_{v \in V_h} \|u - v_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q) \tag{1}$$

with C_h *depending only on the constants* $C_a, C_b, \alpha_h, \beta_h$ *of (a1)–(b1) and (a2h)–(b2h).*

For a proof, see [5] or [4]. The error of the Galerkin method is therefore as good as the best-approximation error, i.e., the Galerkin method is quasi-optimal. Note that, in general, the error in both components will depend on the best-approximation error in both components. For later reference, let us also mention the following specialized result.

Theorem 12. *In addition to (a2h) and (b2h), assume that $N(B_h) \subset N(B)$. Then*

$$\|u - u_h\|_V \leq C_h \inf_{v \in V_h} \|u - v_h\|_V. \tag{2}$$

For the problems under consideration, the condition $N(B_h) \subset N(B)$ means that the numerical approximation of the velocity is exactly divergence free. As we will see, this is valid only for very particular discretizations of flow problems. In particular, stable discretizations of the Stokes problem do often not lead to divergence free approximations.

3.4. **Finite element basics.** According to the results of Theorems 10 and 11, we should select finite dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$ that have

(i) good approximation properties, i.e., such that the best-approximation errors $\inf_{v_h \in V_h} \|u - v_h\|_V$ and $\inf_{q \in Q_h} \|p - q_h\|_Q$ become small;

(ii) good stability properties, i.e., such that (a2h) and (b2h) hold with $\alpha_h, \beta_h \geq c > 0$.

For variational problems stemming from partial differential equations, piecewise polynomials have been proven to be particularly useful for the approximation of solutions. In the following, we recall some basic results and notation about the piecewise polynomial function spaces that will be relevant.

3.4.1. *Mesh.* For ease of presentation, let us assume that $\Omega \subset \mathbb{R}^2$ is a bounded two-dimensional Lipschitz domain with polygonal boundary. We assume that $T_h = \{T\}$ is a partition of Ω into triangles T, i.e., a triangulation. The mesh T_h is called regular, if for two different triangles $T_1, T_2 \in T_h$, their intersection is either a vertex x of of both triangles, or an edge e of both triangles, or empty. We denote by h_T the diameter and by ρ_T the radius of the inner circle of the triangle T. The number $h = \max_T h_T$ is called the meshsize. A mesh T_h is called θ-shape-regular, if $\theta h \leq h_T \leq h$ for all $T \in T_h$ with some $\theta > 0$. We will assume that this is the case, throughout.

3.4.2. *Piecewise polynomials.* Let $P_k(T) = \mathrm{span}\{x^i y^j : 0 \leq i, j, i + j \leq k\}$ be the space of polynomials up to order k on T, and let $P_k(T_h) = \{v \in L^2(\Omega) : v|_T \in P_k(T)\ \forall T \in T_h\}$ be the space of piecewise polynomials over the mesh T_h. For the Galerkin approximation of the flow problems under investigation, we have to require that the approximating functions have some global regularity, i.e., to be elements of $H^1(\Omega)^d$ and $H(\mathrm{div};\Omega)$, respectively.

Lemma 13. *Let $v_h \in P_k(T_h)$. Then*

(i) $\mathbf{v}_h \in H^1(\Omega)^d$ *if, and only if,* $\mathbf{v}_h \subset C(\Omega)^d$, *i.e.,* $\mathbf{v}|_{T_1} = \mathbf{v}|_{T_2}$ *on every* $e = \partial T_1 \cap \partial T_2$.

(ii) $\mathbf{v}_h \in H(\mathrm{div};\Omega)$ *if, and only if,* $\mathbf{v}|_{T_1} \cdot \mathbf{n}_e = \mathbf{v}|_{T_2} \cdot \mathbf{n}_e$ *for all* $e = \partial T_1 \cap \partial T_2$.

Functions with enough global regularity will be called H^1- respectively $H(\mathrm{div})$-conforming. As a next step, let us characterize the approximation properties of the piecewise polynomial spaces introduced above.

Lemma 14 (Approximation)**.**
Let T_h be a θ-shape-regular triangulation. Then
(i) For $Q_h = P_0(T_h) \cap L^2_0(\Omega)$ there holds

$$\inf_{q_h \in Q_h} \|p - q_h\|_{L^2(\Omega)} \leq Ch\|p\|_{H^1(\Omega)} \qquad \text{for all } p \in H^1(\Omega) \cap L^2_0(\Omega).$$

(ii) For $V_h = P_1(T_h)^2 \cap H^1_0(\Omega)^2$ there holds

$$\inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)} \leq Ch\|\mathbf{u}\|_{H^2(\Omega)} \qquad \text{for all } \mathbf{u} \in H^2(\Omega)^2.$$

(iii) For $V_h = P_2(T_h)^2 \cap H_0^1(\Omega)^2$ *there holds*

$$\inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)} \le Ch^l \|\mathbf{u}\|_{H^2(\Omega)} \qquad \text{for all } \mathbf{u} \in H^{l+1}(\Omega)^2, \ l = 1, 2.$$

(vi) For $V_h = P_1(T_h)^d \cap H_0(\mathrm{div};\Omega)$ *there holds*

$$\inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_{H(div,\Omega)} \le Ch \|\mathbf{u}\|_{H^2(\Omega)} \qquad \text{for all } \mathbf{u} \in H^2(\Omega)^2.$$

The operators can be constructed explicitly by construction of certain interpolation operators. The approximation properties follow by a mapping argument and the Bramble-Hilbert lemma. On simple elements, such as triangles, the estimates can also be derived by averaged Taylor approximation.

4. Mixed finite elements methods for flow problems

We will now discuss simple examples of stable and unstable finite element schemes for the Stokes and the Darcy flow problems. As it will turn out, the most obvious choices of spaces sometimes do not work.

4.1. **Stokes problem.** We consider the Galerkin approximation of the variational problem (12a)–12c). Since $a(\mathbf{u}, \mathbf{v}) = \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega$ is elliptic on the whole space $V = H_0^1(\Omega)^d$, the discrete satbility condition (a2h) follows from (a2) on the continuous level. Therefore, only the discrete inf-sup condition

$$\sup_{u_h \in V_h} b(\mathbf{u}_h, q_h)/\|\mathbf{u}_h\|_V \ge \beta \|q_h\|_Q, \qquad \forall q_h \in Q_h, \tag{b2h}$$

has to be taken into account. Spaces V_h and Q_h satisfying such a condition are called an inf-sup stable finite element pair.

4.1.1. *The P1–P0 element.* The most simple non-trivial choice of finite element spaces for the Stokes problem would be

$$V_h = P_1(T_h)^d \cap H_0^1(\Omega)^d \qquad \text{and} \qquad Q_h = P_0(T_h) \cap L_0^2(\Omega),$$

which is typically refered to as the $P_1 - P_0$ element. By construction, these spaces are conforming, i.e., $V_h \subset V$ and $Q_h \subset Q$. For $\mathbf{u} \in H^2(\Omega)^d$, $p \in H^1(\Omega)$, which is a reasonable regularity assumption, the solution could be approximated with an error of $O(h)$; see Lemma 14. According ti Theorem 11, the same rate of convergence should be valid, if the approximation spaces V_h and Q_h satisfy the discrete inf-sup stability condition (b2h). As we will demonstrate now, the condition (b2h) can however not hold for this choice of spaces, except in trivial cases. To see this, let us recall the following relation between the number of inner vertices n_i, of vertices at the boundary n_b, and of elements n_T in a regular two-dimensional mesh, viz.

$$n_T = 2n_i + n_b - 2.$$

By counting the dimensions of the finite element spaces V_h and Q_h, we obtain

$$N = \dim V_h = 2n_i \quad \text{and} \quad M = \dim Q_h = n_t - 1 = 2n_i + n_b - 3.$$

Note that for any non-trivial triangular mesh $n_b > 3$, and thus we always have $M > N$. As a consequence, the matrix $\mathtt{B}$ in (17b) cannot have full row rank, i.e., the operator B_h in (15b) is not surjective, and B_h' is not injective. In particular, (b2h) does not hold.

Since the condition (b2h) is necessary for the well-posedness of the discrete variational problem (14a)–(14b), this finite element approximation doe not yield a well-posed discrete problem. Note that for the Stokes problem a is elliptic on the whole space V, and therefore the operator A_h in (15a) is always regular. Since B_h' is not injective, the pressure can however not be determined uniquely. The functions $p_h \in N(B_h')$ are called *spurious pressure modes.*

4.1.2. *The P2-P0 element.* In order to satisfy the discrete inf-sup condition (b2h), we have to increase the dimension of the discrete velocity space V_h sufficiently. The next simple possible choice of spaces is

$$V_h = P_2(T_h)^d \cap H_0^1(\Omega)^d \quad \text{and} \quad Q_h = P_0(T_h) \cap L_0^2(\Omega),$$

which is usually called the $P_2 - P_0$ element. For this discretization, one can show

Lemma 15. *The P2-P0 element is discrete inf-sup stable, i.e., the condition (b2h) holds with $\beta_h = c_\theta \beta > 0$, where β is the constant of the LBB condition, and c_θ is related only to the shape regularity of the mesh.*

For a proof of this result, let us refer to [15]. Using the a-priori estimate of Theorem 11 and the approximation error estimates (i) and (ii) of Lemma 14, we therefore obtain

Theorem 16. *Assume that the weak solution of (15a)–(15b) satisfies $u \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$. Then the finite element approximation $\mathbf{u}_h$, p_h obtained with the $P^2 - P_0$ element satisfies*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq Ch(\|\mathbf{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}),$$

and the constant C only depends on the Ω, ν, and the shape regularity of the mesh.

On a sequence of uniformly shape regular meshes, we therefore obtain an $O(h)$ convergence of the error. Note that the P_2 finite elemens for the velocities would in principle allow an $O(h^2)$ approximation of the velocity. Since functions $\text{div}\mathbf{v}_h$ are piecewise linear, the condition $B_h\mathbf{v}_h = 0$ does not imply that $\text{div}\mathbf{v}_h = 0$. The condition $N(B_h) \subset N(B)$ is therefore not valid for this discretization, and the $O(h)$ approximation of the pressure limits the overall accuracy.

4.2. **Discretization of the Darcy problem.** We will now discuss the stable discretization of the Darcy equations (16a)–(16c). Besides the discrete inf-sup condition (b2h), also the discrete kernel ellipticity

$$a(\mathbf{u}_h, \mathbf{u}_h) \geq \alpha\|\mathbf{u}_h\|_V^2, \qquad \text{for all } \mathbf{u}_h \in N(B_h) \tag{3}$$

will play a role now; compare with the continuous level. Note that by inceasing the space V_h in order to satisfy (b2h), also the space $N(B_h)$ may increase, so the condition (a2h) may be violated. This explains the failure of the first of our examples.

4.2.1. *Instability of the P2-P0 discretization.* As before, we set

$$V_h = P_2(T_h)^d \cap H_0^1(\Omega)^d \quad \text{and} \quad Q_h = P_0(T_h) \cap L_0^2(\Omega).$$

According to Lemma 15, the discrete inf-sup condition (b2h) is satisfied for this choice, and it remains to prove the discrete kernel ellipticity (a2h). As mentioned before, the condition $B_h\mathbf{u}_h = 0$ does not imply $\text{div}\mathbf{u}_h = 0$, and as a consequence, the P2-P0 element does not satisfy (a2h) with α_h independent of h. By an inverse inequality, one can show however, that (a2h) holds with $\alpha_h = O(h)$. Therefore, the discrete system (16a)–(16b) is uniquely solvable, but the convergence is spoiled by the lack of stability.

4.2.2. *A stable BDM1-P0 discretization.* The discrete kernel ellipticity condition (a2h) directly follows from the stability condition (a2) on the continuous level, if $N(B_h) \subset N(B)$ holds. This is always the case, if we choose $V_h \subset P_k(T_h)^d$ and $Q_h = P_{k-1}(T_h) \cap L^2_0(\Omega)$, since in that case, we have enough constraints. The simplest $H(\mathrm{div})$-conforming choice then reads

$$V_h = P_1(T_h)^d \cap H_0(\mathrm{div};\Omega) \quad \text{and} \quad Q_h = P_0(T_h) \cap L^2_0(\Omega).$$

The space V_h is called the BDM_1 element, after Brezzi, Douglas and Marini. The combination with the P_0 element for the pressure has the following stability property.

Lemma 17. *Let $B_h \mathbf{u}_h = 0$, then $\mathrm{div}\mathbf{u}_h = 0$, i.e., $N(B_h) \subset N(B)$. As a consequence, the discrete kernel ellipticity condition (a2h) with $\alpha_h = \alpha$ from condition (a2).*

As a last step, we now have to verify the discrete inf-sup stability condition (b2h) again. Note that in comparison to the $P_1 - P_0$ element, which was not stable, only the normal components of the velocities are required to be continuous here. Therefore, the BDM_1 velocity space is larger than the H^1-conforming P_1 space. The increase in dimension is in fact sufficient to obtain the inf-sup stability condition. To prove it, we use

Lemma 18 (Fortin operator)**.**
There exists a bounded linear operator $\Pi_h : H_0(\mathrm{div};\Omega) \to V_h$, such that

(i) $b(\Pi_h \mathbf{v}, q_h) = b(\mathbf{v}, q_h)$ *for all* $\mathbf{v} \in V = H_0(\mathrm{div};\Omega)$.

(ii) $\|\Pi_h \mathbf{v}\|_V \le C\|\mathbf{v}\|_V$ *for all* $\mathbf{v} \in V$.

The proof is based on an explicit construction; see [6, 15] for details. Using the Fortin operator, we obtain

Lemma 19. *The BDM_1-P_0 element satisfies the inf-sup condition (b2h) with $\beta_h = \beta/C$.*

Proof. Let $q_h \in P_0(T_h) \cap L^2_0(\Omega)$ be given. By Lemma 6, there exists $\mathbf{u} \in H^1_0(\Omega)^2$ with $b(\mathbf{u}, q_h)/\|\mathbf{u}\|_{H^1(\Omega)} \ge \beta\|q_h\|_{L^2(\Omega)}$. Choosing $\mathbf{u}_h = \Pi_h \mathbf{v}$, we have

$$b(\mathbf{u}_h, q_h)/\|\mathbf{u}_h\|_{H^1(\Omega)} \ge b(\mathbf{u}, q_h)/(C\|\mathbf{u}\|_{H^1(\Omega)}) \ge \beta/C\|q_h\|_{L^2(\Omega)},$$

which already yields the proof of the assertion. Note that both conditions of the Fortin operator were used in the first inequality. □

By combination of the two stability estimates, the a-priori error estimate of Theorem 11, and the approximation error results (i) and (iv) of Lemma 14, we obtain

Theorem 20. *Let the solution $\mathbf{u}$, p of the Darcy problem (17a)–(17b) be regular, i.e., $\mathbf{u} \in H^2(\Omega)^d$ and $p \in H^1(\Omega)$. Then the BDM_1-P_0 approximation $\mathbf{u}_h$, p_h satisfies*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{div};\Omega)} + \|p - p_h\|_{L^2(\Omega)} \le Ch(\|\mathbf{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}),$$

and the constant C only depends on ρ, Ω, and the shape regularity of the mesh.

The BDM$_1$-P_0 element thus yields an order optimal approximation of $O(h)$ again.

4.3. **Discretization of the Darcy-Stokes problem.** For the discretization of the coupled Darcy-Stokes problem, (6a)–(6d), we then consider the following finite element spaces

$$\begin{aligned} V_h = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v}|_{\Omega_S} \in H^1(\Omega_S)^d,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_S,\ \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \Omega_D, \\ \text{such that} \qquad \mathbf{v}|_{\Omega_S} \in P_2(T_h \cap \Omega_S),\ \mathbf{v}|_{\Omega_D} \in P_1(T_h \cap \Omega_D)\}, \\ Q_h = P_0(T_h) \cap L^2_0(\Omega). \end{aligned}$$

A combination of the stability estimates for the Stokes and the Darcy problem yields

Lemma 21. *The above spaces satisfy the discrete kernel ellipticity condition (a2h) and the discrete inf-sup condition (b2h) with* $\alpha_h = c\alpha$ *and* $\beta_h = c\beta$*, where* α *and* β *are the stability constants of the continuous problem, and* c *only depends, on the domain* Ω *and the shape-regularity of the mesh.*

A combination of the a-priori error estimates of Theorem 11 and approximation error estimates similar to those of Lemma 14 then yields

Theorem 22. *Assume that the solution* $\mathbf{u}$*,* p *of the Darcy-Stokes problem (1a)–(3c) is sufficiently smooth. Then the approximation* $\mathbf{u}_h$*,* p_h *for the above spaces satisfies*

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_h\|_{H(\mathrm{div};\Omega)} + \|\mathbf{u} - \mathbf{u}_h\|_{H^1(\Omega_S)} + \|p - p_h\|_{L^2(\Omega)} \\ \leq Ch(\|\mathbf{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}).\end{aligned}$$

The constant C *only depends on the* ρ*,* ν*,* Ω_S*,* Ω_D*, and the shape-regularity of the mesh.*

Let us note that the regularity conditions for the solution can be relaxed to some extent. In particular, continuity of the pressure and the tangential velocities is not required across the interface Γ_{SD}. Also the regularity requirement on the velocities in the Darcy domain could be relaxed.

Further reading

The coupling conditions for Darcy-Stokes problem has been investigated experimentally by Beavers and Joseph [2], and later be simplified and explained theoretically by Saffmann [23]. A rigorous justification based on homogenization has been given in [18]. For modelling issues, see also [1, 8, 9, 20, 17]. Some more practical applications are discussed in [16]. The analysis of saddlepoint problems goeas back to the seminal paper [5], which initiated the research on mixed variational problems. For an extensive treatment, see [6]. Background material on finite element methods, including the theory of saddlepoint problems, can be found [4, 15]. The mixed formulation for the Poisson problem goes back to [21]. Applications in fluid dynamics are discuss in particular in [15], where also a proof of the (LBB) condition can be found; but see also [3] and [12] for other analytical results in the context of flow problems. The coupling of Darcy and Stokes flow in finite element methods has been investigated in [13, 14, 22]. Time dependent problems are investigated, e.g., in [7]. Other aspects of the coupling are treated in [10]. Monolithic discretizations, that use the same discretization spaces for the Darcy and Stokes subproblems can be found in [11, 19]. In [24], a related method is studied for discretization of the Stokes problem.

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