Large time behavior of solutions to the Navier-Stokes system in unbounded domains

Grzegorz Karch
Wroclaw, POLAND

第11回 日独流体数学国際研究集会
LECTURE 1
Scaling method in simplest equations from the fluid dynamic
The Cauchy problem for the heat equation

References
M.-H. Giga, Y. Giga, and J. Saal,

*Nonlinear partial differential equations, Asymptotic behavior of solutions and self-similar solutions.*

The Cauchy problem for the heat equation

First, we consider the Cauchy problem for the heat equation

\[ u_t(x, t) = \Delta u(x, t) \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0 \]  
\[ u(x, 0) = u_0(x). \]  

A solution of the initial value problem (1)-(2) is represented by

\[ u(x, t) = \int_{\mathbb{R}^n} G(x - y, t) u_0(y) \, dy \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0, \]

with the Gauss-Weierstrass kernel

\[ G(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{4t} \right) \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0. \]

In these lectures,

for \( x \in \mathbb{R}^n \), we have always denoted \( |x| = \sqrt{x_1^2 + \ldots + x_n^2} \).
The Cauchy problem for the heat equation

Figure: A few examples of the graph of $G(x, t)$ as a function of $x$ for $n = 1$
The Cauchy problem for the heat equation

The following theorem gathers typical properties of the solution $u = u(x, t)$.

**Theorem**
Assume that $u_0 \in L^1(\mathbb{R}^n)$.
Then

1. $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$,  
2. $u$ satisfies equation (1) for all $x \in \mathbb{R}^n$ and $t > 0$,  
3. $\|u(\cdot, t) - u_0(\cdot)\|_1 \to 0$ when $t \to 0$,  
4. $\|u(\cdot, t)\|_1 \leq \|u_0(\cdot)\|_1$ for all $t \geq 0$.

This is the unique solution of problem (1)-(2) satisfying these properties.

The proof of this theorem can be found in the Evans book.
Decay estimate of solutions

\[ u(x, t) = \int_{\mathbb{R}^n} G(x - y, t)u_0(y) \, dy \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0, \]

with the Gauss-Weierstrass kernel

\[ G(x, t) = \frac{1}{(4 \pi t)^{n/2}} \exp \left( -\frac{|x|^2}{4t} \right) \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0. \]

**DECAY ESTIMATES**

\[ \sup_{x \in \mathbb{R}^n} |u(x, t)| \leq \frac{1}{(4 \pi t)^{n/2}} \int_{\mathbb{R}^n} |u_0(x)| \, dx. \]

One can also prove the following decay estimates of other \( L^p \)-norms.

\[ \|u(\cdot, t)\|_p \leq C(p, q)t^{-\frac{n}{2}}(\frac{1}{q} - \frac{1}{p})\|u_0\|_q \]

for all \( t > 0 \) and every \( 1 \leq q \leq p \leq \infty \).
Self-similar asymptotics of solutions

\[ u(x, t) = \int_{\mathbb{R}^n} G(x - y, t)u_0(y) \, dy \quad \text{for} \quad x \in \mathbb{R}^n \text{ and } t > 0, \]

**Theorem**

Let \( u \) be the solution of the heat equation with initial datum \( u_0 \in L^1(\mathbb{R}^n) \). Let \( M = \int_{\mathbb{R}^n} u_0(y) \, dy \). Then

\[ \lim_{t \to \infty} t^{(n/2)(1-1/p)} \| u(\cdot, t) - MG(\cdot, t) \|_p = 0, \]

where \( G(x, t) \) is the Gauss-Weierstrass kernel.
Scaling and self-similar solutions

Let us notice that the heat equation

$$u_t = \Delta u, \quad x \in \mathbb{R}^n, \ t > 0$$

has the following property:

If $u = u(x, t)$ is a solution of this equation, then the function

$$u_\lambda(x, t) \equiv \lambda^k u(\lambda x, \lambda^2 t)$$

is a solution for each $\lambda > 0$. Here, $k \in \mathbb{R}$ is a fixed parameter.

**Definition**

A solution $u = u(x, t)$ is called **self-similar** if there exists $k \in \mathbb{R}$ such that

$$\lambda^k u(\lambda x, \lambda^2 t) = u(x, t)$$

for all $x \in \mathbb{R}^n, \ t > 0, \text{ and } \lambda > 0$. 
The heat kernel (also called the Gauss-Weierstrass kernel) is a self-similar solution with $k = n$ of the heat equation. Indeed, it is easy to see that

$$\lambda^n G(\lambda x, \lambda^2 t) = \lambda^n \frac{1}{(4\pi \lambda^2 t)^{n/2}} \exp \left( -\frac{\lambda x^2}{4\lambda^2 t} \right)$$

$$= \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{x^2}{4t} \right) = G(x, t).$$
Theorem

Denote

\[ u_\lambda(x, t) = \lambda^n u(\lambda x, \lambda^2 t). \]

Fix \( p \in [1, \infty] \). The following two conditions are equivalent

1. \( \lim_{t \to \infty} t^{(n/2)(1-1/p)} \| u(\cdot, t) - MG(\cdot, t) \|_p = 0 \)

2. for every \( t_0 > 0 \),

\[ u_\lambda(\cdot, t_0) \to MG(\cdot, t_0), \quad \text{as} \quad \lambda \to \infty, \]

where the convergence is in the usual norm of \( L^p(\mathbb{R}^n) \).
Scaling and self-similar solutions

Proof.
Scaling property of the $L^p$-norm:

$$\|v(\lambda \cdot)\|_p = \lambda^{-n/p} \|v\|_p,$$

Now, using this scaling property we obtain

$$\|u(\lambda \cdot, t_0) - MG(\cdot, t_0)\|_p = \|\lambda^n u(\lambda \cdot, \lambda^2 t_0) - M\lambda^n G(\lambda \cdot, \lambda^2 t_0)\|_p$$

$$= \lambda^{n-n/p} \|u(\cdot, \lambda^2 t_0) - MG(\cdot, \lambda^2 t_0)\|_p \quad \text{(substituting} \quad \lambda = \sqrt{t/t_0} \text{)}$$

$$= C(t_0) t^{(n/2)(1-1/p)} \|u(\cdot, t) - MG(\cdot, t)\|_p.$$
**NEW APPROACH: Four steps method**

**Step 1. Scaling.** We introduce the rescaled family of functions

\[ u_\lambda(x, t) = \lambda^n u(\lambda x, \lambda^2 t) \quad \text{for every} \quad \lambda > 0. \]
NEW APPROACH: Four steps method

**Step 1. Scaling.** We introduce the rescaled family of functions

\[ u_\lambda(x, t) = \lambda^n u(\lambda x, \lambda^2 t) \quad \text{for every} \quad \lambda > 0. \]

**Step 2. Estimates and compactness.** We show that the embedding

\[ \{u_\lambda(\cdot, t)\}_{\lambda>0} \subset L^p(\mathbb{R}^n) \quad \text{is compact for every} \quad t > 0. \]
NEW APPROACH: Four steps method

Step 1. Scaling. We introduce the rescaled family of functions

\[ u_\lambda(x, t) = \lambda^n u(\lambda x, \lambda^2 t) \quad \text{for every} \quad \lambda > 0. \]

Step 2. Estimates and compactness. We show that the embedding

\[ \{u_\lambda(\cdot, t)\}_{\lambda > 0} \subset L^p(\mathbb{R}^n) \quad \text{is compact for every} \quad t > 0. \]

Step 3. Passage to the limit. By compactness there exists a sequence \( \lambda_n \to \infty \) and a function \( \bar{u}(x, t) \) such that

\[ u_{\lambda_n}(\cdot, t) \to \bar{u}(\cdot, t) \quad \text{in} \quad L^p(\mathbb{R}^n) \quad \text{for every} \quad t > 0. \]

Since \( u_\lambda \) satisfies the heat equation, one can show that \( \bar{u} \) is a weak solution of the heat equation, as well.
NEW APPROACH: Four steps method

Step 1. Scaling. We introduce the rescaled family of functions

\[ u_\lambda(x, t) = \lambda^n u(\lambda x, \lambda^2 t) \quad \text{for every} \quad \lambda > 0. \]

Step 2. Estimates and compactness. We show that the embedding

\[ \{u_\lambda(\cdot, t)\}_{\lambda > 0} \subset L^p(\mathbb{R}^n) \quad \text{is compact for every} \quad t > 0. \]

Step 3. Passage to the limit. By compactness there exists a sequence \( \lambda_n \to \infty \) and a functions \( \bar{u}(x, t) \) such that

\[ u_{\lambda_n}(\cdot, t) \to \bar{u}(\cdot, t) \quad \text{in} \quad L^p(\mathbb{R}^n) \quad \text{for every} \quad t > 0. \]

Since \( u_\lambda \) satisfies the heat equation, one can show that \( \bar{u} \) is a weak solution of the heat equation, as well.

Step 4. Identification of the limit. The limit function \( \bar{u} \) corresponds usually to singular initial conditions.
Lemma

Let $u_0 \in L^1(\mathbb{R}^n)$. For every test function $\varphi \in C_\infty^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x) \varphi(x) \, dx \to M\varphi(0) \quad \text{as } \lambda \to \infty,$$

where $M = \int_{\mathbb{R}^n} u_0(x) \, dx$. 

Proof.

This is an immediate consequence of the Lebesgue dominated convergence theorem, because

$$\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x) \varphi(x) \, dx = \int_{\mathbb{R}^n} u_0(x) \varphi(x/\lambda) \, dx,$$

by a simple change of variables.
Lemma
Let \( u_0 \in L^1(\mathbb{R}^n) \). For every test function \( \varphi \in C^\infty_c(\mathbb{R}^n) \) we have

\[
\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x) \varphi(x) \, dx \rightarrow M \varphi(0) \quad \text{as} \quad \lambda \rightarrow \infty,
\]

where \( M = \int_{\mathbb{R}^n} u_0(x) \, dx \).

Proof.
This is an immediate consequence of the Lebesgue dominated convergence theorem, because

\[
\int_{\mathbb{R}^n} \lambda^n u_0(\lambda x) \varphi(x) \, dx = \int_{\mathbb{R}^n} u_0(x) \varphi(x/\lambda) \, dx,
\]

by a simple change of variables. \( \square \)
Application to convection-diffusion equation
We are going to show the *Four Step Method* “in action”, by applying it to the initial value problem for the nonlinear convection diffusion equation

\[
  u_t - u_{xx} + (u^q)_x = 0 \quad \text{for } x \in \mathbb{R}, \ t > 0, \tag{3}
\]

\[
  u(x, 0) = u_0(x), \tag{4}
\]

where \( q > 1 \) is a fixed parameter.
Theorem (Existence of global-in-time solution)

Assume that

$$u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R}).$$

Suppose $u_0 \geq 0$.

Then the initial value problem (3)--(4) has a nonnegative, global-in-time solution $u \in C^{2,1}(\mathbb{R} \times (0, \infty))$.

This solution satisfies

$$u \in C^1((0, \infty), L^p(\mathbb{R})) \cap C((0, \infty), W^{2,p}(\mathbb{R})))$$

for each $p \in [1, \infty]$. Moreover,

$$M \equiv \|u(t)\|_1 = \int_{\mathbb{R}} u(x, t) \, dx = \int_{\mathbb{R}} u_0(x) \, dx = \|u_0\|_1 \quad \text{for all} \quad t \geq 0.$$
Local existence via the Banach contraction principle

Local-in-time *mild* solutions:

\[
    u(t) = G(\cdot, t) * u_0 + \int_0^t \partial_x G(\cdot, t - s) * u^q(s) \, ds
\]

with the heat kernel \( G(x, t) = (4\pi t)^{-1/2} \exp \left( -|x|^2/(4t) \right). \)

By the Young inequality for the convolution:

\[
    \| G(\cdot, t) * f \|_p \leq C t^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right)} \| f \|_q, \tag{5}
\]

\[
    \| \partial_x G(\cdot, t) * f \|_p \leq C t^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{1}{2}} \| f \|_q \tag{6}
\]

for every \( 1 \leq q \leq p \leq \infty \), each \( f \in L^q(\mathbb{R}) \), and \( C = C(p, q) \) independent of \( t, f \).

Notice that \( C = 1 \) in inequality (5) for \( p = q \) because \( \| G(\cdot, t) \|_{L^1} = 1 \) for all \( t > 0 \).
Lemma (Local existence)

Assume that

\[ u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R}). \]

Then there exists \( T = T(\|u_0\|_1, \|u_0\|_\infty) > 0 \) such that the integral equation (18) has the unique solution in the space

\[ \mathcal{Y}_T = C([0, T], L^1(\mathbb{R})) \cap C([0, T], L^\infty(\mathbb{R})), \]

supplemented with the norm \( \|u\|_{\mathcal{Y}_T} = \sup_{0 \leq t \leq T} \|u\|_1 + \sup_{0 \leq t \leq T} \|u\|_\infty. \)

**Proof.**

The Banach contraction principle.

For sufficiently small \( T \) the right hand side of this equation defines the contraction in the space \( \mathcal{Y}_T. \)
Regularity and comparison principle

One can show that a local in time solution is nonnegative, if an initial condition is so. Moreover, this solution has the following regularity property

\[ u \in C^1((0, \infty), L^p(\mathbb{R})) \cap C((0, \infty), W^{2,p}(\mathbb{R})) \]

for each \( p \in [1, \infty] \).

Below, we show that the solution satisfies the following a priori estimates for each \( p \in [1, \infty] \):

\[ \| u(\cdot, t) \|_p \leq \| u_0 \|_p \quad \text{for all} \quad t > 0. \]

Hence, by a standard reasoning, we can show that the solution is global in time.
Theorem (LINEAR Self-similar asymptotics)

Let 

$$q > 2.$$ 

Every solution $$u = u(x, t)$$ of problem (3)–(4) satisfies 

$$t^{(1-1/p)/2} \|u(t) - MG(t)\|_p \to 0 \quad \text{as } t \to \infty$$ 

for every $$p \in [1, \infty]$$, where 

$$M = \int_{\mathbb{R}} u_0(x) \, dx$$ 

and 

$$G(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t}\right)$$

is the heat kernel.
Theorem (NONLINEAR Self-similar asymptotics)

If

$$q = 2,$$

we have

$$t^{(1-1/p)/2} \| u(t) - U_M(t) \|_p \to 0 \quad \text{as} \quad t \to \infty$$

for every $p \in [1, \infty]$, where

$$U_M(x, t) = \frac{1}{\sqrt{t}} U_M \left( \frac{x}{\sqrt{t}}, 1 \right)$$

is the so-called nonlinear diffusion wave:

$$U_t - U_{xx} + (U^2)_x = 0, \quad \text{for} \quad x \in \mathbb{R}, \quad t > 0,$$

$$U(x, 0) = M \delta_0,$$

where $\delta_0$ is the Dirac measure.
Remark on nonlinear diffusion waves

The Hopf-Cole transformation allows us to solve this problem:

\[ U_M(x, t) = \frac{t^{-1/2} \exp\left(-|x|^2/(4t)\right)}{\sqrt{x/\sqrt{t}}} \times \bigg( C_M + \frac{1}{2} \int_0^\infty \exp\left(-\xi^2/4\right) d\xi \bigg), \]

where \( C_M \) is a constant which is determined uniquely as a function of \( M \) by the condition \( \int_{\mathbb{R}} U_{M,A}(\eta, 1) \ d\eta = M. \)

For every \( M \in \mathbb{R} \) the function \( U_M \) is a unique solution of the Burgers equation in the space \( C((0, \infty); L^1(\mathbb{R})) \) having the properties

\[ \int_{\mathbb{R}} U_M(x, t) \ dx = M \quad \text{for all} \quad t > 0 \]

and

\[ \int_{\mathbb{R}} U_M(x, t)\varphi(x) \ dx \to M\varphi(0) \quad \text{as} \quad t \to 0 \]

for all \( \varphi \in C_c^\infty(\mathbb{R}). \)
Idea of the proof.
Rescaled family of functions

We study the behavior, as $\lambda \to \infty$, of the rescaled family of functions

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t) \quad \text{for every} \quad \lambda > 0,$$

which satisfy

$$\partial_t u_\lambda - \partial_x^2 u_\lambda + \lambda^{2-q} \partial_x u_\lambda^q = 0,$$

$$u_\lambda(x, 0) = u_{0, \lambda}(x) = \lambda u_0(\lambda x).$$

Notice that, by a simple change of variables, we have the following identity

$$\|u_\lambda(t)\|_1 = \|u_0\|_1$$

holds true for all $t > 0$ and all $\lambda > 0$. 
Theorem

Under the assumptions of Theorem 6, the solution of problem (3)–(4) satisfies

$$\|u(\cdot, t)\|_p \leq C t^{-(1-1/p)/2} \|u_0\|_1$$

for each $p \in [1, \infty]$, a constant $C = C(p)$ and all $t > 0$.

We sketch the proof for $p = 2$, only. Multiplying the equation by $u$ and integrating the resulting equation over $\mathbb{R}$ we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 \, dx = - \int_{\mathbb{R}} |u_x|^2 \, dx.$$

Here, we have used an elementary equalities

$$\int_{\mathbb{R}} u^q(x, t)u(x, t) \, dx = \frac{1}{q + 1} \int_{\mathbb{R}} (u^{q+1}(x, t))_x \, dx = 0$$

if $u(x, t) \to 0$ when $|x| \to \infty$. 

Optimal $L^p$-decay of solutions
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u^2 \, dx = -\int_{\mathbb{R}} |u_x|^2 \, dx
\]

Now, by the Nash inequality

\[
\|u\|_2 \leq C \|u_x\|_2^{1/3} \|u\|_1^{2/3},
\]

which is valid for all \(u \in L^1(\mathbb{R})\) such that \(u_x \in L^2(\mathbb{R})\), (since the \(L^1\)-norm of the solution is constant in time) we obtain the differential inequality

\[
\frac{d}{dt} \|u(t)\|_2^2 + C \|u_0\|_1^{-4} (\|u(t)\|_2^2)^3 \leq 0,
\]

which implies

\[
\|u(t)\|_2 \leq Ct^{-1/4}
\]

for all \(t > 0\) and \(C > 0\) independent of \(t\).
Estimates of the rescaled family of solutions

Lemma
For each $p \in [1, \infty]$ there exists $C = C(\|K'\|_1, \|u_0\|_1) > 0$, independent of $t$ and of $\lambda$, such that

$$\|u_\lambda(t)\|_p \leq Ct^{-\frac{1}{2}}(1 - \frac{1}{p})$$

for all $t > 0$ and all $\lambda > 0$.

Proof.
By the change of variables and the decay estimate we obtain

$$\|u_\lambda(t)\|_p = \lambda^{1 - \frac{1}{p}} \|u(\cdot, \lambda^2 t)\|_p \leq C \lambda^{1 - \frac{1}{p}} (\lambda^2 t)^{-\frac{1}{2}}(1 - \frac{1}{p}) = Ct^{-\frac{1}{2}}(1 - \frac{1}{p}).$$
Lemma
For each $p \in [1, \infty)$ there exists $C = C(p, \|K'\|_1, \|u_0\|_1) > 0$, independent of $t$ and of $\lambda$, such that

$$\|\partial_x u_{\lambda}(t)\|_p \leq Ct^{-\frac{1}{2}}(1 - \frac{1}{p})^{-\frac{1}{2}}$$

for all $t > 0$ and all $\lambda > 0$.

Identical estimates hold true for $u_{\lambda}$. 
Aubin-Lions-Simon’s compactness result

**Theorem**

Let $X$, $B$ and $Y$ be Banach spaces satisfying

$$X \subset B \subset Y$$

with **compact** embedding $X \subset B$ and **continuous** embedding $B \subset Y$. Assume, for $1 \leq p \leq \infty$ and $T > 0$, that

- $F$ is bounded in $L^p(0, T; X)$,
- $\{\partial_t f : f \in F\}$ is bounded in $L^p(0, T; Y)$.

Then $F$ is relatively compact in $L^p(0, T; B)$ and in $C(0, T; B)$ if $p = \infty$. 
Compactness in $L^1_{loc}(\mathbb{R})$

**Lemma**
*For every $0 < t_1 < t_2 < \infty$ and every $R > 0$, the set*

$$\{u_{\lambda}\}_{\lambda > 0} \subseteq C([t_1, t_2], L^1([-R, R]))$$

*is relatively compact.*

**Proof.**
We apply Theorem with $p = \infty$, $F = \{u_{\lambda}\}_{\lambda > 0}$, and

$$X = W^{1,1}([-R, R]), \quad B = L^1([-R, R]), \quad Y = W^{-1,1}([-R, R]),$$

where $R > 0$ is fixed and arbitrary, and $Y$ is the dual space of $W^{1,1}_0([-R, R])$.

Obviously, the embedding $X \subseteq B$ is compact by the Rellich-Kondrashov theorem. \qed
Compactness in $L^1(\mathbb{R})$

**Lemma**
For every $0 < t_1 < t_2 < \infty$, the set

$$\{u_\lambda\}_{\lambda > 0} \subseteq C([t_1, t_2], L^1(\mathbb{R}))$$

is relatively compact.

**Proof.**
Let $\psi \in C^\infty(\mathbb{R})$ be nonnegative and satisfy $\psi(x) = 0$ for $|x| < 1$ and $\psi(x) = 1$ for $|x| > 2$.
Put $\psi_R(x) = \psi(x/R)$ for every $R > 0$. It suffices to show that

$$\sup_{t \in [t_1, t_2]} \|u_\lambda(t)\psi_R\|_1 \to 0 \quad \text{as} \quad R \to \infty, \quad \text{uniformly in} \quad \lambda \geq 1.$$
Lemma
For every test function $\phi \in C^\infty_c(\mathbb{R})$, there exists $C = C(\phi, \| K' \|_1, \| u_0 \|_1)$ independent of $\lambda$ such that

$$\left| \int_{\mathbb{R}} u_\lambda(x, t)\phi(x) \, dx - \int_{\mathbb{R}} u_{0,\lambda}(x)\phi(x) \, dx \right| \leq C \left( t + t^{1/2} \right).$$

(9)
Proof of the main result

By compactness, there exists a subsequence of \( \{ u_\lambda \}_{\lambda > 0} \) (not relabeled) and a function \( \bar{u} \in C((0, \infty), L^1(\mathbb{R})) \) such that

\[
    u_\lambda \to \bar{u} \quad \text{in} \quad C([t_1, t_2], L^1(\mathbb{R})) \quad \text{as} \quad \lambda \to \infty.
\]

Passing to a subsequence, we can assume that

\[
    u_\lambda(x, t) \to \bar{u}(x, t) \quad \text{as} \quad \lambda \to \infty
\]

almost everywhere in \( (x, t) \in \mathbb{R} \times (0, \infty) \).

Now, multiplying the equation by a test function \( \phi \in C_c^\infty(\mathbb{R} \times (0, \infty)) \) and integrating over \( \mathbb{R} \times (0, \infty) \), we obtain

\[
- \int_0^\infty \int_\mathbb{R} u_\lambda \phi_t \, ds \, dx = \int_0^\infty \int_\mathbb{R} u_\lambda \phi_{xx} \, ds \, dx + \lambda^{2-q} \int_0^\infty \int_\mathbb{R} u_\lambda^q \phi_x \, ds \, dx.
\]
We obtain that $\bar{u}(x, t)$ is a weak solution of the equation

$$\bar{u}_t = \bar{u}_{xx} - (\bar{u}^2)_x$$

if $q = 2$.

Initial conditions:

$$\int_{\mathbb{R}} u_{0,\lambda}(x) \phi(x) \, dx = \int_{\mathbb{R}} u_0(x) \phi(x/\lambda) \, dx \to M\phi(0)$$

as $\lambda \to \infty$. Hence,

$$\bar{u}(x, 0) = M\delta_0.$$

Thus, $\bar{u}$ is a weak solution of the initial value problem

$$\bar{u}_t = \bar{u}_{xx} - (\bar{u}^2)_x, \quad \bar{u}(x, 0) = M\delta_0. \quad (10)$$

Since problem (10)-(11) has a unique solution, we obtain that $\bar{u} = \mathcal{U}_M$. 
Obviously, if $q > 2$, this limit function is a solution to the linear problem

$$\tilde{u}_t = \tilde{u}_{xx}, \quad (12)$$

$$\tilde{u}(x, 0) = M\delta_0. \quad (13)$$

So, it is the multiple of Gauss-Weierstrass kernel

$$\tilde{u}(x, t) = MG(x, t) = M \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Hence, we have

$$\lim_{\lambda \to \infty} \|u_\lambda(1) - \tilde{u}(1)\|_1 = 0$$

and, after setting $\lambda = \sqrt{t}$ and using the self-similar form of $\tilde{u}(x, t) = t^{-1/2}\tilde{u}(xt^{-1/2}, 1)$, we obtain

$$\lim_{t \to \infty} \|u(t) - \tilde{u}(t)\|_1 = 0.$$
The convergence of $u(\cdot, t)$ in the $L^p$-norms for $p \in (1, \infty)$:

$$\|u(t) - \bar{u}(t)\|_p \leq \left(\|u(t)\|_\infty + \|\bar{u}(t)\|_\infty\right)^{1-1/p} \|u(t) - \bar{u}(t)\|_1^{1/p} = o(t^{-(1-1/p)/2})$$
as $t \to \infty$.

The convergence in the $L^\infty$-norm. Here, by the Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$\|u(t) - \bar{u}(t)\|_\infty \leq C\left(\|u_x(t)\|_2 + \|\bar{u}_x(t)\|_2\right)^{1/2} \|u(t) - \bar{u}(t)\|_2^{1/2} = o(t^{-1/2})$$
as $t \to \infty$. 

Summary on $u_t - u_{xx} + (u^q)_x = 0$.

Suppose that $u_0 \in L^1(\mathbb{R})$. Put

$$M = \int_{\mathbb{R}} u_0(x) \, dx.$$ 

Then

- **Case I:** linear asymptotics
  
  For $q > 2$ and for every $p \in [1, \infty]$,

  $$t^{\frac{1}{2} \left(1 - \frac{1}{p} \right)} \| u(\cdot, t) - M G(\cdot, t) \|_{L^p(\mathbb{R})} \to 0$$

  as $t \to \infty$, where

  $$G(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

  is the Gauss-Weierstrass kernel.

- **Case II:** balance case

  For $q = 2$, the large time asymptotics of solutions is described as $t \to \infty$ by a self-similar solution of the convection-diffusion equation with the initial datum

  $$u_0(x) = M \delta_0.$$
Hyperbolic asymptotics

\[ u_t - u_{xx} + (u^q)_x = 0 \]

**Case III:** For \(1 < q < 2\), the asymptotics is given by a self-similar solution of the convection equation

\[ v_t + (|v|^q)_x = 0 \]

with the initial datum

\[ v_0(x) = M\delta_0. \]
Hyperbolic asymptotics

\[ u_t - u_{xx} + (u^q)_x = 0 \]

**Case III:** For \( 1 < q < 2 \), the asymptotics is given by a self-similar solution of the convection equation

\[ \nu_t + (|\nu|^q)_x = 0 \]

with the initial datum

\[ \nu_0(x) = M\delta_0. \]

One should use the following different scaling

\[ u_\lambda(x, t) = \lambda u(\lambda x, \lambda^q t). \]

and the rescaled equation

\[ \partial u_\lambda - \lambda^{q-2} \partial_x u_\lambda + \partial(u_\lambda)^q = 0. \]
Zero mass initial conditions

The main assumption

\[ u_0 \in L^1(\mathbb{R}, (1 + |x|) \, dx), \quad u_0 \not\equiv 0 \]

and

\[ \int_{\mathbb{R}} u_0(x) \, dx = 0. \]

Define

\[ U_0(x) = \int_{-\infty}^{x} u_0(y) \, dy = -\int_{x}^{\infty} u_0(y) \, dy. \]
Diffusion-dominated case

Assume that one of the following assertions hold true:

(i) $U_0 \geq 0$ and $q > 3/2$,
(ii) $U_0 \leq 0$ and $q \geq 2$,
(iii) $U_0 \leq 0$, $q \in (3/2, 2)$ the quantity

$$\left| \int_{\mathbb{R}} x u_0(x) \, dx \right| \|u_0\|_{L^\infty(\mathbb{R})}^{2q-3}$$

is sufficiently small.

Then

$$t^{\frac{1}{2}(1-\frac{1}{p})+\frac{1}{2}} \left\| u(\cdot, t) - l_\infty \partial_x G(\cdot, t) \right\|_{L^p(\mathbb{R})} \rightarrow 0,$$

as $t \rightarrow \infty$, where

$$l_\infty \equiv - \lim_{t \rightarrow \infty} \int_{\mathbb{R}} x u(x, t) \, dx$$

$$= - \int_{\mathbb{R}} x u_0(x) \, dx - \int_{0}^{\infty} \int_{\mathbb{R}} |u(x, s)|^q \, dx \, ds.$$
Other results

- the convergence towards very singular solutions (special self-similar solutions of the convection-diffusion equation)
- the convergence towards hyperbolic waves

Conclusion:
The large time asymptotics of zero mass solutions depends not only on the exponent of the nonlinearity $q > 1$ but also on the size, sign, and shape of the initial datum.