Large time behavior of solutions to the Navier-Stokes system in unbounded domains

Grzegorz Karch
Wroclaw, POLAND

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LECTURE 2
Self-similar solutions to the Navier-Stokes system

Navier–Stokes equations

Navier–Stokes equations for $x \in \mathbb{R}^3$:

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = F, \tag{1}$$
$$\nabla \cdot u = 0, \tag{2}$$
$$u(0) = u_0. \tag{3}$$

The external force F and initial velocity u_0 are assigned.

Equivalent form for sufficiently regular solutions:

$$u_t - \Delta u + \nabla \cdot (u \otimes u) + \nabla p = F, \quad \nabla \cdot u = 0.$$

The Leray projector on solenoidal vector fields:

$$\mathbb{P} v = v - \nabla \Delta^{-1} (\nabla \cdot v).$$

We formally transform the system into

$$u_t - \Delta u + \mathbb{P} \nabla \cdot (u \otimes u) = \mathbb{P} F, \quad \nabla \cdot u = 0.$$

The Leray projector $\mathbb{P}$

The Riesz transforms

$$\widehat{R_k f}(\xi) = \frac{i\xi_k}{|\xi|}\widehat{f}(\xi).$$

The Fourier transform: $\widehat{v}(\xi) \equiv (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} e^{-ix\cdot\xi} v(x)\, dx.$

Using these well-known operators we define

$$(\mathbb{P}v)_j = v_j + \sum_{k=1}^{3} R_j R_k v_k.$$

Note that

$$(\widehat{\mathbb{P}}(\xi))_{j,k} = \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2},$$

hence

$$\max_{1\leqslant j,k\leqslant 3} \sup_{\xi\in\mathbb{R}^3\setminus\{0\}} |(\widehat{\mathbb{P}}(\xi))_{j,k}| = 1.$$

Statement of the problem

The Duhamel principle:

$$u(t) = S(t)u_0 - \int_0^t S(t-\tau)\mathbb{P}\nabla(u \otimes u)(\tau)\, d\tau + \int_0^t S(t-\tau)\mathbb{P}F(\tau)\, d\tau,$$

where $S(t)$ is the heat semigroup given as the convolution with the Gauss–Weierstrass kernel:

$$G(x,t) = (4\pi t)^{-3/2} \exp(-|x|^2/(4t))$$

ABSTRACT LEMMA

Lemma

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and $B : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ a bounded bilinear form satisfying

$$\|B(x_1, x_2)\|_{\mathcal{X}} \leqslant \eta \|x_1\|_{\mathcal{X}} \|x_2\|_{\mathcal{X}}$$

for all $x_1, x_2 \in \mathcal{X}$ and a constant $\eta > 0$. Then, if

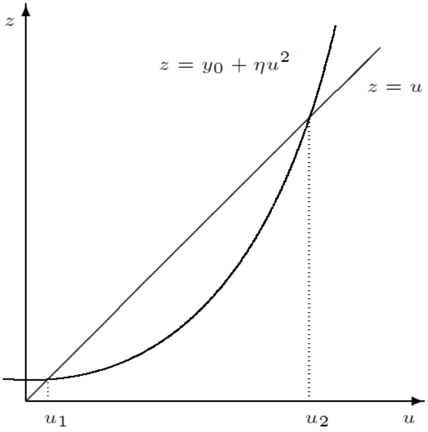
$$0 < \varepsilon < 1/(4\eta)$$

and if $y \in \mathcal{X}$ such that $\|y\| < \varepsilon$, the equation

$$x = y + B(x, x)$$

has a solution in $\mathcal{X}$ such that $\|x\|_{\mathcal{X}} \leqslant 2\varepsilon$. This solution is the only one in the ball $\bar{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the following sense: if $\|\tilde{y}\|_{\mathcal{X}} \leqslant \varepsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})$, and $\|\tilde{x}\|_{\mathcal{X}} \leqslant 2\varepsilon$, then

$$\|x - \tilde{x}\|_{\mathcal{X}} \leqslant \frac{1}{1 - 4\eta\varepsilon} \|y - \tilde{y}\|_{\mathcal{X}}.$$



Two solutions u_1 and u_2 of the quadratic equation $u = y_0 + \eta u^2$.

Function spaces

$$\mathcal{PM}^a \equiv \{v \in \mathcal{S}'(\mathbb{R}^3) : \widehat{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^3), \quad \|v\|_{\mathcal{PM}^a} \equiv \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^3} |\xi|^a |\widehat{v}(\xi)| < \infty\},$$

where $a \in [0, 3)$ is a given parameter.

Definition

By a solution we mean in this chapter a function $u = u(x,t)$ belonging to the space

$$\mathcal{X} = \mathcal{C}_w([0,T); \mathcal{PM}^2),$$

$0 < T \leq \infty$, and such that

$$\begin{aligned}\widehat{u}(\xi,t) &= e^{-t|\xi|^2}\widehat{u}(\xi,0) + \int_0^t e^{-(t-\tau)|\xi|^2} i\xi \cdot \widehat{\mathbb{P}}(\xi)\left(\widehat{u \otimes u}\right)(\xi,\tau)\,d\tau \\ &\quad + \int_0^t e^{-(t-\tau)|\xi|^2}\widehat{\mathbb{P}}(\xi)\widehat{F}(\xi,\tau)\,d\tau\end{aligned}$$

for all $0 \leqslant t \leqslant T$.

Remark
Given $f \in \mathcal{S}'(\mathbb{R}^3) \cap L^1_{loc}(\mathbb{R}^3)$ we denote the rescaling

$$f_\lambda(x) = f(\lambda x).$$

It follows from elementary calculations that

$$\widehat{f_\lambda}(\xi) = \lambda^{-3}\widehat{f}(\lambda^{-1}\xi).$$

Hence, for every $\lambda > 0$, we obtain the scaling property of the norm in $\mathcal{PM}^a$

$$\|f_\lambda\|_{\mathcal{PM}^a} = \lambda^{a-3}\|f\|_{\mathcal{PM}^a}.$$

In particular, the norm $\mathcal{PM}^2$ is invariant under rescaling $f \mapsto \lambda f(\lambda \cdot)$.

Moreover, for $a = 3(1 - 1/p)$, the norms $\|\cdot\|_{\mathcal{PM}^a}$ and $\|\cdot\|_{L^p(\mathbb{R}^3)}$ have the same scaling property.

To simplify the notation, the quadratic term in (4) will be denoted by

$$B(u, v)(t) = -\int_0^t S(t-\tau)\mathbb{P} \cdot \nabla(u(\tau) \otimes v(\tau))\, d\tau,$$

where $u = u(t)$ and $v = v(t)$ are functions defined on $[0, T)$ with values in a vector space (here most frequently $\mathcal{PM}^2$).

The space

$$\mathcal{X} = \mathcal{C}_w([0,\infty), \mathcal{PM}^2)$$

The integral equation

$$u = y + B(u,u),$$

where the bilinear form is as above and

$$y = S(t)u_0 + \int_0^t S(t-\tau)F(\tau)\, d\tau.$$

Lemma

Given $u_0 \in \mathcal{PM}^2$, we have $S(\cdot)u_0 \in \mathcal{X}$.

Proof.

By the definition of the norm in $\mathcal{PM}^2$, it follows that

$$\|S(t)u_0\|_{\mathcal{PM}^2} = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^3} |\xi|^2 \left| e^{-t|\xi|^2}\widehat{u}_0(\xi)\right| \leqslant \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^3} |\xi|^2 |\widehat{u}_0(\xi)| = \|u_0\|_{\mathcal{PM}^2},$$

so, $S(\cdot)u_0 \in L^\infty([0,\infty), \mathcal{PM}^2)$.

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so, $S(\cdot)u_0 \in L^\infty([0,\infty),\mathcal{PM}^2)$.

The weak continuity with respect to t. For every $\varphi \in \mathcal{S}(\mathbb{R}^3)$, by the Plancherel formula, we obtain

$$\begin{aligned} |\langle S(t)u_0 - u_0, \varphi\rangle| &= \left| \int \left(e^{-t|\xi|^2} - 1\right) \widehat{u}_0(\xi)\widehat{\varphi}(\xi)\, d\xi \right| \\ &\leqslant t \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^3} \left| \frac{e^{-t|\xi|^2} - 1}{t|\xi|^2} \right| \|u_0\|_{\mathcal{PM}^2} \|\widehat{\varphi}\|_{L^1\mathbb{R}^3} \to 0 \end{aligned}$$

as $t \searrow 0$. $\square$

Lemma

Given $F \in \mathcal{C}_w([0,\infty),\mathcal{PM})$*, it follows that*

$$w \equiv \int_0^t S(t-\tau)F(\tau)\,d\tau \in \mathcal{X}.$$

Moreover, $\|w\|_{\mathcal{X}} \leqslant \|F\|_{\mathcal{C}_w([0,\infty),\mathcal{PM})}$.

Proof.

$$\begin{aligned} \|w(t)\|_{\mathcal{PM}^2} &= \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^3} |\xi|^2 \left| \int_0^t e^{-(t-\tau)|\xi|^2} \widehat{F}(\xi,\tau)\,d\tau \right| \\ &\leqslant |\xi|^2 \int_0^t e^{-(t-\tau)|\xi|^2}\,d\tau \|F\|_{\mathcal{C}_w([0,\infty),\mathcal{PM})} \\ &\leqslant \|F\|_{\mathcal{C}_w([0,\infty),\mathcal{PM})}. \end{aligned}$$

□

$\mathcal{X} = \mathcal{C}_w([0,\infty), \mathcal{PM}^2)$

Lemma

There exists a constant $\eta > 0$ such that for every $u, v \in \mathcal{X}$, it follows

$$\|B(u,v)\|_{\mathcal{X}} \leqslant \eta \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}.$$

Proof

Using elementary properties of the Fourier transform we obtain

$$\begin{aligned} \left|\widehat{(u \otimes v)}(\xi,\tau)\right| &\leqslant C \int_{\mathbb{R}^3} \frac{dz}{|\xi - z|^2 |z|^2} \|u(\tau)\|_{\mathcal{PM}^2} \|v(\tau)\|_{\mathcal{PM}^2} \\ &= \frac{\eta}{|\xi|} \|u(\tau)\|_{\mathcal{PM}^2} \|v(\tau)\|_{\mathcal{PM}^2}. \end{aligned}$$

In the computations above, we use the equality

$$|\xi|^{-2} * |\xi|^{-2} = \pi^3 |\xi|^{-1}.$$

Now, the boundedness of the bilinear form on $\mathcal{X}$ results from the following estimates

$$\begin{aligned}
&|\xi|^2 \left| \int_0^t e^{-(t-\tau)|\xi|^2} \, i\widehat{\mathbb{P}}\xi \cdot \widehat{(u \otimes v)}(\xi,\tau) \right| \, d\tau \\
&\qquad \leqslant \quad \eta|\xi|^2 \int_0^t e^{-(t-\tau)|\xi|^2} \, d\tau \, \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} \\
&\qquad \leqslant \quad \eta\|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}.
\end{aligned}$$

Main theorem

Theorem

Assume that $u_0 \in \mathcal{PM}^2$ *and* $F \in \mathcal{C}_w([0,\infty),\mathcal{PM})$ *satisfy*

$$\|u_0\|_{\mathcal{PM}^2} + \|F\|_{\mathcal{C}_w([0,\infty),\mathcal{PM})} < \varepsilon$$

for some $0 < \varepsilon < 1/(4\eta)$ *where* η *is defined above.*
There exists a global-in-time solution of the Navier-Stokes system in the space

$$\mathcal{X} = \mathcal{C}_w([0,\infty),\mathcal{PM}^2).$$

This is the unique solution satisfying the condition

$$\|u\|_{\mathcal{C}_w([0,\infty),\mathcal{PM}^2)} \leqslant 2\varepsilon.$$

Moreover, this solution depends continuously on initial data and external forces in the sense of Abstract Lemma.

Self-similar solutions

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0$$

If u solves the Cauchy problem, then the rescaled functions

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

is also a solution for each $\lambda > 0$ (scaling of the pressure $\lambda^2 p(\lambda x, \lambda^2 t)$).

Forward self-similar solution: $U_\lambda \equiv U$ for all $\lambda > 0$.

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Forward self-similar solution: $U_\lambda \equiv U$ for all $\lambda > 0$.

Take $t = 1$ and $\lambda = t^{1/2}$ in the equation $U \equiv U_\lambda$ to obtain

$$U(x,t) = t^{-1/2} U(x/t^{1/2}, 1)$$

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The initial condition:

$$\lim_{t \searrow 0} U(x,t)$$

has to be a distribution homogeneous of degree -1 at the origin.

Self-similar solutions

Theorem

Assume that $u_0 \in \mathcal{PM}^2$ *and* $F \in \mathcal{C}_w([0,\infty), \mathcal{PM})$ *satisfy*

$$\|u_0\|_{\mathcal{PM}^2} + \|F\|_{\mathcal{C}_w([0,\infty),\mathcal{PM})} < \varepsilon$$

for some $0 < \varepsilon < 1/(4\eta)$ *where* η *is defined above.*

Suppose that the initial condition $u_0 \in \mathcal{PM}^2$ *is homogeneous of degree* -1 *and* $F \in \mathcal{C}_w([0,\infty), \mathcal{PM})$ *satisfies*

$$\lambda^3 F(\lambda x, \lambda^2 t) = F(x,t) \quad \text{for all} \quad \lambda > 0$$

Then, the corresponding unique solution is self-similar.

Stationary solutions

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = F, \qquad \nabla \cdot u = 0$$

Theorem

Assume that $u = u(x) \in \mathcal{PM}^2$ and $F = F(x) \in \mathcal{PM}$. The following two facts are equivalent

1) *$u = u(x)$ is a stationary mild solution of the Navier-Stokes system in our sense. Hence, u is the solution of the integral equation*

$$u = S(t)u - \int_0^t S(t-\tau)\nabla\mathbb{P}(u \otimes u)(\tau)\, d\tau + \int_0^t S(\tau)F\, d\tau$$

for every $t > 0$;

2) *u satisfies the integral equation*

$$u = -\int_0^\infty S(\tau)\nabla\mathbb{P}(u \otimes u)(\tau)\, d\tau + \int_0^\infty S(\tau)F\, d\tau.$$

Proof.

$$\begin{aligned}
\widehat{u}(\xi) &= e^{-t|\xi|^2}\widehat{u}(\xi) - \int_0^t e^{-(t-\tau)|\xi|^2}\, d\tau\ i\xi\widehat{\mathbb{P}}(\xi)(\widehat{u \otimes u})(\xi) \\
&\quad + \int_0^t e^{-(t-\tau)|\xi|^2}\, d\tau\ \widehat{\mathbb{P}}(\xi)\widehat{F}(\xi) \\
&= e^{-t|\xi|^2}\widehat{u}(\xi) - \frac{1-e^{-t|\xi|^2}}{|\xi|^2} i\xi\widehat{\mathbb{P}}(\xi)(\widehat{u \otimes u})(\xi) + \frac{1-e^{-t|\xi|^2}}{|\xi|^2}\widehat{\mathbb{P}}(\xi)\widehat{F}(\xi).
\end{aligned}$$

for every $t > 0$. Passing to the limit as $t \to \infty$ and using the identity

$$\frac{1}{|\xi|^2} = \int_0^\infty e^{-\tau|\xi|^2}\, d\tau \quad \text{for } \xi \neq 0,$$

we complete the proof. □

Existence of stationary solutions

Theorem

Assume that $F \in \mathcal{PM}$ satisfies

$$\|F\|_{\mathcal{PM}} < \varepsilon < 1/(4\eta).$$

There exists a stationary solution u_∞ to the Navier–Stokes system in the space $\mathcal{PM}^2$ with F as the external force.

This is the unique solution satisfying the condition $\|u\|_{\mathcal{PM}^2} \leqslant 2\varepsilon$.

Smooth solutions

Solutions of the Cauchy problem for the Navier-Stokes system constructed in the space

$$\mathcal{X} = \mathcal{C}_w([0, \infty), \mathcal{PM}^2)$$

are, in fact, more regular for sufficiently regular external forces.

Lemma

For every $p \in (3, \infty]$, there exists a constant $C = C(p)$ such that

$$\sup_{t>0} t^{(1-3/p)/2} \|S(t)u_0\|_{L^p(\mathbb{R}^3)} \leqslant C\|u_0\|_{\mathcal{PM}^2}$$

for all $t > 0$ and $u_0 \in \mathcal{PM}^2$.

Proof.

Here, our tool is the Hausdorff–Young inequality:

$$\begin{aligned}
\|S(t)u_0\|^q_{L^p(\mathbb{R}^3)} &\leqslant C \int_{\mathbb{R}^3} \left| e^{-t|\xi|^2} \widehat{u}_0(\xi) \right|^q \, d\xi \\
&\leqslant C \left(\operatorname*{ess\,sup}_{\xi \in \mathbb{R}^3} |\xi|^2 |\widehat{u}_0(\xi)| \right)^q \int_{\mathbb{R}^3} \left| \frac{e^{-qt|\xi|^2}}{|\xi|^{2q}} \right| \, d\xi \\
&= C\|u_0\|^q_{\mathcal{PM}^2} t^{-3/2+q} \int_{\mathbb{R}^3} \left| \frac{e^{-q|w|^2}}{|w|^{2q}} \right| \, dw.
\end{aligned}$$

□

Remark for experts

Now, given $u_0 \in \mathcal{PM}^2$ with sufficiently small $\mathcal{PM}^2$-norm and for $F \equiv 0$ we may apply the above theory to get the solution $u = u(x,t)$ which is unique in the space

$$\mathcal{C}_w([0,\infty), \dot{B}_p^{-1+3/p,\infty}(\mathbb{R}^3)) \cap \{v \ : \ t^{(3/p-1)/2}\|v(t)\|_{L^p(\mathbb{R}^3)} < \infty\}.$$

Regularizing effect

Definition
Let $2 \leqslant a < 3$. We define the Banach space

$$\begin{aligned} \mathcal{Y}^a \quad &\equiv \quad \mathcal{C}_w([0,\infty), \mathcal{PM}^2) \\ &\cap \ \{v : (0,\infty) \to \mathcal{PM}^a : |||v|||_a \equiv \sup_{t>0} t^{a/2-1} \|v(t)\|_{\mathcal{PM}^a} < \infty\}. \end{aligned}$$

The space $\mathcal{Y}^a$ is normed by the quantity $\|v\|_{\mathcal{Y}^a} = |||v|||_2 + |||v|||_a$.

Remark
The norm $|||\cdot|||_a$ is invariant under the rescaling

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

for every $\lambda > 0$.

Theorem

Let $2 \leqslant a < 3$. There exists a constant $\eta_a > 0$ such that for every $u \in \mathcal{C}_w([0,\infty), \mathcal{PM}^2)$ and $v \in \{v(t) \in \mathcal{PM}^a \ : \ \|v\|_a < \infty\}$ we have

$$\|B(u,v)\|_a \leqslant \eta_a \|u\|_2 \|v\|_a.$$

Proof.

We have

$$\begin{aligned} |\widehat{(u \otimes v)}(\xi,t)| &\leqslant \int_{\mathbb{R}^3} \frac{1}{|\xi - z|^2 |z|^a} \, dz \, \|u(t)\|_{\mathcal{PM}^2} \|v(t)\|_{\mathcal{PM}^a} \\ &= C|\xi|^{1-a} \|u(t)\|_{\mathcal{PM}^2} \|v(t)\|_{\mathcal{PM}^a}. \end{aligned}$$

For every $a \geqslant 2$

$$t^{a/2-1} \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \tau^{1-a/2} \, d\tau \leqslant C$$

independent of ξ and t. □

Soothing of the heat semi-group

Lemma

For every $u_0 \in \mathcal{PM}^2$ and $t > 0$, it follows that $S(t)u_0 \in \mathcal{PM}^a$ with $a \geqslant 2$. Moreover, there exists C depending on the exponent a only such that

$$\sup_{t>0}\left(t^{a/2-1}\|S(t)u_0\|_{\mathcal{PM}^a}\right) \leqslant C\|u_0\|_{\mathcal{PM}^2}.$$

Proof.

Simple estimates give

$$\begin{aligned}\sup_{t>0}\left(t^{a/2-1}\|S(t)u_0\|_{\mathcal{PM}^a}\right) &\leqslant \|u_0\|_{\mathcal{PM}^2}\sup_{\xi\in\mathbb{R}^3}\left(t^{a/2-1}|\xi|^{a-2}e^{-t|\xi|^2}\right)\\ &= C\|u_0\|_{\mathcal{PM}^2}\end{aligned}$$

where $C = \sup_{w\in\mathbb{R}^3}\left(|w|^{a-2}e^{-|w|^2}\right)$.

□

Estimates of the external force

Lemma

Let $2 \leqslant a < 3$. Assume that $F(t) \in \mathcal{PM}^{a-2}$ for all $t > 0$ and

$$\sup_{t>0} t^{a/2-1} \|F(t)\|_{\mathcal{PM}^{a-2}} < \infty.$$

There exists a constant C such that for $w(t) = \int_0^t S(t-\tau)F(\tau)\, d\tau$ it follows that

$$|||w|||_a \leqslant C \sup_{t>0} t^{a/2-1} \|F(t)\|_{\mathcal{PM}^{a-2}}.$$

Existence of regular solutions

Theorem
Let $a \in [2,3)$. There exists $\varepsilon > 0$ such that for every $u_0 \in \mathcal{PM}^2$ and $F \in \mathcal{C}_w([0,\infty), \mathcal{PM})$ satisfying

$$\|u_0\|_{\mathcal{PM}^2} + \|F\|_{\mathcal{C}_w([0,\infty),\mathcal{PM})} + \sup_{t>0} t^{a/2-1} \|F(t)\|_{\mathcal{PM}^{a-2}} < \varepsilon,$$

the Cauchy problem for the Navier-Stokes system has a solution in the space

$$\mathcal{Y}^a = \mathcal{C}_w([0,\infty), \mathcal{PM}^2) \cap \{u \ : \ \sup_{t>0} t^{a/2-1} \|u(t)\|_{\mathcal{PM}^a} < \infty\}.$$

This is the unique solution under the condition $\|\!|u|\!\|_a \leqslant 2\varepsilon$.

Interpolation inequality involving L^q and $\mathcal{PM}^a$ norms

Lemma

Fix $a \in (2,3)$. For every $q \in \left(3, \frac{3}{3-a}\right)$ there exists a constant $C = C(a,q)$ such that

$$\|v\|_{L^q(\mathbb{R}^3)} \leqslant C\|v\|_{\mathcal{PM}^2}^{1-\beta}\|v\|_{\mathcal{PM}^a}^{\beta}$$

for all $v \in \mathcal{PM}^2 \cap \mathcal{PM}^a$, where $\beta = \frac{1}{a-2}\left(1 - \frac{3}{q}\right)$.

Proof.

The Hausdorff–Young inequality (with $1/p + 1/q = 1$ and $p \in [1,2)$):

$$\begin{aligned}
\|v\|_q^p &\leqslant C\|\widehat{v}\|_p^p \leqslant C\|v\|_{\mathcal{PM}^2}^p \int_{|\xi|\leqslant R} \frac{1}{|\xi|^{2p}}\, d\xi + C\|v\|_{\mathcal{PM}^a}^p \int_{|\xi|>R} \frac{1}{|\xi|^{ap}}\, d\xi \\
&\leqslant C\|v\|_{\mathcal{PM}^2}^p R^{3-2p} + C\|v\|_{\mathcal{PM}^a}^p R^{3-ap}
\end{aligned}$$

for all $R > 0$ and C independent of v and R.

□

Corollary

Under the above assumptions, the constructed solution satisfies

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leqslant Ct^{-(1-3/q)/2}$$

for each $3 < q < 3/(3-a)$, *all* $t > 0$, *and* C *independent of* t.

Singular solutions

The Navier-Stokes system

$$\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad \operatorname{div} u = 0,$$

for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$.

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for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}^+$.

explicit stationary SINGULAR SOLUTIONS

$$u_1^c(x) = 2\frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, \qquad u_2^c(x) = 2\frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2},$$

$$u_3^c(x) = 2\frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \qquad p^c(x) = 4\frac{cx_1 - |x|}{|x|(c|x| - x_1)^2},$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $c \in \mathbf{R}$ is an arbitrary constant such that $|c| > 1$.

These explicit stationary solutions seem to be discovered first by Slezkin and described by Landau.

Independently, they were obtained by Tian and Xin (1998).

These are solutions of the singular Navier-Stokes problem

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p &= \kappa(c)(\delta_0, 0, 0), \quad (x,t) \in \mathbf{R}^3 \times (0, \infty), \\ \operatorname{div} u &= 0. \end{aligned}$$

Here, $\kappa(c) \to 0$ as $|c| \to \infty$.

Note that

$$(\delta_0, 0, 0) \in \mathcal{PM}^0.$$

It is easy to show that

$$u_k^c \in \mathcal{PM}^2 \qquad k \in 1, 2, 3.$$

Singular solution on a curve

We focus on the following initial value problem for the incompressible Navier-Stokes system with a singular force

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p &= \kappa\delta_{\gamma(t)}\bar{e}_1, \quad (x,t)\in \mathbf{R}^3\times(0,\infty),\\ \operatorname{div} u &= 0,\\ u|_{t=0} &= 0, \end{aligned}$$

where $\kappa \in \mathbf{R}$, $\bar{e}_1 = (1,0,0)$ and $\delta_{\gamma(t)}$ is the Dirac measure on $\mathbf{R}^3$ concentrated at the point $x = \gamma(t)$.

Singular solution on a curve

Theorem

Assume that $\gamma : [0,\infty) \to \mathbf{R}^3$ is Hölder continuous with an exponent $\alpha \in \left(\frac{1}{2}, 1\right]$ and denote by Γ the curve $\{(\gamma(t), t) \in \mathbf{R}^3 \times \mathbf{R}^+ : \ t > 0\}$.

There exists a vector field $u(x,t) = (u_1, u_2, u_3) \in L^\infty\big([0,\infty); L^{3,\infty}(\mathbf{R}^3)\big)$ and a pressure $p \in L^\infty\big([0,\infty); L^{\frac{3}{2},\infty}(\mathbf{R}^3)\big)$ such that

1. $u(x,t)$ and $p(x,t)$ satisfy the homogeneous Navier-Stokes system for all $(x,t) \in (\mathbf{R}^3 \times \mathbf{R}^+)\setminus\Gamma$ in the sense of distributions,
2. $u \in L^\infty_{\rm loc}\big((\mathbf{R}^3 \times \mathbf{R}^+)\setminus\Gamma\big)$,
3. for every $t > 0$
   - $u(\cdot,t) - V^c(\cdot - \gamma(t)) \in L^q(\mathbf{R}^3)$ for each $q \in \left(3, \frac{3}{2-2\alpha}\right)$,
   - $p(\cdot,t) - Q^c(\cdot - \gamma(t)) \in L^q(\mathbf{R}^3)$ for each $q \in \left(\frac{3}{2}, \frac{3}{3-2\alpha}\right)$,

   where (V^c, Q^c) denotes the Slezkin-Landau solution with fixed and sufficiently large $|c| > 1$.

Proof of Theorem

1. Work space: For every fixed $a \geqslant 0$, we set

$$\mathcal{PM}^a \equiv \left\{ v \in \mathcal{S}'(\mathbf{R}^3) : \ \widehat{v} \in L^1_{\mathrm{loc}}(\mathbf{R}^3), \ \|v\|_{\mathcal{PM}^a} \equiv \operatorname*{ess\,sup}_{\xi \in \mathbf{R}^3} |\xi|^a |\widehat{v}(\xi)| < \infty \right\}.$$

and the Banach space

$$\begin{aligned} \mathcal{Y}^a_T \quad &\equiv \quad C_{\mathrm{w}}\big([0,T), \mathcal{PM}^2\big) \\ &\qquad \cap \ \Big\{ v : (0,T) \to \mathcal{PM}^a : \ |||v|||_{a,T} \equiv \sup_{0<t\leqslant T} t^{a/2-1} \|v(t)\|_{\mathcal{PM}^a} < \infty \end{aligned}$$

for each $a \geqslant 2$ and $T \in (0, \infty]$. The space $\mathcal{Y}^a_T$ is normed by the quantity $\|v\|_{\mathcal{Y}^a_T} = |||v|||_{2,T} + |||v|||_{a,T}$ and of course, $\mathcal{Y}^2_\infty = C_{\mathrm{w}}\big([0,\infty), \mathcal{PM}^2\big)$ with this definition.

2. The main method: The Banach fixed point theorem.

By the Duhamel formula, we know that

$$\widehat{u}(\xi,t) = \int_0^t e^{-(t-\tau)|\xi|^2}\widehat{\mathbb{P}}(\xi)\; i\xi\cdot\big(\widehat{u\otimes u}\big)(\xi,\tau)\,\mathrm{d}\tau + \kappa\int_0^t e^{-(t-\tau)|\xi|^2}\widehat{\mathbb{P}}(\xi)e^{i\gamma(\tau)\cdot\xi}\bar{e}_1\,\mathrm{d}\tau$$

for all $t \geqslant 0$ and almost all $\xi \in \mathbf{R}^3$, where for two tempered distributions $u,\ v \in \big(\mathcal{PM}^2\big)^3$.

Back to the heat equation

Yanagida and his collaborators studied solutions of the linear heat equation with time-dependent singularities

$$\partial_t u - \Delta u = \delta_{\gamma(t)}, \quad (x,t) \in \mathbf{R}^3 \times \mathbf{R}^+, \tag{4}$$

with the singular force $\delta_{\gamma(t)}$ for every $t > 0$. It is shown that if a singularity is weaker than the order of the fundamental solution of the Laplace equation, then it is removable. Now, we denote by

$$\Phi(x,t) = (4\pi t)^{-\frac{3}{2}}\, e^{-\frac{|x|^2}{4t}}$$

the fundamental solution of the heat equation. Moreover, we define F in $\mathbf{R}^3 \times (0, T)$ by

$$F(x,t) = \int_0^t \Phi\big(x - \gamma(\tau), t - \tau\big)\,\mathrm{d}\tau,$$

which satisfies (4) in the distribution sense.

Theorem
Let $\Gamma = \{(\gamma(t), t),\ t > 0\} \subset \mathbf{R}^3 \times \mathbf{R}^+$ be a curve, where $\gamma = \gamma(t)$ is Hölder continuous of exponent $\alpha \in (\frac{1}{2}, 1]$. There exists a function $u(x, t)$ such that

1. u is regular in $(\mathbf{R}^3 \times \mathbf{R}^+)\backslash\Gamma$;
2. $\partial_t u = \Delta u$, for all $(x, t) \in (\mathbf{R}^3 \times \mathbf{R}^+)\backslash\Gamma$;
3. we have the decomposition $u(x, t) = \omega_0(x, t) + \frac{1}{4\pi|x-\gamma(t)|}$, where the function ω_0 satisfies $\|\omega_0(\cdot, t)\|_{L^q(\mathbf{R}^3)} \leqslant Ct^{\frac{1}{2}\left(\frac{3}{q}-1\right)}$ for every $q \in (3, \frac{3}{2-2\alpha})$ and $t \in (0, T]$ with a constant $C = C(q, \alpha, T) > 0$.