Nonlinear dispersive equations

Herbert Koch

Universität Bonn

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Linear dispersion

\[ iu_t + p\left(\frac{1}{i} \nabla\right)u = 0 \]

where \( p\left(\frac{1}{i} \partial\right) \) is self-adjoint differential operator. Fourier transform:

\[ i\partial_t \hat{u} + p(\xi)\hat{u} = 0 \]

\[ \hat{u}(t, \xi) = e^{itp(\xi)}\hat{u}(0, \xi) \]
Examples

1. The Schrödinger equation in $\mathbb{R} \times \mathbb{R}^n \ni (t, x)$, $p(\xi) = -|\xi|^2$

$$i\partial_t u + \Delta u = 0$$

2. The Airy equation in $\mathbb{R} \times \mathbb{R} \ni (t, x)$, $p(\xi) = \xi^3$

$$\partial_t u + \partial_{xxx} u - 3uu_x = 0$$

3. The linear Kadomtsev-Petviashvili equations in $\mathbb{R} \times \mathbb{R}^2 \ni (t, x, y)$, $p(\xi, \eta) = \xi^3 \pm \frac{\eta^2}{\xi}$, $v = -(3\xi^2 \mp \frac{\eta^2}{\xi^2}, \frac{\eta}{\xi})$

$$\partial_x (\partial_t u + \partial_{xxx} u - 3u\partial_x u) \pm u_{yy} = 0$$

4. The half wave equation

$$i\partial_t u + \sqrt{1 + \frac{1}{i} \nabla|^2} u = 0.$$
What is dispersion?

1. The group velocity depends on the frequency. For compactly supported smooth initial data the wave decays pointwise, despite the conservation of the $L^2$ norm.

2. The characteristic set $\{(\tau, \xi) : \tau = p(\xi)\}$ is curved. Stationary phase: Curvature leads to pointwise decay of the fundamental solution. The main contribution to

$$\int e^{i(x \cdot \xi + t p(\xi))} d\xi$$

comes from points with stationary phase $\frac{x}{t} = -\nabla p(\xi)$ - the group velocity at frequency $\xi$ is $-\nabla p(\xi)$

3. Decay of the fundamental solution.
Bow Wave
Bow Wave, Berry (Bristol)
The nonlinearity may cooperate with dispersion (defocusing), or work against it (focusing). (KPII)
Outline

1. The spaces $U^p$ and $V^p$
2. Strichartz estimates and bilinear estimates
3. Nonlinear dispersive equations
4. Dynamics near solitons
Motivation

We consider

\[ iu_t + Au = f(u) \]

where \( A \) is self adjoint and \( f \) is nonlinear. Search function spaces \( X \) and \( Y \) so that

- \( X \ni u \to f(u) \in Y \) is smooth (polynomial)
- There is a unique solution \( u \in X \) to data \( f \in Y \) and \( u_0 \in H \) such that \( \|u\|_X \leq c(\|u_0\|_H + \|f\|_Y) \).

Then a solution can be constructed as fixed point of the map which maps \( u \) to the solution \( J(u) \) with initial data \( u_0 \) and right hand side \( f(u) \).
Motivation

One needs

- A radius $R$ so that, if $\|u\|_X \leq R$,
  \[ \|J(u)\|_X \leq c (\|u_0\|_H + \|f(u)\|_Y) \leq R \]

- The map $J(u)$ has a small Lipschitz constant
  \[ \|J(u) - J(v)\|_X \leq c \|f(u) - f(v)\|_Y \leq \mu \|u - v\|_X. \]

There are usually (but not always) two ways to achieve that: Small time, or small data. In this setting the assumptions of the implicit function theorem are satisfied.
Example: The nonlinear Schrödinger equation

We begin with the nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u = |u|^{\frac{4}{n}} u. \]

Here

\[ H = L^2(\mathbb{R}^n), \quad X = L^{\frac{2(n+2)}{n}}(\mathbb{R} \times \mathbb{R}^n), \quad Y = L^{\frac{2(n+2)}{n+4}}(\mathbb{R} \times \mathbb{R}^n). \]

The estimates are

- The estimate of the nonlinearity

\[ \| |u|^{\frac{4}{n}} u \|_{L^2}^{\frac{2(n+2)}{n+4}} = \| u \|_{L^\frac{2(n+2)}{n+4}}^{\frac{4}{n} + 1}. \]

- Strichartz estimate

\[ \| u \|_{L^\frac{n+2}{n} \left( \mathbb{R} \times \mathbb{R}^n \right)} \leq c \left( \| u_0 \|_{L^2(\mathbb{R}^n)} + \| i\partial_t u + \Delta u \|_{L^\frac{2(n+2)}{n+4}} \right). \]
Desired properties

We want the following properties of the function spaces.

1. Heritates Strichartz estimates as embeddings, bilinear estimates, and 'high modulation estimates' in the elliptic part.

2. Allow duality arguments.

Example:

\[ u_t = f \]

Would want: \( X = \dot{H}^{1/2} \) and \( Y = \dot{H}^{-1/2} \). Then

\[ \int u f \, dt \leq \| u \|_{\dot{H}^{1/2}} \| f \|_{\dot{H}^{-1/2}} \]

would give everything. Not true! How close can we get? This is relevant in probability and dispersive equations.
**Bounded \( p \) variation**

A partition is defined by an increasing (finite) sequence \( \tau = (t_j)_{0 \leq j \leq N} \),

\[
t_0 < t_1 \cdots < t_n \leq \infty.
\]

We denote the set of all partition by \( \mathcal{T} \). Let \( \mathcal{S} \) be the set of all step functions with finitely many discontinuities (*test functions*) and \( \mathcal{R} \) the set of ruled functions with left and right limits everywhere including \( \pm \infty \) (*distributions*). We define \( v(\infty) = 0 \). All these functions are bounded.

For a function space \( X \) we denote by \( X_{rc} \) the subspace of right continuous functions with limit 0 at \( -\infty \).
The regulated Stieltjes integral

\[ \mathcal{R} \times S_{rc} \ni (v, u) \rightarrow \int v \, du = \sum_j v(t_j)(u(t_j) - u(t_{j-1})) \]

defines a duality. We equip \( S_{rc} \) with the norm

\[ \|u\|_{BV} = \sum_j |u(t_j) - u(t_{j-1})|. \]

and \( \mathcal{R} \) with the supremum norm. If \( L : S_{rc} \rightarrow \mathbb{R} \) is continuous we define

\[ v(t) = L(\chi_{[t, \infty)}) \]

It is in \( \mathcal{R} \).
The spaces $U^p$ and $V^p$

Definition of $V^p$

Let $1 \leq p < \infty$.

**Definition (Definition of $V^p$)**

We define the space $V^p$ as the space of all functions such that the norm

$$
\|u\|_{V^p} = \sup_\tau \left( \sum |u(t_{i+1}) - u(t_i)|^p \right)^{1/p}
$$

is finite.
The definition of $U^p$

We call $a$ a $p$ atom if there exist a partition $\tau$ and $\phi_i$ with $\sum |\phi_i|^p \leq 1$ and

$$a = \sum \phi_i \chi_{[t_i, t_{i+1})}.$$

Definition

The $p$ atoms define an atomic space $U^p$ by

$$\|u\|_{U^p} = \inf \left\{ \sum |\lambda_j| : u = \sum \lambda_j a_j \right\}.$$

(there exist atoms $a_j$ and numbers $\lambda_j$ with $u = \sum_j \lambda_j a_j$).
Density

Theorem

We have

\[ S_{rc} \subset U^p \subset V^p \subset \mathcal{R}. \]

Step functions are dense in \( U^p \). If \( p \leq q \) then

\[ U^p \subset U^q, \quad V^p \subset V^q. \]

Step functions are also dense in \( V^p \), but the proof requires duality.
Interpolation

Lemma

Let $1 \leq p < q < \infty$. There exists $C > 0$ so that for all $M > 0$ and $v \in V^p_{rc}$ there exist $u \in U^p$ and $w \in U^q$ with

$$v = u + w, \quad \frac{1}{M} \|u\|_{U^p} + e^M \|w\|_{U^q} \leq C \|v\|_{V^p}$$

Corollary

Let $1 \leq p < q < \infty$. Then

$$U^p \subset U^q, \quad V^p \subset V^q$$

and

$$V^p_{rc} \subset U^q \subset V^q$$
The spaces $U^p$ and $V^p$

Proof

This is proven via a sort of parametrization invariant Littlewood-Paley decomposition. Without loss of generality we assume that $\|v\|_{V^p} = 1$. If $t_1 \leq t_2 \leq t_3$ then

$$\|\chi_{[t_1,t_2)}(v - v(t_2))\|_{V^p}^p + \|\chi_{[t_2,t_3)}(v - v(t_3))\|_{V^p}^p \leq \|\chi_{[t_1,t_2)}(v - v(t_3))\|_{V^p}^p.$$ 

We choose

$$t_{k,0} = -\infty, t_{k,1} = \infty, u^0 = \lim_{t \to -\infty} u(t)$$

$$t_{k,2j} = t_{k-1,j}$$

$$t_{k,2j-1} = \sup \left\{ t : t < t_{k,2j}, \|\chi_{(-\infty,t)}(v - v(t))\|_{V^p}^p \geq \|\chi_{(-\infty,t_{k,2j})}(v - v(t_{k,2j}))\|_{V^p}^p \right\}$$

$$u^k(t) = \sum_j \left( v(t_{k,2j-1}) - v(t_{k,2(j-1)}) \right) \chi_{[t_{k,2j-1},t_{k,2j})}. $$
Proof of interpolation II

Let

\[ u = \sum_{k=1}^{k_0} u^k, \quad w = v - u \]

Then

\[ \| v - \sum_{j=0}^{k_0} u^j \|_{\sup} \leq 2^{-k_0/p} \]

\[ \| u^k \|_{\sup} \leq 2^{-k/p}, \quad \# \tau(u^k) \leq 2^k. \]

\[ \| u^k \|_{Ur} \leq 2^{-k(\frac{1}{p} - \frac{1}{r})}. \]

and we arrive at

\[ \| u \|_{U^p} \leq k_0, \quad \| w \|_{U^q} \leq \frac{1}{1 - 2^{\frac{1}{q} - \frac{1}{p}}} 2^{-k_0(\frac{1}{p} - \frac{1}{q})}. \]
Duality

Recall

\[ B(v, u) = \sum v(t_j)(u(t_j) - u(t_{j-1})) \].

Theorem

Let \( \frac{1}{q} + \frac{1}{p} = 1 \), \( 1 < p, q < \infty \). The bilinear map defines a unique bilinear map \( B : V^p \times U^q \) such that

\[ V^p \ni v \rightarrow (u \rightarrow B(v, u)) \in (U^q)^* \]

is an isometric isomorphism.
Proof of Duality

Proof.

For atoms (after an integration/summation by parts, with \( t_N = \infty \))

\[
B(v, a) \leq \sum |v(t_{j+1}) - v(t_j)||a(t_j)| \leq \|v\|_{V^p}.
\]

This gives the bound. If \( L \in (U^q)^* \) define

\[
v(t) = L([t, \infty)).
\]
This is a generalization of Young's integral (1912).

\[ \int_{a}^{b} uv' dt \]

with \( u \in V^p \cap C, v \in V^q \cap C, \frac{1}{p} + \frac{1}{q} > 1. \)

**Lemma**

*Step functions are dense in \( V^p \). Test functions are weak* dense in \( V^p \).*

**Proof.**

Let \( \tilde{V}^p \) be the closure of the step functions in \( V^p \). Let \( X \subset U^q \) be the set of all functions for which \( B(v, u) = 0 \) for all \( v \in \tilde{V}^p \). Since \( u(t) = -B(\chi_{[t, \infty)}, u) \) the set \( X \) is trivial. Then \( \{0\} = X^* = V^p / \tilde{V}^p \).

Let \( \tilde{V}^p \) be the weak closure of \( C_0^\infty \) and let \( (\tilde{V}^p)^\perp \subset U^q \) be the functions which are orthogonal. It suffices to show that \( u(0) = 0 \) for all functions in \( (\tilde{V}^p)^\perp \). This requires a simple explicit construction. \(\square\)
The spaces $U^p$ and $V^p$

**Duality 2**

**Theorem**

*The space $U^q$ is the dual space to*

\[ V^p_C := \{ v \in V^p \cap C(\mathbb{R}) : v(t) \to 0 \text{ as } t \to \infty \} \]

$C^\infty_0 \subset U^p$ *is weak*\(^*\) *dense.*

In particular is suffices to test by smooth functions.

**Proof.**

It suffices to find a representation of a linear functional $L$. We reverse time. Then $U^p_C = U^p \cap C \subset V^p_C$. Thus $L \in (U^p_C)^*$. By Hahn Banach there is an extension $\tilde{L} \in (U^p)^*$. By the duality theorem it can be represented by a function $g \in V^q$ which we can choose to be right continuous. Now we integrate by parts. The weak* density is almost obvious. \(\square\)
We can consider sequence spaces $u_p$ and $v_p$ on sequences $(u_j)_{j \in \mathbb{N}}$. Let $v_p^0$ be the subspace of sequences converging to 0. Then

- $(v_p^0)^* = u_p$
- $u_p^* = v_p$
- $v_p^0 \subset v_p$ has codimension 1

The space $v_2^0$ has been introduced by James (1951) because of this property. These spaces played a role in the study of Banach spaces by Pisier.
Relation to function spaces

An almost trivial computation implies

$$\|u\|_{V^p([0,1])} \leq \|u\|_{\dot{C}^{1/p}([0,1])}.$$ 

Lemma

Let $\phi \in C_0^\infty(\mathbb{R})$ with $\int \phi = 1$. Then

$$\|f \ast \phi\|_{L^p} \leq c\|f\|_{V^p}.$$ 

Moreover

$$B_{p,1}^{1/p} \subset U^p \subset V_{rc}^p \subset B_{p,\infty}^{1/p}.$$ 

In particular

$$\|u^\Lambda\|_{L^p} \leq c\Lambda^{-1/p}\|u^\Lambda\|_{V^p}.$$
Solving ODEs

We consider the initial value problem \( u_t = f, \ u(0) = u_0 \).

**Theorem**

*Suppose that \( f \) is a distribution and

\[
\| f \|_{DV^p} = \sup \left\{ \int f \phi dx : \phi \in C_0^\infty, \| \phi \|_{U^q} \leq 1 \right\} < \infty.
\]

Then there exists a unique solution \( u \in V^p \) with

\[
\| u \|_{V^p} \leq \| f \|_{DV^p} + |u_0|.
\]

*Then there exists a unique solution \( u \in U^p \) for \( t \geq 0 \) with

\[
\| u \|_{U^p} \leq \| f \|_{DU^p} + |u_0|.
\]
The Brownian $B_t$ motion satisfies for all $p, q > 2$

$$\left\| \left\| B_t \chi_{[0,1]} \right\|_{V^p} \right\|_{L^q} \leq c_{p,q}$$

The space $V^p$ are invariant under reparametrization, and reparametrizations of the Brownian motion are in $V^p$. 
Stochastic differential equation lead to integrals

$$\int f \, dg$$

where typically $g$ is the Brownian motion and $f$ is a local martingal. The integrals are pathwise defined if $g \in U^2$. This is not the case, and we need the Ito- or the Stratonovitch integral to integrate the Brownian motion.
T. Lyons has observed that one may enhance the Young integral by defining the Levy process by stochastic integration, and then a rough path integral depending only on the path and the Levy area process. Hairer and Gubinelli have extended these ideas to partial differential equations.
Modifications

- Functions spaces on bounded intervals: Extend functions in $V^p$ by zero to the right, and constant to the left, and functions in $U^p$ constant to the right, and by zero to the left.
- Values in Hilbert/Banach spaces. If $q \geq p$ by Minkowski’s inequality
  \[
  \|v\|_{V^q(L^p)} \leq \|v\|_{L(V^q)}
  \]
  \[
  \|u\|_{L^q(U^p)} \leq \|u\|_{U^p(L^q)}.
  \]
- Pull back a unitary evolution
  \[
  \|u\|_{U^p} = \|e^{-itP(D)}u\|_{U^p}
  \]
The spaces $U^p$ and $V^p$
The spaces $U^p$ and $V^p$

- $V^p$: Bounded $p$ variation.
- $p$- atom: $a = \sum \phi_j \chi_{[t_j, t_{j+1})}$, $\sum |\phi_j|^p = 1$.
- $U^p$: $u = \sum \lambda_j a_j$.
- $T : U^p \to X$, $\|T\|_{L(U^p, X)} = \sup \|Ta\|$.
- Duality: $V^p \times U^q \ni (v, u) \to B(v, u) = \int vdu$ defines an isometric isomorphism $V^q \to (U^p)^*$ and $U^p \to (V^q_C)^*$.
- Embeddings
  $$B^{1/p}_{p,1} \subset U^p \subset V^p_{rc} \subset B^{1/p}_{p,\infty}.$$ 
- High modulation estimate
  $$\|u^{>\Lambda}\|_{L^p} \leq c \Lambda^{-\frac{1}{p}} \|u\|_{V^p}$$
- Step functions are dense. Test functions are weak* dense.
Adaptation to operator

- Values in $L^2$.

$$\sup_t \|u(t)\|_{L^2} \leq \|u\|_{V^p} \leq c\|u\|_{U^p} \leq c\|u\|_{BV}.$$  

- Consider

$$i\partial_t u + Au = 0$$

- Pull back

$$\|u\|_{U^p_A} = \|e^{-itA}u(t)\|_{U^p}$$

$$\|v\|_{V^p_A} = \|e^{-itA}v(t)\|_{V^p}.$$
Solving differential equations

To solve

\[ i\partial_t u + Au = f \]

in \( V^p \) prove

\[ \int_0^\infty \langle f, \phi \rangle_{L^2} dt \leq C_1 \]

for \( \phi \in C_0^\infty \) with \( \| \phi \|_{U^q} \leq 1 \). Then there exists a unique solution \( u \) (distributional with values in \( L^2 \)) with

\[ \| u \|_{V^p} \leq \| u_0 \|_{L^2} + C_1. \]

Similarly with \( U^p \).
The linear Schrödinger equation

\[ i\partial_t u + \Delta u = 0 \]

has a fundamental solution

\[ g_t(x) = ((4\pi it)^{1/2})^{-n} e^{-\frac{|x|^2}{4it}} \]

with Fourier transform

\[ \hat{g}_t(x) = e^{it|\xi|^2} \]

hence

\[ \|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad \|u(t)\|_{L^\infty} \leq |4\pi t|^{-n/2} \|u_0\|_{L^1} \]
Strichartz estimates

1. Interpolation: \( \| u(t) \|_{L^p'} \leq |4\pi t|^{-n\left(\frac{1}{p} - \frac{1}{2}\right)} \| u_0 \|_{L^p(\mathbb{R}^n)}. \)

2. Duhamel’s formula, weak Young (Keel-Tao)

\[
\| u \|_{L^{r'}_t L^p'(\mathbb{R}^n)} \leq c \| i\partial_t u + \Delta u \|_{L^r_t L^p(\mathbb{R}^n)}
\]

if \( r, p \geq 2, (r, p, n) \neq (2, 1, 2) \) and

\[
2\left(\frac{1}{r} - \frac{1}{2}\right) + n\left(\frac{1}{p} - \frac{1}{2}\right) = 1
\]

3. A \( TT^* \) argument gives

\[
\| u \|_{L^\infty L^2} + \| u \|_{L^{r'}_t L^p'} \leq c \left( \| u_0 \|_{L^2} + \| i\partial_t u + \Delta u \|_{L^r L^p} \right)
\]
Strichartz estimates

The pointwise decay follows typically by stationary phase. For the Airy equation

\[ u_t + u_{xxx} = 0 \]

there is a fundamental solution (up to constants)

\[ g(t, x) = t^{-1/3} \text{Ai}(x/t^{1/3}) \]

where

\[ \text{Ai}(x) = \int e^{i(x\xi + \xi^3)} \]

The Lemma of van der Corput implies that \( \text{Ai} \) is bounded. Stationary phase implies that half a derivative is bounded.

The half wave equation

\[ i\partial_t u \pm |D|u = 0 \]

(and hence the wave equation) has a characteristic set with \( n - 1 \) nonvanishing curvatures. This implies a shift of the exponents compared to the Schrödinger equation.
Strichartz estimates and embedding for $U^p$ and $V^p$

Let $A$ be a selfadjoint operator on $L^2$. Then

$$i\partial_t u + Au = 0$$

generates the unitary group $S(t) = e^{itA}$.

**Definition**

$$U^p_A = S(t)U^p, V^p_A = S(t)V^p$$

$$\|u\|_{U^p_A} = \|S(-t)u(t)\|_{U^p_A}, \quad \|u\|_{V^p_A} = \|S(-t)u(t)\|_{V^p_A}.$$ 

**Theorem**

*Suppose the unitary group admits Strichartz estimates with the exponents $r, p$. Then*

$$\|u\|_{L^{r'} L^p} \leq c \|u\|_{U^{r'}_A}$$

*and*

$$\|u\|_{V^r_A} \leq c \|(i\partial_t + \Delta)u\|_{L^r L^p}.$$
Proof

It suffices to prove the first estimate for atoms, and hence for free solutions. Similarly for the second part we choose a partition. Then the estimate reduces to proving them for a fixed interval of the partition. But this is equivalent to the Strichartz estimate for free solutions.
The Fourier transform of free waves

Consider

\[ i\partial_t u - \phi(D) u = 0 \]

where \( \phi \) is a real valued function, and \( \phi(D) \) is the Fourier multiplier. Then

\[ i\partial_t \hat{u} = \phi(\xi) \hat{u} \]

and

\[ \hat{u}(t, \xi) = e^{-it\phi(\xi)} \hat{u}(0, \xi) \]

Thus

\[ \int F_{t,x} u \psi d\xi d\tau = \int_{\mathbb{R}^n} \psi(\phi(\xi), \xi) u_0(\xi) d\tau d\xi \]

hence

\[ F_{t,x} u = \hat{u}_0(\xi) \delta_{\tau - \phi}. \]
Convolution estimates

Let $\Sigma_j$, $j = 1, 2$ be two hypersurfaces defined by $\Xi_j(\xi) = 0$. We search estimates

$$\|uv\|_{L^2} \leq C \|\hat{u}\|_{L^2(\delta_{\Xi_1})} \|\hat{v}\|_{L^2(\delta_{\Xi_2})}$$

where by and abuse of notation the Fourier transform of $\hat{u}$ resp $\hat{v}$ is $\hat{u}\delta_{\Xi_1}$ resp. $\hat{u}\delta_{\Xi_1}$. By Plancherel this reduces to

$$\| (\hat{u}\delta_{\Xi_1}) \ast (\hat{v}\delta_{\Xi_2}) \|_{L^2} \leq C \|\hat{u}\|_{L^2(\delta_{\Xi_1})} \|\hat{v}\|_{L^2(\delta_{\Xi_2})}.$$
The calculation

We approximate the Dirac function by smooth functions. Then, with nonnegative functions $h_1$ and $h_2$,

$$
\|uh_1 * vh_2\|_{L^2}^2
= \int \left| \int u(\xi - \eta)h_1^{1/2}(\xi - \eta)h_2^{1/2}(\eta)v(\eta)h_2^{1/2}(\eta)h_1^{1/2}(\xi - \eta)d\eta \right|^2 d\xi
\leq \int \int |u(\eta)|^2h_1(\eta)h_2(\xi - \eta)d\eta \int |v(\eta)|^2h_2(\eta)h_1(\xi - \eta)d\eta d\xi
\leq C_h^2\|h_1^{1/2}u\|_{L^2}^2\|h_1^{1/2}v\|_{L^2}^2(\delta_{\Xi_2})
$$

where

$$
C_h^2 = \sup_{\xi_1,\xi_2} \int h_1(\xi - \xi_1)h_2(\xi - \xi_2)d\xi.
$$
The calculation

We set $h_i = j_k \circ \Xi_i$ for a Dirac sequence and obtain in the limit by the coarea formula with respect to $h_i$

$$C^2 = \sup_{\xi_1 \in \Sigma_1, \xi_2 \in \Sigma_2} \int \delta_{\Xi_1}(\xi - \xi_2) \delta_{\Xi_2}(\xi - \xi_1) d\xi. \quad (1)$$

For this limit we used the coarea formula: $\phi : U \to V \subset \mathbb{R}^m, m < n$.

$$\int_U \det(D\phi D\phi^T)^{1/2} f(x) dm^n(x) = \int_V \int_{\phi^{-1}(y)} f(x) dH^{n-m} dm^m(y)$$
A reduction

Consider

\[ \Xi_1(\tau, \xi) = \tau - \phi_1(\xi), \quad \Xi_2(\tau, \xi) = \tau - \phi_2(\xi) \]

In this case the formula (1) can be considerably simplified.

Lemma

\[ \int_{-\infty}^{\infty} \delta_{\Xi_1} \delta_{\Xi_2} \, dt = \delta_{\phi_1 - \phi_2} \]

Proof.

By the calculation of Gram determinants

\[ \delta_{\Xi_1} \delta_{\Xi_2} = \delta_{\Xi_1} \delta_{\Xi_2 - \Xi_1} \]

and hence

\[ \delta_{\tau - \phi_1} \delta_{\tau - \phi_2} = \delta_{\tau - \phi_1} \delta_{\phi_2 - \phi_1} \]

Now the formula follows by an application of Fubini’s theorem.
Local smoothing

Consider $d = 1$, $\Xi_1 = \tau - \xi^3$, $\Xi_2 = \tau$. The equation $\xi_1 = (\xi - \xi_2)^3$ with the unique solution $\xi = \xi_2 - \xi_1$. The gradients are 0 and $3\xi_2^2$. Hence

$$\int (\xi f)\delta_{\tau - \xi^3} \ast (g\delta_{\tau}) d\eta \frac{1}{\sqrt{3}} \leq \|f\|_{L^2[\delta(\tau - \xi^3)]} \|g\|_{L^2(\mathbb{R})}.$$ 

This gives the local smoothing estimate below.

**Theorem**

*Let $u$ be the solution to the Airy equation with initial data $u_0$ given by the Fourier transform. Then*

$$\sup_x \left( \int |u_x(t, x)|^2 dt \right)^{1/2} \leq \frac{1}{\sqrt{3}} \|u_0\|_{L^2}$$
Local smoothing

Proof.

We apply the previous formula with $u$ and a sequence $g_j(x)$ so that $g_j^2$ is Dirac measure. Then

$$
\| (\partial_x u(x, t)) g_j(x) \|_{L^2} \to \left( \int |\partial_x u(t, 0)|^2 \, dx \right)^{1/2}
$$

and

$$
\| (\partial_x u) g_j \|_{L^2(\mathbb{R} \times \mathbb{R})} \leq c \| u(0) \|_{L^2(\mathbb{R})}
$$

In the limit we obtain the bound at $x = 0$, and by translation we obtain the general bound.
One space dimension

Consider $\Xi_1 = \Xi_2 = \tau - \xi^3$.

**Theorem**

*Suppose that $u$ and $v$ satisfy the Airy equation. Then*

$$|||D_1|^2 - |D_2|^2|^{1/2}uv||_{L^2} \leq c||u_0||_{L^2}||v_0||_{L^2}.$$

**Proof.**

This is roughly a gain of one derivative. The proof requires going through the proof of the bilinear estimate with a bilinear multiplier,

$$\int |\hat{\eta}_1^2 - \hat{\eta}_2^2|^{1/2} |\hat{u}(t, \eta_1)\hat{u}(t, \eta_2)| d\eta_1.$$

This exactly compensates for $C$ without further changes.
Schrödinger equations

Theorem

Let $a, b \in \mathbb{R}\setminus 0$ and

$$ia\partial_t u + \Delta u = 0$$

$$ib\partial_t v + \Delta v = 0$$

Suppose that the Fourier transform of $u$ is supported in $B_R(\xi_0)$. Then, if $a = b$

$$\| |D_1 - D_2|^{1/2}(uv)\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq cR^{n-1/2} \| u \|_{L^2(\mathbb{R}^n)} \| v \|_{L^2(\mathbb{R}^n)}.$$
Proof of bilinear estimates for Schrödinger 1

Proof.

Let $\tau_1 = |\xi_1|^2$ and $\tau_2 = |\xi_2|^2$. The equations

$$\tau - |\xi_2|^2 - |\xi - \xi_2|^2 = 0 = \tau - |\xi_1|^2 - |\xi - \xi_1|^2$$

lead to

$$\langle \xi, \xi_2 - \xi_1 \rangle = 0.$$ 

We integrate out $\tau$ in the definition of $C$ and get with

$$\phi(\xi) = 2\langle \xi, \xi_2 - \xi_1 \rangle$$

$$C^2 \leq \sup_{\xi_1, \xi_2} \int_{B_R(\xi_0 - \xi_2)} \delta_2 \langle \xi, \xi_2 - \xi_1 \rangle \, d\xi = 2|\xi_2 - \xi_1|^{-1}|B_1| R^{1-n}. $$
It is worthwhile to explore the case $a \neq b$. Then we obtain a codimension 2 parabola.
**Kadomtsev-Petviashvili II, 2d**

The Kadomtsev-Petviashvili II equation is

\[
\partial_t u + u_{xxx} + \partial_x^{-1} u_{yy} + \partial_x u^2 = 0.
\]

where \((t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}\), and \((\tau, \xi, \eta)\) are the corresponding Fourier variables. The Strichartz estimate is

\[
\|u\|_{L^4(\mathbb{R}^3)} \leq c \|u\|_{U^4}.
\]

The bilinear estimate is (with \(u_\lambda\) the Littlewood Paley projection to \(\lambda \leq |\xi| \leq 2\lambda\))

\[
\|u_\mu u_\lambda\|_{L^2} \leq c \left(\frac{\mu}{\lambda}\right)^{1/2} \|u_\mu\|_{L^2} \|u_\lambda\|_{L^2}.
\]

Here

\[
\hat{u}_\lambda = \chi_{\lambda \leq |\xi| < 2\lambda} \hat{u}
\]

is a Littlewood-Paley decomposition with respect to \(x\).
Kadomtsev-Petviashvili II, 3d

With two $y$ directions for $\mu \leq \lambda/4$

$$\|u_\lambda\|_{L^4(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^2)} \leq c\lambda^{1/2}\|u_\lambda(0)\|_{L^2(\mathbb{R} \times \mathbb{R}^2)}$$

and

$$\|u_\mu u_\lambda\|_{L^2} \leq c\mu\|u_\mu\|_{L^2}\|u_\lambda\|_{L^2}.$$
Proof of the bilinear estimate

By the reduction

\[ C = \sup \int \delta \phi(\xi_1, \eta_1) + \phi(\xi - \xi_1, \eta - \eta_1) - \phi(\xi_2, \eta_2) - \phi(\xi - \xi_2, \eta - \eta_2) \]

with \( \mu \leq |\xi_1| \leq 2\mu \), \( \lambda \leq |\xi_2| \leq 2\lambda \) and \( \phi = \xi^3 - \frac{|\eta|^2}{\xi} \). The set is given by

\[ \phi(\xi_1, \eta_1) + \phi(\xi - \xi_1, \eta - \eta_1) = \phi(\xi_2, \eta_2) + \phi(\xi - \xi_2, \eta - \eta_2) \]

An algebraic calculation gives

\[ \phi(\xi_2, \eta_2) - \phi(\xi_1, \eta_1) - \Phi(\xi_1 - \xi_2, \eta_1 - \eta_2) + 3(\xi_1 - \xi_2)(\xi - \xi_1)(\xi - \xi_2) \]

\[ = \phi(\xi - \xi_1, \eta - \eta_1) - \Phi(\xi - \xi_2, \eta - \eta_2) + \Phi(\xi_1 - \xi_2, \eta_1 - \eta_2) \]

\[ + 3(\xi_1 - \xi_2)(\xi - \xi_1)(\xi - \xi_2) \]

\[ = (\xi_2 - \xi_1)(\xi - \xi_1)(\xi - \xi_2) \left( \frac{|\eta - \eta_1|}{\xi - \xi_1} - \frac{|\eta - \eta_2|}{\xi - \xi_2} \right)^2 \]

\[ \leq \left( \frac{|\xi_1 - \xi_2|}{|\xi_1 - \xi_2|} \right)^2 \]
Proof

This describes a circle in the $\eta$ variables, or a point, or the empty set, for fixed $\xi_1, \xi_2, \xi, \eta_1$ and $\eta_2$. We apply Fubini’s theorem and integrate in the $\eta$ variables. This integral does not depend on the radius:

$$\int_{\mathbb{R}^2} \delta_a(|x|^2 - R^2) \, dx = |a|^{-1} \pi.$$ 

We are left with

$$\frac{\pi}{2|\xi_2 - \xi_1|} \int_{|\xi - \xi_2| \leq \mu} |\xi - \xi_2| |\xi - \xi_1| \, d\xi \leq \pi \mu^2.$$
**Connection to $U^2$**

There is a general argument that deduces bilinear estimates with respect to $U^2$.

**Theorem**

*Suppose that $u$ and $v$ are solutions to the dispersive equations*

\[
i \partial_t u - p_1(D)u = 0 = i \partial_t v - p_2(D)v
\]

\[
\left\| \int k(\xi, \eta) \hat{u}(t, \xi - \eta) \hat{v}(t, \eta) \right\|_{L^2} \leq c \|u_0\|_{L^2} \|v_0\|_{L^2}
\]

*Then*

\[
\left\| \int k(\xi, \eta) \hat{u}(t, \xi - \eta) \hat{v}(t, \eta) \right\|_{U^2_{p_1}} \leq c \|u_0\|_{U^2_{p_1}} \|v_0\|_{U^2_{p_2}}.
\]
The spaces $U^p$ and $V^p$

- **$V^p$:** Bounded $p$ variation.
- **$p$-atom:** $a = \sum \phi_j \chi_{[t_j, t_{j+1})}$, $\sum |\phi_j|^p = 1$.
- **$U^p$:** $u = \sum \lambda_j a_j$.
- **$T: U^p \to X$, $\|T\|_{L(U^p, X)} = \sup \|Ta\|$.
- **Duality:** $V^p \times U^q \ni (v, u) \to B(v, u) = \int v du$ defines an isometric isomorphism $V^q \to (U^p)^*$ and $U^p \to (V^q_C)^*$.
- **Embeddings**
  
  $$B^{1/p}_{p,1} \subset U^p \subset V^p_{rc} \subset B^{1/p}_{p,\infty}.$$

- **High modulation estimate**
  
  $$\|u^{>\Lambda}\|_{L^p} \leq c\Lambda^{-\frac{1}{p}} \|u\|_{V^p}$$

- **Step functions are dense. Test functions are weak* dense.**
Adaptation to operator

- Values in $L^2$.

$$\sup_t \|u(t)\|_{L^2} \leq \|u\|_{V^p} \leq c\|u\|_{U^p} \leq c\|u\|_{BV}. $$

- Consider

$$i\partial_t u + Au = 0$$

- Pull back

$$\|u\|_{U^p_A} = \|e^{-itA}u(t)\|_{U^p}$$

$$\|v\|_{V^p_A} = \|e^{-itA}v(t)\|_{V^p}.$$
Solving differential equations

To solve

\[ i\partial_t u + Au = f \]

in \( V^p \) prove

\[ \int_0^\infty \langle f, \phi \rangle_{L^2} \, dt \leq C_1 \]

for \( \phi \in C_0^\infty \) with \( \|\phi\|_{U^q} \leq 1 \). Then there exists a unique solution \( u \) (distributional with values in \( L^2 \)) with

\[ \|u\|_{V^p} \leq \|u_0\|_{L^2} + C_1. \]

Similarly with \( U^p \).
Strichartz estimates

The linear Schrödinger equation

\[ i\partial_t u + \Delta u = 0 \]

has a fundamental solution

\[ g_t(x) = ((4\pi it)^{1/2})^{-n} e^{-\frac{|x|^2}{4it}} \]

with Fourier transform

\[ \hat{g}_t(x) = e^{it|\xi|^2} \]

hence

\[ \|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad \|u(t)\|_{L^\infty} \leq |4\pi t|^{-n/2} \|u_0\|_{L^1} \]
Strichartz estimates and bilinear estimates

Strichartz estimates for free waves

\[ \| |D|^s u\|_{L_t^p L_x^q} \leq c \| u(0) \|_{L_2} \]

imply

\[ \| D^s u \|_{L_t^p L_x^q} \leq c \| u(0) \|_{U_p} \]

Bilinear estimates for free solutions

\[ \left\| \int_{\eta_1 + \eta_2 = \eta} k(\eta_1, \eta_2) \hat{u}(t, \eta_1) \hat{v}(t, \eta_2) d\eta_1 \right\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq c \| u(0) \|_{L^2} \| v(0) \|_{L^2} \]

imply

\[ \left\| \int_{\eta_1 + \eta_2 = \eta} k(\eta_1, \eta_2) \hat{u}(t, \eta_1) \hat{v}(t, \eta_2) d\eta_1 \right\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq c \| u \|_{U^2} \| v \|_{U^2}. \]
Consider in $\mathbb{R} \times \mathbb{R}^2 \ni (t, x)$

$$i\partial_t u + \Delta u = \partial_{x_1} \bar{u}^2$$

with initial condition $u(0, x) = u_0(x)$.

**Theorem**

There exists $\varepsilon > 0$ such that for all $u_0$ with $\|u_0\|_{L^2} < \varepsilon$ there exists a unique global in time solution $u$. It scatters at $\infty$: The limit

$$\lim_{t \to \infty} e^{-it\Delta} u(t)$$

exists in $L^2$. 
Step 1: Littlewood-Paley decomposition, duality

Let \( \lambda \in 2^\mathbb{Z} \) and \( \hat{u}_\lambda = \chi_{\lambda \leq |\xi| < 2\lambda} \hat{u} \). Let

\[
\|u\|_X = \left( \sum_{\lambda \in 2^\mathbb{Z}} \|u_\lambda\|_{V^2}^2 \right)^{1/2}.
\]

Then

\[
v(t) = \begin{cases} 
    e^{it\Delta}u_0 & \text{if } t > 0 \\
    0 & \text{otherwise}
\end{cases}
\]

satisfies

\[
\|v\|_X \leq \sqrt{2}\|u_0\|_{L^2}.
\]
We claim

\[ \left| \int_{\mathbb{R} \times \mathbb{R}^2} \bar{u} \bar{v} \partial_x \bar{w} \, dx \, dt \right| \leq c \| u \|_X \| v \|_X \| w \|_X. \]  

(2)

Then, by duality

\[ \left\| \int_0^t e^{i(t-s)\Delta} \partial_x \bar{u} \bar{v} \, ds \right\|_X \leq c \| u \|_X \| v \|_X \]

and the theorem follows by standard arguments.
Littlewood-Paley reduction

We expand

\[ u = \sum_{\lambda \in 2^\mathbb{Z}} u_\lambda \]

where

\[ \hat{u}_\lambda = \chi_{\lambda \leq |\xi| < 2\lambda} \hat{u} \]

and expand the integral. We claim

\[
\sum_{\mu \leq \lambda} \left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dx dt \right| \leq c \lambda^{-1} \left( \sum_{\mu \leq \lambda} \| u_\mu \|_{V^2}^2 \right)^{1/2} \| v_\lambda \|_{V^2} \| v_\lambda \|_{V^2}. \quad (3)
\]
Dyadic implies full estimate

We expand (with sums over $2^\mathbb{Z}$)

$$\int \bar{u} \bar{v} \partial_{x_1} \bar{w} dx dt \leq \sum_{\lambda_1, \lambda_2, \lambda_2} \left| \int \bar{u}_{\lambda_1} \bar{v}_{\lambda_2} \partial_{x_1} \bar{w}_{\lambda_3} . dx dt \right| .$$

Since the integral of the product is the evaluation of the Fourier transform of the triple convolution at 0, there is only a contribution if there are

$$\xi_1 + \xi_2 + \xi_3 = 0, \lambda_j \leq |\xi_j| \leq 2\lambda_j.$$

Then necessarily the two larger numbers of $\lambda_j$ are of similar size. To simplify the notation we assume that they are equal and we denote them by $\lambda$ and the smaller number by $\mu$.

Moreover

$$\| \partial_{x_1} w_{\lambda_3} \|_{V^2} \leq 2\lambda_3 \| w_{\lambda_3} \|_{V^2}$$

and we may replace the derivative with a multiplication by $\lambda_3$. 
The bound

We bound using (3)

$$
\sum_{\lambda} \sum_{\mu \leq \lambda} \lambda \left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dxdt \right| \leq c \sum_{\lambda} \left( \sum_{\mu \leq \lambda} \| u_\mu \|_{V^2}^2 \right)^{1/2} \| u_\lambda \|_{V^2} \| w_\lambda \|_{V^2}
$$

$$
\leq c \| u \|_X \| v \|_X \| w \|_X
$$

and

$$
\sum_{\lambda} \sum_{\mu \leq \lambda} \mu \left| \int \bar{u}_\lambda \bar{v}_\lambda \bar{w}_\mu dxdt \right| \leq c \sum_{\lambda} \sum_{\mu \leq \lambda} \mu \| u_\lambda \|_{V^2} \| v_\lambda \|_{V^2} \| w_\mu \|_{V^2}.
$$

$$
\leq c \| u \|_X \| v \|_X \| w \|_X
$$
Modulation

Step 2. We want to bound the left hand side of (3), in particular

$$\left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dx dt \right|.$$  

The integral is zero unless there are points in the support which add up to 0. If $\tau_1 = |\xi_1|^2$ and $\tau_2 = |\xi_2|^2$ and $\tau_3 = -\tau_1 - \tau_2$ and $\xi_3 = -\xi_1 - \xi_2$ then

$$\tau_3 - |\xi_3|^2 = -|\xi_1|^2 - |\xi_2|^2 - |\xi_1 + \xi_2|^2$$

Thus, with $\mu \leq \lambda$, in

$$\int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dx \, dt$$

at least one of the terms has high modulation - i.e. vertical distance $\lambda^2 / 3$ to the characteristic set, otherwise the integral is zero.
High modulation on low frequency

We denote this term by \( h \) and we have to bound

\[
\left| \int u^h_{\mu} \bar{v}_\lambda \bar{w}_\lambda \, dx \, dt \right| \leq \| u^h_{\mu} \|_{L^2} \| (v_\lambda w_\lambda)_\mu \|_{L^2}
\]

\[
\leq \lambda^{-1} \| u_{\mu} \|_{V^2} \| v_\lambda \|_{L^4} \| w_\lambda \|_{L^4}
\]

\[
\leq \lambda^{-1} \| u_{\mu} \|_{V^2} \| v_\lambda \|_{U^4} \| w_\lambda \|_{U^4}
\]

This completes the estimate in this case since

\[
\| v_\lambda \|_{U^4} \leq c \| v_\lambda \|_{V^2}
\]

and

\[
\left( \sum_{\mu \leq \lambda} \| (v_\lambda w_\lambda)_\mu \|_{L^2}^2 \right)^{1/2} \leq \| v_\lambda w_\lambda \|_{L^2}
\]
High modulation on high frequency

Here we bound

\[ \left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda^h dx dt \right| \leq c \| u_\mu v_\lambda \|_{L^2} \| w_\lambda^h \|_{L^2}. \]

The bilinear estimate gives

\[ \| u_\mu v_\lambda \|_{L^2} \leq c (\mu/\lambda)^{1/2} \| u_\mu \|_{U^2} \| v_\lambda \|_{U^2} \]

and the Strichartz estimate implies

\[ \| u_\mu v_\lambda \|_{L^2} \leq c \| u_\mu \|_{U^4} \| v_\lambda \|_{U^4} \]

which is not good enough. How do we replace $U^2$ by $V^2$?
Logarithmic interpolation

For $M > 1$ we write

$$u_\mu = u^1_\mu + u^2_\mu$$

with

$$\frac{1}{M} \|u^1_\mu\|_{U^2} + e^M \|u^2_\mu\|_{U^4} \leq c \|u_\mu\|_{V^2}$$

and similarly $v_\lambda = v^1_\lambda + v^2_\lambda$. Then

$$\|u_\mu v_\lambda\|_{L^2} \leq c \left( \left( \frac{\mu}{\lambda} \right)^{1/2} M^2 + Me^{-M} + e^{-2M} \right) \|u_\mu\|_{V^2} \|v_\lambda\|_{V^2}.$$ 

Now we optimize $M$ so that

$$\|u_\mu v_\lambda\|_{L^2} \leq c \left( \frac{\mu}{\lambda} \right)^{1/2} \ln(1 + \lambda/\mu) \|u_\mu\|_{V^2} \|v_\lambda\|_{V^2}.$$
Conclusion

We obtain

$$\lambda \left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}^h_\chi dxdt \right| \leq c \left( (\mu/\lambda)^{1/2} \ln(1 + \lambda/\mu) \right) \| u_\mu \|_{V^2} \| v_\lambda \|_{V^2} \| w_\lambda \|_{V^2}.$$

This allows to sum to get the desired estimate.
The Kadomtsev Petviashvili II equation in 2d

\[ u_t + u_{xxx} + \partial_x^{-1} u_{yy} + \partial_x u^2 = 0 \]

Symmetries: Scaling \( \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y) \)
Galilean: \( u(t, x - cy - |c|^2t, y + 2ct) \)

Critical space:
\[ |D_x|^{-1/2} u_0 \in L^2. \]

Estimates: Strichartz \( \|u\|_{L^4} \leq c \|u\|_{U^4} \)
Bilinear: \( \|u_\mu u_\lambda\|_{L^2} \leq (\mu/\lambda)^{1/2} \|u_\mu\|_{U^2} \|u_\lambda\|_{U^2}. \)

Here
\[ \hat{u}_\lambda = \chi_{\lambda \leq |\xi| < 2\lambda} \hat{u}. \]
Theorem (Hadac, Herr, Koch 2008)

There exists $\delta > 0$ so that there is a unique global solution for all initial data with $\|D_x|^{-1/2}u_0\|_{L^2} \leq \delta$. The solution scatters.

The proof is almost verbatim the same as for the nonresonant Schrödinger equation.

The Kadomtsev Petviashvili II equation in 3d

\[ u_t + u_{xxx} + \partial_x^{-1}u_{yy} + \partial_x u^2 = 0 \]

Critical space: \(|D_x|^{1/2}u_0 \in L^2\).

Strichartz estimate: \(\|u_\lambda\|_{L^4} \leq c\lambda^{1/2}\|u_\lambda\|_{U^4}\)

Bilinear: \(\|u_\mu u_\lambda\|_{L^2} \leq c\mu\|u_\mu\|_{U^2}\|u_\lambda\|_{U^2}\).

Theorem (Koch, Li 2015)

*There exists \(\delta > 0\) so that the equation is well posed for*

\[ \|D_x^{1/2}u_0\|_{L^2_\ast} < \delta \]

*and the solution scatters.*

The space \(L^2_\ast\) differs from \(L^2\) but it reflects to full symmetry of the equation.
The geometry
Nonlinear Klein Gordon

Theorem (Schottdorf)

Let \( n \geq 2, \ s \geq \max\{\frac{1}{2}, \frac{n-2}{2}\} \). Then there exists \( \delta > 0 \) so that for initial data
\[
\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} < \delta
\]
there is a unique global solution to the quadratic Klein-Gordon equation
\[
u_{tt} - \Delta u + u = u^2
\]
\[
\begin{align*}
  u(0, x) &= u_0 \\
  u_t(0, x) &= u_1
\end{align*}
\]

Nonresonant systems. Without decay assumptions on the initial data.
We consider the generalized Korteweg-de Vries equation

\[ u_t + u_{xxx} + (u^p)_x = 0, \quad u(0) = u_0 \]

Tools:

- \[ \| D^{1/r} u \|_{L_t^r L_x^q} \leq c \| u \|_{U^r} \] for \( \frac{2}{r} + \frac{1}{q} = \frac{1}{2} \) \((\infty, 2), (6, 6), (4, \infty), (8, 4)\) (Strichartz)
- \[ \| (|D_1|^2 - |D_2|^2)^{1/2}(uv) \|_{L^2(\mathbb{R} \times \mathbb{R})} \leq c\lambda^{-1} \| u \|_{U^2} \| v \|_{U^2} \] (bilinear)
- High modulation
Wellposedness for gKdV

- $p = 5$. Scaling: $L^2$. Wellposedness $L^2$ (Kenig, Ponce, Vega) ($\dot{B}^0_{2,\infty}$)

- $p = 4$. Scaling $\dot{H}^{-1/6}$. Wellposedness $\dot{H}^{-1/6}, \dot{B}^{-1/6}_{2,\infty}$ (Tao, Koch & Marzuola)

- Stability of soliton: Martel & Merle ($H^1$), scattering at soliton in $\dot{H}^{-1/6}$ (Tao, smallness assumption in $H^1 \cap H^{-1/6}$, Koch & Marzuola smallness in $\dot{H}^{-1/6}$).

- $p = 3$. Scaling $\dot{H}^{-1/2}$. Global wellposedness $H^{1/4}$ (Kenig, Ponce, Vega)

Nonlinear Schrödinger equation on torus

Consider

\[ i\partial_t u + \Delta u = |u|^4 u, \quad u(0) = u_0 \in H^1(M) \]

on a three dimensional manifold $M$. Globally wellposed in $\mathbb{R}^3$ (Colliander, Keel, Tao, Staffilani, Takaoka, Tao 2008).

Theorem (Herr & Tataru & Tzvetkov, Pausader & Ionescu, Pausader & Tzvetkov & Wang 12)

*Quintic equation wellposed in $H^1(\mathbb{T}^3)$.*

Theorem (Herr, Pausader & Tzvetkov & Wang 13)

*Quintic equation wellposed in $H^1(\mathbb{S}^3)$.*

Theorem (Strunk 14)

*Quintic equation wellposed on rectangular torus for small data.*
Introduction

In this section we consider the Korteweg-de Vries equation

\[ u_t + u_{xxx} - 6uu_x = 0 \]

with the soliton solution

\[-2 \text{sech}^2(x - 4t)\]

and the modified Korteweg-de Vries equation

\[ v_t + v_{xxx} - 6v^2v_x = 0 \]

for which the kinks

\[ \pm \tanh(x + 2t) \]

are special solutions.
Global stability of the kink

Conjecture (Global orbital stability)

Suppose that

\[ v_0 - \tanh(x) \in H^1. \]

Then there exists \( \gamma < -2 \) and \( y \in C^1(\mathbb{R}) \) so that

\[ \|v(t,.) - \tanh(x - y(t))\|_{H^1(\gamma t, \infty)} \to 0 \]

and

\[ \lim_{t \to \infty} y' = -2. \]
Corollary (Soliton resolution)

*Suppose that the conjecture is true, \( u_0 \in L^2 \) and \( \varepsilon > 0 \). Let \( (\lambda_j) \) be the eigenvalues of*

\[
\psi \rightarrow -\psi'' + u\psi
\]

*in increasing order. There exists \( N \in \mathbb{N} \) and functions \( y_j \in C^1 \) so that*

\[
y_j' \rightarrow 4\lambda_j^2
\]

\[
\|u(t) - \sum_{j=0}^{N} (-2\lambda_j^2 \text{sech}^2(\lambda_j(x - y)))\|_{L^2(\varepsilon t, \infty)} \rightarrow 0
\]
Remarks

- Different regimes
- Via inverse scattering under much stronger conditions (implying at most finite number of eigenvalues, decay and integrability).
- Here: Local stability of the kink in $L^2$ and local asymptotic stability of solitons in $H^{-1}$ (with T. Buckmaster, 2015).
- Wellposedness in $H^{-1}$ is not known. We obtain global in time estimates in $H^{-1}$ without conserved quantity.
- Baoping Liu 2014: Apriori estimates in $H^s$ for some $-\frac{5}{6} > s > -1$, not uniform in time. (As for cubic nonlinear Schrödinger in $H^{-1/4}$).
The kink and the soliton are connected via the Miura map.

Lemma

Suppose that \( v \) satisfies mKdV. Then

\[
    u(t, x) = v_x(t, x - 6t) + v^2(t, x - 6t) - 1
\]

satisfies the Korteweg-de Vries equation.

Proof.

Calculation.
Diffeomorphism in function spaces

Example: \( v = \tanh(x), \ v_x + v^2 = 1, \ (-v)_x + (-v)^2 = 1 - 2 \text{sech}^2(x) \).

\[
u(t, x) = v_x(t + 6t, ) + v^2(t + 6t, x) - 1
\]

The Miura map relates:

1. A neighborhood of 0 for KdV
2. A neighborhood of \(\tanh\) for mKdV
3. A neighborhood of \(-\tanh\) for mKdV and
4. A neighborhood of \(-2 \text{sech}^2\) for KdV
Lax pair

Suppose that $u$ satisfies the KdV equation. Let

$$L_u = -\partial_x^2 + u$$

be the time dependent Schrödinger operator with potential $u$. Let

$$P = -4\partial_x^3 + 3(u\partial_x + \partial_x u)$$

Then

$$L_t = [P, L].$$

We may factor

$$-\partial_{xx}^2 + u = (\partial_x + v)(-\partial_x + v)$$

provided

$$u = v_x + v^2.$$
Lemma (Perry, Kappeler, Shubin, Topalov)

The potential $u$ is in the range of the Miura map if and only if $L \geq 0$.

Proof.

Let $u = v_x + v^2$. Then

$$\langle Lu \psi, \psi \rangle = \|\psi_x - v \psi\|_{L^2}^2 \geq 0$$

Vice versa, if $Lu \geq 0$ and $t_0 < 0$ then there is a unique nonnegative solution to

$$-\psi_{xx} + u \psi = 0$$

in $[t_0, \infty)$ with $\psi(t_0) = 0$ and $\psi(0) = 1$. This yields a nonnegative solution $\psi$ as $t_0 \to -\infty$. We define

$$v = \partial_x \ln \psi$$

Then

$$\partial_x v + v^2 = \frac{\psi_{xx}}{\psi} = u.$$
Shifts

Let \( u \in H^{-1} \). There exists \( C \) so that \( L_{u+C} \) is positive. This is in the range of the Miura map. We consider the Miura map applied to \( \pm(\lambda \tanh(\lambda x) + r) \) and define

\[
F^{+}_{\lambda}(r) = r_x + (2\lambda \tanh(\lambda x) + r)r \\
F^{-}_{\lambda}(r) = -r_x + (2\lambda \tanh(\lambda x) + r)r - 2\lambda^2 \text{sech}^2(\lambda x)
\]

**Theorem**

Let \( s \geq 0 \) and \( \lambda > 0 \). Then

\[
F^{+}_{\lambda} : H^s \to H^{s-1}
\]

is analytic. The range of \( F^{+}_{\lambda} \) is the set of all potentials such that the corresponding Schrödinger operator has spectrum in \((-\lambda^2, \infty)\). The null space of its derivative has dimension 1. Let \( r_0 \in L^2 \). The map

\[
(F^{+})^{-1}(r_0) \ni r \to \lim_{\epsilon \to 0} \int e^{-\epsilon |x|^2} (r - r_0) \, dx
\]
Theorem

The map

\[ F^{-1}(\lambda, r) \rightarrow F_\lambda(r) \in H^{s-1} \]

is an analytic diffeomorphism to its image. Its range consists of all potentials in \( H^s \) with at least one negative eigenvalue. The lowest eigenvalue of the potential \( F_\lambda(r) \) is \(-\lambda^2\).
The spectrum and the Miura map

The proof relies on a study of the Riccati type differential equation

\[ v_x + v^2 = u. \]

The linearization at \( \tanh(x) \) is

\[ w_x + 2 \tanh(x) w = f \]

hence

\[ w(x) = w(0) e^{-2 \int_0^x \tanh(t) dt} + \int_0^x e^{\int_s^x -2 \tanh(t) dt} f(s) ds. \]

The operator has a one dimensional kernel and it is surjective.
Similarly, if

$$w_x - 2 \tanh(x)w = f$$

then

$$w(x) = \begin{cases} \int_x^\infty e^{\int_x^s -2 \tanh(t)dt} f(s)ds & \text{if } x > 0 \\ \int_x^\infty e^{\int_s^x 2 \tanh(t)dt} f(s)ds & \end{cases}$$

which yields a solution only under the compatibility condition of continuity at 0. The operator is injective, with a range of codimension 1. The nonlinear equation requires further similar considerations.
Let $u \in C(\mathbb{R}, L^2)$ be a solution to the Korteweg-de Vries equation. The Lax pair

$$L_t = [P, L]$$

implies that the spectrum does not change: $P$ defines a unitary evolution $S(s, t)$ and

$$S(s, t)L(t) = L(s)S(s, t).$$

For matrices this implies similarity.
We can derive soliton resolution from the conjecture above. Suppose that the smallest eigenvalue is \( -\lambda_0^2 \). Then there exists a unique \( v \) with
\[
v + \lambda_0 \tanh(\lambda_0 x) \in H^1
data satisfying
\[
v_x + v^2 = u
\]
Let
\[
u_1 = -v_x + v^2
\]
Since

\[ (-\partial_{xx} + u_1 + \lambda_0^2)(-\partial_x + v) = (-\partial_x + v)(\partial_x + v)(-\partial_x + v) \]

\[ = (-\partial_x + v)(-\partial_{xx} + u + \lambda_0^2) \]

the effect on the spectrum is simple: It removes the lowest eigenvalues. 
\((-\partial_x + v\) is surjective and it has a one dimensional null space. Thus the operator on the right is surjective. Every other eigenfunction is mapped to an eigenfunction).

We iterate and arrive at an operator which has only small eigenvalues. This procedure commutes with the evolution - since \(v\) is unique.

If we wait long then we can isolate and remove the eigenvalues (and solitons) one by one. With the reverse procedure we regain the solitons.
The linear problem

The linearization at the kink in moving coordinates is

\[ u_t + u_{xxx} - 4u_x = \partial_x (6 \text{sech}^2(x) u) + \alpha(t) \text{sech}^2(x). \]

The spectrum of the generator is the imaginary axis. It has an eigenvalue 0 with eigen function \( \text{sech}^2(x) \) imbedded in the continuous spectrum. The red term is added in order to mode out the motion in the null space. The effect of this mode on the dynamics is simple: It corresponds to translations. Now

\[ \frac{d}{dt} \int e^x u^2 dx = -3B(e^{x/2} u) + \alpha \langle e^x \text{sech}^2(x), u \rangle \]

where

\[ B(f) := \int f_x^2 + \left( \frac{5}{4} - 2 \text{sech}^2(x) - 4 \text{sech}^2(x) \tanh(x) \right) f(x)^2 dx. \]
The quadratic form

Lemma

\[ B(f) + 2 \langle f, e^{\cdot/2} \text{sech}^2(\cdot) \rangle^2 \geq \frac{1}{3} \| f \|_{L^2}^2, \]

holds for all \( f \in H^1 \); moreover we also have the estimate

\[ B(f) + 2 \langle f, e^{\cdot/2} \text{sech}^2(\cdot) \rangle^2 \geq \frac{1}{20} \| f \|_{H^1}^2. \]
The quadratic form

Proof.

Lieb-Thirring inequality: \(-\Delta + V\)

\[
\sum_j |\lambda_j|^\gamma \leq L_{\gamma,n} \int |V_-|^\gamma+\frac{n}{2} dx.
\]

\[
L_{\gamma,1} = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 3/2)}
\]

\[
\sum |\lambda_j|^{\frac{3}{2}} \leq \frac{3}{16} \int |V_-|^2 dx = \frac{567}{320}
\]

Now we test functions to get upper bounds on \(\lambda_0\), and get control on the lowest eigen function. *Geometrie in Hilbert space.*
The linear equation

Consider again

\[ u_t + u_{xxx} - 4u_x = \partial_x (6 \sech^2(x) u) + \alpha(t) \sech^2(x) \]

with the orthogonality condition

\[ \langle u, e^x \sech^2(x) \rangle = 0 \quad (4) \]

This determines \( \alpha \).

**Lemma**

Let \( u_0 \) satisfy

\[ \int e^x u_0^2 dx < \infty, \quad \langle u_0, e^x \sech^2(x) \rangle = 0. \]

Then there exists a unique solution in that space which satisfies

\[ \int e^x u(t)^2 dx \leq e^{-t} \int e^x u_0^2 dx. \]
Ansatz:

\[ v = \tanh(x - y(t)) + u \]

Then

\[
\begin{align*}
    u_t + u_{xxx} - 6u_x - 2\partial_x (u^3 + 3 \tanh(x - y(t))u^3 - 3 \text{sech}^2(x - y(t))u) \\
    = (\dot{y} + 2) \text{sech}^2(x - y).
\end{align*}
\]

We choose

\[ \eta(x) = \varepsilon + 1 + \tanh((x - A)/2) \]

for some large \( A \) and require (4). Then

\[
\frac{d}{dt} \int \eta(x - y(t))u^2 \, dx + 3B((\eta')^{1/2}u) \leq \text{l.o.t.}
\]

This gives stability.
Theorem (Buckmaster and K', 2015)

Let $\gamma > 0$. There exists $\varepsilon > 0$ so that if $u_0 \in H^k(\mathbb{R})$, $k \geq -1$,

$$\|u_0 + 2 \text{sech}^2(x)\|_{H^{-1}} \leq \varepsilon$$

then there exists $y(t) \sim 4t$ so that the solution to KdV satisfies

$$\lim_{t \to \infty} \|u(t) + 2 \text{sech}^2(x - y(t))\|_{H^k([\gamma t, \infty))} = 0.$$