Large time behavior of solutions to the Navier-Stokes system in unbounded domains

Grzegorz Karch
Wroclaw, POLAND

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LECTURE 1

Scaling method in simplest equations from the fluid dynamic

The Cauchy problem for the heat equation

References
M.-H. Giga, Y. Giga, and J. Saal,
Nonlinear partial differential equations, Asymptotic behavior of solutions and self-similar solutions.
Progress in Nonlinear Differential Equations and their Applications, 79, Birkhuauser Boston Inc., Boston, MA, 2010.

The Cauchy problem for the heat equation

First, we consider the *Cauchy problem* for the heat equation

$$u_t(x,t) = \Delta u(x,t) \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0 \tag{1}$$
$$u(x,0) = u_0(x). \tag{2}$$

A solution of the initial value problem (1)-(2) is represented by

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)u_0(y)\,dy \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

with the *Gauss-Weierstrass kernel*

$$G(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

In these lectures,

for $x \in \mathbb{R}^n$, we have always denoted $|x| = \sqrt{x_1^2 + ... + x_n^2}$.

The Cauchy problem for the heat equation

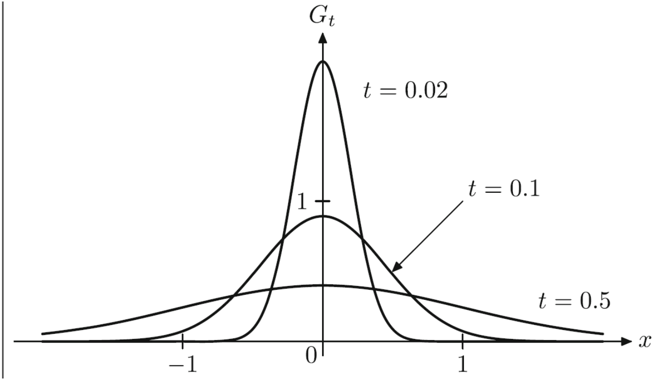

Figure : A few examples of the graph of $G(x, t)$ as a function of x for $n = 1$

The Cauchy problem for the heat equation

The following theorem gathers typical properties of the solution $u = u(x,t)$.

Theorem
Assume that $u_0 \in L^1(\mathbb{R}^n)$.
Then

1. $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$,
2. u *satisfies equation* (1) *for all* $x \in \mathbb{R}^n$ *and* $t > 0$,
3. $\|u(\cdot,t) - u_0(\cdot)\|_1 \to 0$ *when* $t \to 0$,
4. $\|u(\cdot,t)\|_1 \leqslant \|u_0(\cdot)\|_1$ *for all* $t \geqslant 0$.

This is the unique solution of problem (1)-(2) *satisfying these properties.*

The proof of this theorem can be found in the Evans book.

Decay estimate of solutions

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)u_0(y)\,dy \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

with the *Gauss-Weierstrass kernel*

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DECAY ESTIMATES

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leqslant \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |u_0(x)|\,dx.$$

One can also prove the following decay estimates of other L^p-norms.

$$\|u(\cdot,t)\|_p \leqslant C(p,q) t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_q$$

for all $t > 0$ and every $1 \leqslant q \leqslant p \leqslant \infty$.

Self-similar asymptotics of solutions

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)u_0(y)\,dy \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

Theorem

Let u be the solution of the heat equation with initial datum $u_0 \in L^1(\mathbb{R}^n)$. Let $M = \int_{\mathbb{R}^n} u_0(y)\,dy$. Then

$$\lim_{t\to\infty} t^{(n/2)(1-1/p)} \|u(\cdot,t) - MG(\cdot,t)\|_p = 0,$$

where $G(x,t)$ is the Gauss-Weierstrass kernel.

Scaling and self-similar solutions

Let us notice that the heat equation

$$u_t = \Delta u, \qquad x \in \mathbb{R}^n, t > 0$$

has the following property:

If $u = u(x,t)$ is a solution of this equation, then the function

$$u_\lambda(x,t) \equiv \lambda^k u(\lambda x, \lambda^2 t)$$

is a solution for each $\lambda > 0$. Here, $k \in \mathbb{R}$ is a fixed parameter.

Definition

A solution $u = u(x,t)$ is called **self-similar** if there exists $k \in \mathbb{R}$ such that

$$\lambda^k u(\lambda x, \lambda^2 t) = u(x,t)$$

for all $x \in \mathbb{R}^n$, $t > 0$, and $\lambda > 0$.

Scaling and self-similar solutions

The heat kernel (also called the Gauss-Weierstrass kernel) is a self-similar solution with $k = n$ of the heat equation. Indeed, it is easy to see that

$$\begin{aligned}
\lambda^n G(\lambda x, \lambda^2 t) &= \lambda^n \frac{1}{(4\pi\lambda^2 t)^{n/2}} \exp\left(-\frac{|\lambda x|^2}{4\lambda^2 t}\right) \\
&= \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) = G(x, t).
\end{aligned}$$

Scaling and self-similar solutions

Theorem

Denote

$$u_\lambda(x,t) = \lambda^n u(\lambda x, \lambda^2 t).$$

Fix $p \in [1,\infty]$. The following two conditions are equivalent

1. $\lim_{t\to\infty} t^{(n/2)(1-1/p)} \|u(\cdot,t) - MG(\cdot,t)\|_p = 0$
2. *for every $t_0 > 0$,*

$$u_\lambda(\cdot,t_0) \to MG(\cdot,t_0), \quad \text{as} \quad \lambda \to \infty,$$

where the convergence is in the usual norm of $L^p(\mathbb{R}^n)$.

Scaling and self-similar solutions

Proof.
Scaling property of the L^p-norm:

$$\|v(\lambda\cdot)\|_p = \lambda^{-n/p}\|v\|_p,$$

Now, using this scaling property we obtain

$$\begin{aligned}
\|u_\lambda(\cdot,t_0) - MG(\cdot,t_0)\|_p &= \|\lambda^n u(\lambda\cdot,\lambda^2 t_0) - M\lambda^n G(\lambda\cdot,\lambda^2 t_0)\|_p \\
&= \lambda^{n-n/p}\|u(\cdot,\lambda^2 t_0) - MG(\cdot,\lambda^2 t_0)\|_p \quad (\text{substituting} \quad \lambda = \sqrt{t/t_0}) \\
&= C(t_0)t^{(n/2)(1-1/p)}\|u(\cdot,t) - MG(\cdot,t)\|_p.
\end{aligned}$$

□

NEW APPROACH: Four steps method

Step 1. Scaling. We introduce the rescaled family of functions

$$u_\lambda(x,t) = \lambda^n u(\lambda x, \lambda^2 t) \qquad \text{for every} \quad \lambda > 0.$$

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Step 2. Estimates and compactness. We show that the embedding

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Step 3. Passage to the limit. By compactness there exists a sequence $\lambda_n \to \infty$ and a functions $\bar{u}(x,t)$ such that

$$u_{\lambda_n}(\cdot,t) \to \bar{u}(\cdot,t) \quad \text{in } L^p(\mathbb{R}^n) \text{ for every } t > 0.$$

Since u_λ satisfies the heat equation, one can show that $\bar{u}$ is a weak solution of the heat equation, as well.

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Since u_λ satisfies the heat equation, one can show that $\bar{u}$ is a weak solution of the heat equation, as well.

Step 4. Identification of the limit. The limit function $\bar{u}$ corresponds usually to singular initial conditions.

Initial condition

Lemma

Let $u_0 \in L^1(\mathbb{R}^n)$. For every test function $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x)\varphi(x)\, dx \to M\varphi(0) \qquad \text{as } \lambda \to \infty,$$

where $M = \int_{\mathbb{R}^n} u_0(x)\, dx$.

Initial condition

Lemma

Let $u_0 \in L^1(\mathbb{R}^n)$*. For every test function* $\varphi \in C_c^\infty(\mathbb{R}^n)$ *we have*

$$\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x)\varphi(x)\, dx \to M\varphi(0) \qquad \text{as } \lambda \to \infty,$$

where $M = \int_{\mathbb{R}^n} u_0(x)\, dx$*.*

Proof.

This is an immediate consequence of the Lebesgue dominated convergence theorem, because

$$\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x)\varphi(x)\, dx = \int_{\mathbb{R}^n} u_0(x)\varphi(x/\lambda)\, dx,$$

by a simple change of variables. □

Application to convection-diffusion equation

Self-similar asymptotics of solutions to convection-diffusion equation

We are going to show the *Four Step Method* “in action”, by applying it to the initial value problem for the nonlinear convection diffusion equation

$$u_t - u_{xx} + (u^q)_x = 0 \quad \text{for } x \in \mathbb{R},\ t > 0, \tag{3}$$

$$u(x,0) = u_0(x), \tag{4}$$

where $q > 1$ is a fixed parameter.

Theorem (Existence of global-in-time solution)

Assume that

$$u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R}).$$

Suppose $u_0 \geqslant 0$*.*

Then the initial value problem (3)–(4) *has a nonnegative, global-in-time solution* $u \in C^{2,1}(\mathbb{R} \times (0,\infty))$*.*

This solution satisfies

$$u \in C^1((0,\infty), L^p(\mathbb{R})) \cap C((0,\infty), W^{2,p}(\mathbb{R}))$$

for each $p \in [1,\infty]$*. Moreover,*

$$M \equiv \|u(t)\|_1 = \int_{\mathbb{R}} u(x,t)\, dx = \int_{\mathbb{R}} u_0(x)\, dx = \|u_0\|_1 \quad \textit{for all} \quad t \geqslant 0.$$

Local existence via the Banach contraction principle

Local-in-time *mild* solutions:

$$u(t) = G(\cdot, t) * u_0 + \int_0^t \partial_x G(\cdot, t-s) * u^q(s)\, ds$$

with the heat kernel $G(x,t) = (4\pi t)^{-1/2} \exp\big(-|x|^2/(4t)\big)$.
By the Young inequality for the convolution:

$$\|G(\cdot,t) * f\|_p \le Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|f\|_q, \tag{5}$$

$$\|\partial_x G(\cdot,t) * f\|_p \le Ct^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \|f\|_q \tag{6}$$

for every $1 \le q \le p \le \infty$, each $f \in L^q(\mathbb{R})$, and $C = C(p,q)$ independent of t, f.

Notice that $C = 1$ in inequality (5) for $p = q$ because $\|G(\cdot,t)\|_{L^1} = 1$ for all $t > 0$.

Lemma (Local existence)

Assume that

$$u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R}).$$

Then there exists $T = T(\|u_0\|_1, \|u_0\|_\infty) > 0$ *such that the integral equation* (18) *has the unique solution in the space*

$$\mathcal{Y}_T = C([0,T], L^1(\mathbb{R})) \cap C([0,T], L^\infty(\mathbb{R})),$$

supplemented with the norm $\|u\|_{\mathcal{Y}_T} = \sup_{0\le t\le T} \|u\|_1 + \sup_{0\le t\le T} \|u\|_\infty$.

Proof.

The Banach contraction principle.
For sufficiently small T the right hand side of this equation defines the contraction in the space $\mathcal{Y}_T$. □

Regularity and comparison principle

One can show that a local in time solution is nonnegative, if an initial condition is so. Moreover, this solution has the following regularity property

$$u \in C^1((0,\infty), L^p(\mathbb{R})) \cap C((0,\infty), W^{2,p}(\mathbb{R})))$$

for each $p \in [1,\infty]$.

Below, we show that the solution satisfies the following a priori estimates for each $p \in [1,\infty]$:

$$\|u(\cdot,t)\|_p \leqslant \|u_0\|_p \quad \text{for all} \quad t > 0.$$

Hence, by a standard reasoning, we can show that the solution is global in time.

Self-similar large time behavior of solutions

Theorem (LINEAR Self-similar asymptotics)

Let

$$q > 2.$$

Every solution $u = u(x,t)$ *of problem* (3)–(4) *satisfies*

$$t^{(1-1/p)/2} \|u(t) - MG(t)\|_p \to 0 \quad \text{as } t \to \infty$$

for every $p \in [1,\infty]$, *where* $M = \int_{\mathbb{R}} u_0(x)\,dx$ *and*

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{|x|^2}{4t}\Big)$$

is the heat kernel.

Self-similar large time behavior of solutions

Theorem (NONLINEAR Self-similar asymptotics)

If

$$q = 2,$$

we have

$$t^{(1-1/p)/2} \|u(t) - \mathcal{U}_M(t)\|_p \to 0 \quad \text{as } t \to \infty$$

for every $p \in [1, \infty]$, *where*

$$\mathcal{U}_M(x,t) = \frac{1}{\sqrt{t}} \mathcal{U}_M\left(\frac{x}{\sqrt{t}}, 1\right)$$

is the so-called nonlinear diffusion wave:

$$U_t - U_{xx} + \left(U^2\right)_x = 0, \quad \text{for } x \in \mathbb{R}, \ t > 0, \tag{7}$$

$$U(x,0) = M\delta_0, \tag{8}$$

where δ_0 *is the Dirac measure.*

Remark on nonlinear diffusion waves

The Hopf-Cole transformation allows us to solve this problem:

$$\mathcal{U}_M(x,t) = \frac{t^{-1/2}\exp\left(-|x|^2/(4t)\right)}{C_M + \frac{1}{2}\int\limits_0^{x/\sqrt{t}} \exp\left(-\xi^2/4\right)d\xi},$$

where C_M is a constant which is determined uniquely as a function of M by the condition $\int\limits_{\mathbb{R}} \mathcal{U}_{M,A}(\eta,1)\,d\eta = M$.

For every $M \in \mathbb{R}$ the function $\mathcal{U}_M$ is a unique solution of the Burgers equation in the space $C((0,\infty); L^1(\mathbb{R}))$ having the properties

$$\int\limits_{\mathbb{R}} \mathcal{U}_M(x,t)\,dx = M \quad \text{for all} \quad t > 0$$

and

$$\int\limits_{\mathbb{R}} \mathcal{U}_M(x,t)\varphi(x)\,dx \to M\varphi(0) \quad \text{as} \quad t \to 0$$

for all $\varphi \in C_c^\infty(\mathbb{R})$.

Idea of the proof.
Rescaled family of functions

We study the behavior, as $\lambda \to \infty$, of the rescaled family of functions

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \quad \text{for every} \quad \lambda > 0,$$

which satisfy

$$\partial_t u_\lambda - \partial_x^2 u_\lambda + \lambda^{2-q} \partial_x u_\lambda^q = 0,$$

$$u_\lambda(x,0) = u_{0,\lambda}(x) = \lambda u_0(\lambda x).$$

Notice that, by a simple change of variables, we have the following identity

$$\|u_\lambda(t)\|_1 = \|u_0\|_1$$

holds true for all $t > 0$ and all $\lambda > 0$.

Optimal L^p-decay of solutions

Theorem
Under the assumptions of Theorem 6, the solution of problem (3)–(4) *satisfies*

$$\|u(\cdot,t)\|_p \leqslant Ct^{-(1-1/p)/2}\|u_0\|_1$$

for each $p \in [1,\infty]$, *a constant* $C = C(p)$ *and all* $t > 0$.

We sketch the proof for $p = 2$, only.
Multiplying the equation by u and integrating the resulting equation over $\mathbb{R}$ we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2\,dx = -\int_{\mathbb{R}} |u_x|^2\,dx.$$

Here, we have used an elementary equalities

$$\int_{\mathbb{R}} u_x^q(x,t)u(x,t)\,dx = \frac{1}{q+1}\int_{\mathbb{R}}(u^{q+1}(x,t))_x\,dx = 0$$

if $u(x,t) \to 0$ when $|x| \to \infty$.

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2\,dx = -\int_{\mathbb{R}} |u_x|^2\,dx$$

Now, by the Nash inequality

$$\|u\|_2 \le C\|u_x\|_2^{1/3}\|u\|_1^{2/3},$$

which is valid for all $u \in L^1(\mathbb{R})$ such that $u_x \in L^2(\mathbb{R})$, (since the L^1-norm of the solution is constant in time) we obtain the differential inequality

$$\frac{d}{dt}\|u(t)\|_2^2 + C\|u_0\|_1^{-4}\left(\|u(t)\|_2^2\right)^3 \le 0,$$

which implies

$$\|u(t)\|_2 \le Ct^{-1/4}$$

for all $t > 0$ and $C > 0$ independent of t.

Estimates of the rescaled family of solutions

Lemma

For each $p \in [1, \infty]$ there exists $C = C(\|K'\|_1, \|u_0\|_1) > 0$, independent of t and of λ, such that

$$\|u_\lambda(t)\|_p \leq C t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$ and all $\lambda > 0$.

Proof.

By the change of variables and the decay estimate we obtain

$$\|u_\lambda(t)\|_p = \lambda^{1-\frac{1}{p}} \|u(\cdot, \lambda^2 t)\|_p \leq C \lambda^{1-\frac{1}{p}} \left(\lambda^2 t\right)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} = C t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}.$$

□

Estimates of the rescaled family of solutions

Lemma

For each $p \in [1, \infty)$ there exists $C = C(p, \|K'\|_1, \|u_0\|_1) > 0$, independent of t and of λ, such that

$$\|\partial_x u_\lambda(t)\|_p \leq C t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}$$

for all $t > 0$ and all $\lambda > 0$.

Identical estimates hold true for u_λ.

Aubin-Lions-Simon's compactness result

Theorem

Let X, B and Y be Banach spaces satisfying

$$X \subset B \subset Y$$

with **compact** *embedding $X \subset B$ and* **continuous** *embedding $B \subset Y$.*
Assume, for $1 \leq p \leq \infty$ and $T > 0$, that

- *F is bounded in $L^p(0, T; X)$,*
- *$\{\partial_t f \,:\, f \in F\}$ is bounded in $L^p(0, T; Y)$.*

Then F is relatively compact in $L^p(0, T; B)$
and in $C(0, T; B)$ if $p = \infty$.

Compactness in $L^1_{loc}(\mathbb{R})$

Lemma

For every $0 < t_1 < t_2 < \infty$ *and every* $R > 0$*, the set*

$$\{u_\lambda\}_{\lambda>0} \subseteq C([t_1, t_2], L^1([-R, R]))$$

is relatively compact.

Proof.

We apply Theorem with $p = \infty$, $F = \{u_\lambda\}_{\lambda>0}$, and

$$X = W^{1,1}([-R, R]), \qquad B = L^1([-R, R]), \qquad Y = W^{-1,1}([-R, R]),$$

where $R > 0$ is fixed and arbitrary, and Y is the dual space of $W^{1,1}_0([-R, R])$.

Obviously, the embedding $X \subseteq B$ is compact by the Rellich-Kondrashov theorem. □

Compactness in $L^1(\mathbb{R})$

Lemma

For every $0 < t_1 < t_2 < \infty$*, the set*

$$\{u_\lambda\}_{\lambda>0} \subseteq C([t_1, t_2], L^1(\mathbb{R}))$$

is relatively compact.

Proof.

Let $\psi \in C^\infty(\mathbb{R})$ be nonnegative and satisfy $\psi(x) = 0$ for $|x| < 1$ and $\psi(x) = 1$ for $|x| > 2$.
Put $\psi_R(x) = \psi(x/R)$ for every $R > 0$. It suffices to show that

$$\sup_{t\in[t_1,t_2]} \|u_\lambda(t)\psi_R\|_1 \to 0 \quad \text{as} \quad R \to \infty, \quad \text{uniformly in} \quad \lambda \geq 1.$$

□

Initial condition

Lemma

For every test function $\phi \in C_c^\infty(\mathbb{R})$*, there exists* $C = C(\phi, \|K'\|_1, \|u_0\|_1)$ *independent of* λ *such that*

$$\left| \int_{\mathbb{R}} u_\lambda(x,t)\phi(x)\,dx - \int_{\mathbb{R}} u_{0,\lambda}(x)\phi(x)\,dx \right| \leq C\left(t + t^{1/2}\right). \qquad (9)$$

Proof of the main result

By compactness, there exists a subsequence of $\{u_\lambda\}_{\lambda>0}$ (not relabeled) and a function $\bar{u} \in C((0,\infty), L^1(\mathbb{R}))$ such that

$$u_\lambda \to \bar{u} \quad \text{in } C([t_1,t_2], L^1(\mathbb{R})) \quad \text{as } \lambda \to \infty.$$

Passing to a subsequence, we can assume that

$$u_\lambda(x,t) \to \bar{u}(x,t) \quad \text{as } \lambda \to \infty$$

almost everywhere in $(x,t) \in \mathbb{R} \times (0,\infty)$.
Now, multiplying the equation by a test function $\phi \in C_c^\infty(\mathbb{R} \times (0,\infty))$ and integrating over $\mathbb{R} \times (0,\infty)$, we obtain

$$-\int_{\mathbb{R}} \int_0^\infty u_\lambda \phi_t \, dsdx = \int_{\mathbb{R}} \int_0^\infty u_\lambda \phi_{xx} \, dsdx + \lambda^{2-q} \int_{\mathbb{R}} \int_0^\infty u_\lambda^q \phi_x \, dsdx.$$

We obtain that $\bar{u}(x,t)$ is a weak solution of the equation

$$\bar{u}_t = \bar{u}_{xx} - (\bar{u}^2)_x$$

if $q = 2$.
Initial conditions:

$$\int_{\mathbb{R}} u_{0,\lambda}(x)\phi(x)\,dx = \int_{\mathbb{R}} u_0(x)\phi(x/\lambda)\,dx \to M\phi(0)$$

as $\lambda \to \infty$. Hence,

$$\bar{u}(x,0) = M\delta_0.$$

Thus, $\bar{u}$ is a weak solution of the initial value problem

$$\bar{u}_t = \bar{u}_{xx} - (\bar{u}^2)_x, \tag{10}$$

$$\bar{u}(x,0) = M\delta_0. \tag{11}$$

Since problem (10)-(11) has a unique solution, we obtain that $\bar{u} = \mathcal{U}_M$.

Obviously, if $q > 2$, this limit function is a solution to the linear problem

$$\bar{u}_t = \bar{u}_{xx}, \tag{12}$$

$$\bar{u}(x,0) = M\delta_0. \tag{13}$$

So, it is the multiple of Gauss-Weierstrass kernel

$$\bar{u}(x,t) = MG(x,t) = M\frac{1}{\sqrt{4\pi t}}\exp\big(-\frac{|x|^2}{4t}\big).$$

Hence, we have

$$\lim_{\lambda\to\infty}\|u_\lambda(1) - \bar{u}(1)\|_1 = 0$$

and, after setting $\lambda = \sqrt{t}$ and using the self-similar form of $\bar{u}(x,t) = t^{-1/2}\bar{u}(xt^{-1/2},1)$, we obtain

$$\lim_{t\to\infty}\|u(t) - \bar{u}(t)\|_1 = 0.$$

The convergence of $u(\cdot, t)$ in the L^p-norms for $p \in (1, \infty)$:

$$\|u(t)-\bar{u}(t)\|_p \leqslant \big(\|u(t)\|_\infty+\|\bar{u}(t)\|_\infty\big)^{1-1/p}\|u(t)-\bar{u}(t)\|_1^{1/p} = o(t^{-(1-1/p)/2})$$

as $t \to \infty$.

The convergence in the L^∞-norm. Here, by the Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$\|u(t) - \bar{u}(t)\|_\infty \leqslant C\big(\|u_x(t)\|_2 + \|\bar{u}_x(t)\|_2\big)^{1/2}\|u(t) - \bar{u}(t)\|_2^{1/2} = o(t^{-1/2})$$

as $t \to \infty$.

Summary on $u_t - u_{xx} + (u^q)_x = 0$.

Suppose that $u_0 \in L^1(\mathbb{R})$. Put

$$M = \int_{\mathbb{R}} u_0(x)\, dx.$$

Then

- *Case I:* linear asymptotics

  For $\boxed{q > 2}$ and for every $p \in [1, \infty]$

  $$t^{\frac{1}{2}\left(1-\frac{1}{p}\right)} \| u(\cdot, t) - MG(\cdot, t) \|_{L^p(\mathbb{R})} \to 0$$

  as $t \to \infty$, where

  $$G(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

  is the Gauss-Weierstrass kernel.

- *Case II:* balance case

  For $\boxed{q = 2}$, the large time asymptotics of solutions is described as $t \to \infty$ by a self-similar solution of the convection-diffusion equation with the initial datum

  $$u_0(x) = M\delta_0.$$

Hyperbolic asymptotics

$$u_t - u_{xx} + (u^q)_x = 0$$

Case III: For $\boxed{1 < q < 2}$, the asymptotics is given by a self-similar solution of the convection equation

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with the initial datum

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One should use the following different scaling

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^q t).$$

and the rescaled equation

$$\partial u_\lambda - \lambda^{q-2}\partial_x u_\lambda + \partial(u_\lambda)^q = 0.$$

Zero mass initial conditions

The main assumption

$$u_0 \in L^1(\mathbb{R}, (1 + |x|)\, dx), \quad u_0 \not\equiv 0$$

and

$$\int_{\mathbb{R}} u_0(x)\, dx = 0.$$

Define

$$U_0(x) = \int_{-\infty}^{x} u_0(y)\, dy = -\int_{x}^{\infty} u_0(y)\, dy.$$

Diffusion-dominated case

Assume that one of the following assertions hold true:

(i) $U_0 \geq 0$ and $q > 3/2$,

(ii) $U_0 \leq 0$ and $q \geqslant 2$,

(iii) $U_0 \leq 0$, $q \in (3/2, 2)$ the quantity

$$\left| \int_{\mathbb{R}} x u_0(x)\, dx \right| \|u_0\|_{L^\infty(\mathbb{R})}^{2q-3}$$

is sufficiently small.

Then

$$t^{\frac{1}{2}\left(1-\frac{1}{p}\right)+\frac{1}{2}} \left\| u(\cdot,t) - I_\infty \partial_x G(\cdot,t) \right\|_{L^p(\mathbb{R})} \to 0,$$

as $t \to \infty$, where

$$\begin{aligned} I_\infty &\equiv -\lim_{t\to\infty} \int_{\mathbb{R}} x u(x,t)\, dx \\ &= -\int_{\mathbb{R}} x u_0(x)\, dx - \int_0^\infty \int_{\mathbb{R}} |u(x,s)|^q\, dx\, ds. \end{aligned}$$

Other results

- the convergence towards very singular solutions (special self-similar solutions of the convection-diffusion equation)
- the convergence towards hyperbolic waves

Conclusion:
The large time asymptotics of zero mass solutions depends not only on the exponent of the nonlinearity $q > 1$ but also on the

size, sign, and shape

of the initial datum.