Nonlinear dispersive equations

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Linear dispersion

$$iu_t + p(\frac{1}{i}\nabla)u = 0$$

where $p(\frac{1}{i}\partial)$ is self-adjoint differential operator. Fourier transform:

$$i\partial_t \hat{u} + p(\xi)\hat{u} = 0$$

$$\hat{u}(t,\xi) = e^{itp(\xi)}\hat{u}(0,\xi)$$

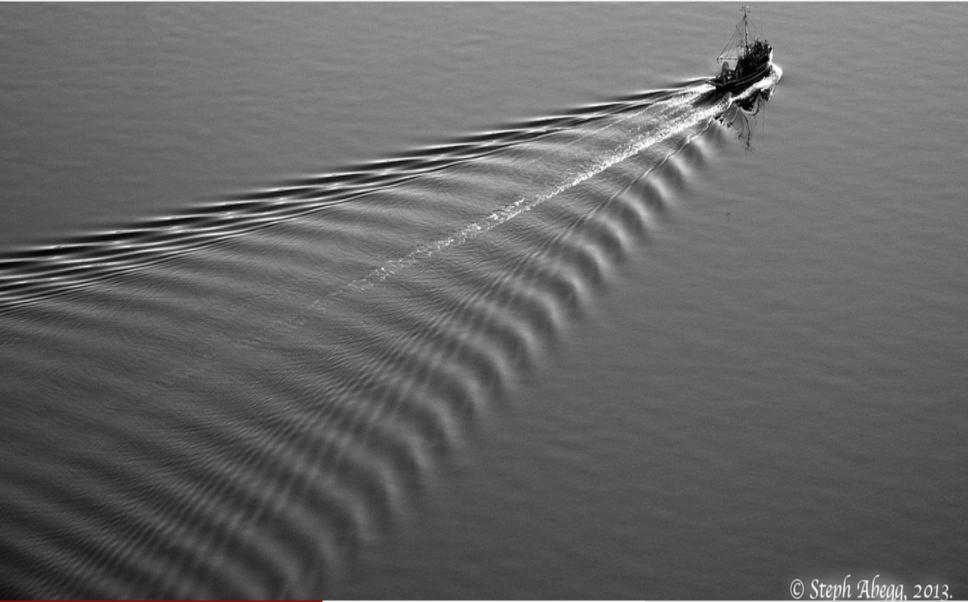
Examples

1. The Schrödinger equation in $\mathbb{R} \times \mathbb{R}^n \ni (t,x)$, $p(\xi) = -|\xi|^2$
$$i\partial_t u + \Delta u = 0$$
2. The Airy equation in $\mathbb{R} \times \mathbb{R} \ni (t,x)$, $p(\xi) = \xi^3$
$$\partial_t u + \partial_{xxx} u - 3uu_x = 0$$
3. The linear Kadomtsev-Petviashvili equations in $\mathbb{R} \times \mathbb{R}^2 \ni (t,x,y)$, $p(\xi,\eta) = \xi^3 \pm \frac{\eta^2}{\xi}$, $v = -(3\xi^2 \mp \frac{\eta^2}{\xi^2}, \frac{\eta}{\xi})$
$$\partial_x(\partial_t u + \partial_{xxx} u - 3u\partial_x u) \pm u_{yy} = 0$$
4. The half wave equation
$$i\partial_t u + \sqrt{1 + |\frac{1}{i}\nabla|^2} u = 0.$$

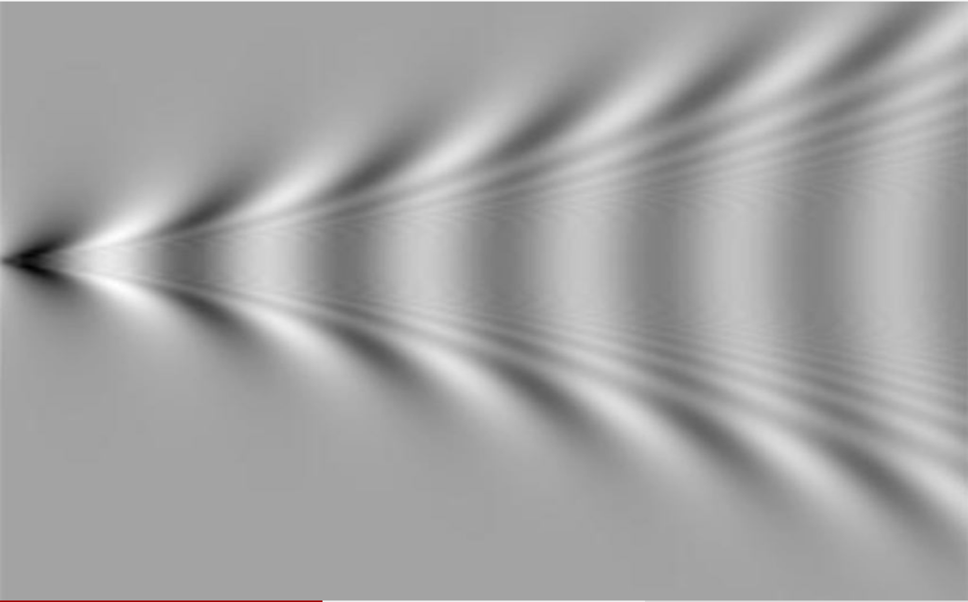
What is dispersion?

1. The group velocity depends on the frequency. For compactly supported smooth initial data the wave decays pointwise, despite the conservation of the L^2 norm.
2. The characteristic set $\{(\tau, \xi) : \tau = p(\xi)\}$ is curved. Stationary phase: Curvature leads to pointwise decay of the fundamental solution. The main contribution to
$$\int e^{i(x\cdot\xi + tp(\xi))} d\xi$$
comes from points with stationary phase $\frac{x}{t} = -\nabla p(\xi)$ - the group velocity at frequency ξ is
$$-\nabla p(\xi)$$
3. Decay of the fundamental solution.

Bow Wave



Bow Wave, Berry (Bristol)



Nonlinear waves

The nonlinearity may cooperate with dispersion (defocusing) , or work against it (focusing). (KPII)

Outline

1. The spaces U^p and V^p
2. Strichartz estimates and bilinear estimates
3. Nonlinear dispersive equations
4. Dynamics near solitons

Motivation

We consider

$$iu_t + Au = f(u)$$

where A is self adjoint and f is nonlinear. Search function spaces X and Y so that

- $X \ni u \to f(u) \in Y$ is smooth (polynomial)
- There is a unique solution $u \in X$ to data $f \in Y$ and $u_0 \in H$ such that $\|u\|_X \leq c(\|u_0\|_H + \|f\|_Y)$.

Then a solution can be constructed as fixed point of the map which maps u to the solution $J(u)$ with initial data u_0 and right hand side $f(u)$.

Motivation

One needs

- A radius R so that, if $\|u\|_X \le R$,

$$\|J(u)\|_X \le c\,(\|u_0\|_H + \|f(u)\|_Y) \le R$$

- The map $J(u)$ has a small Lipschitz constant

$$\|J(u) - J(v)\|_X \le c\|f(u) - f(v)\|_Y \le \mu\|u - v\|_X.$$

There are usually (but not always) two ways to achieve that: Small time, or small data. In this setting the assumptions of the implicit function theorem are satisfied.

Example: The nonlinear Schrödinger equation

We begin with the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = |u|^{\frac{4}{n}} u.$$

Here

$$H = L^2(\mathbb{R}^n), \quad X = L^{\frac{2(n+2)}{n}}(\mathbb{R}\times\mathbb{R}^n), \quad Y = L^{\frac{2(n+2)}{n+4}}(\mathbb{R}\times\mathbb{R}^n).$$

The estimates are

- The estimate of the nonlinearity

$$\big\| |u|^{\frac{4}{n}} u \big\|_{L^{\frac{2(n+2)}{n+4}}} = \|u\|^{\frac{4}{n}+1}_{L^{\frac{2(n+2)}{n}}}.$$

- Strichartz estimate

$$\|u\|_{L^{2\frac{n+2}{n}}(\mathbb{R}\times\mathbb{R}^n)} \le c\Big(\|u_0\|_{L^2(\mathbb{R}^n)} + \|i\partial_t u + \Delta u\|_{L^{\frac{2(n+2)}{n+4}}} \Big).$$

Desired properties

We want the following properties of the function spaces.

1. Heritates Strichartz estimates as embeddings, bilinear estimates, and 'high modulation estimates' in the elliptic part.
2. Allow duality arguments.

Example:

$$u_t = f$$

Would want: $X = \dot H^{1/2}$ and $Y = \dot H^{-1/2}$. Then

$$\int u f \, dt \le \|u\|_{\dot H^{1/2}} \|f\|_{\dot H^{-1/2}}$$

would give everything. Not true! How close can we get? This is relevant in probability and dispersive equations.

Bounded p variation

A partition is defined by an increasing (finite) sequence $\tau = (t_j)_{0 \le j \le N}$,

$$t_0 < t_1 \cdots < t_n \le \infty.$$

We denote the set of all partition by $\mathcal{T}$. Let
$\mathcal{S}$ be the set of all step functions with finitely many discontinuities (*test functions*) and
$\mathcal{R}$ the set of ruled functions with left and right limits everywhere including $\pm\infty$ (*distributions*).
We define $v(\infty) = 0.$ All these functions are bounded.
For a function space X we denote by X_{rc} the subspace of right continuous functions with limit 0 at $-\infty$.

Stieltjes integral

The *regulated Stieltjes integral*

$$\mathcal{R} \times \mathcal{S}_{rc} \ni (v,u) \to \int v du = \sum_j v(t_j)(u(t_j) - u(t_{j-1}))$$

defines a *duality*. We equip $\mathcal{S}_{rc}$ with the norm

$$\|u\|_{BV} = \sum_j |u(t_j) - u(t_{j-1})|.$$

and $\mathcal{R}$ with the supremum norm. If $L : \mathcal{S}_{rc} \to \mathbb{R}$ is continuous we define

$$v(t) = L(\chi_{[t,\infty)})$$

It is in $\mathcal{R}$.

Definition of V^p

Let $1 \le p < \infty$.

Definition (Definition of V^p)

We define the space V^p as the space of all functions such that the norm

$$\|u\|_{V^p} = \sup_{\tau} \Big(\sum |u(t_{i+1}) - u(t_i)|^p \Big)^{1/p}$$

is finite.

The definition of U^p

We call a a p atom if there exist a partition τ and ϕ_i with $\sum |\phi_i|^p \le 1$ and

$$a = \sum \phi_i \chi_{[t_i,t_{i+1})}.$$

Definition

The p atoms define an atomic space U^p by

$$\|u\|_{U^p} = \inf \Big\{ \sum |\lambda_j| : u = \sum \lambda_j a_j \Big\}.$$

(there exist atoms a_j and numbers λ_j with $u = \sum_j \lambda_j a_j$).

Density

Theorem

We have

$$\mathcal{S}_{rc} \subset U^p \subset V^p \subset \mathcal{R}.$$

Step functions are dense in U^p. If $p \le q$ then

$$U^p \subset U^q, \quad V^p \subset V^q.$$

Step functions are also dense in V^p, but the proof requires duality.

Interpolation

Lemma

Let $1 \le p < q < \infty$. *There exists* $C > 0$ *so that for all* $M > 0$ *and* $v \in V^p_{rc}$ *there exist* $u \in U^p$ *and* $w \in U^q$ *with*

$$v = u + w, \quad \frac{1}{M}\|u\|_{U^p} + e^M \|w\|_{U^q} \le C\|v\|_{V^p}$$

Corollary

Let $1 \le p < q < \infty$. *Then*

$$U^p \subset U^q, \qquad V^p \subset V^q$$

and

$$V^p_{rc} \subset U^q \subset V^q$$

Proof

This is proven via a sort of parametrization invariant Littlewood-Paley decomposition. Without loss of generality we assume that $\|v\|_{V^p} = 1$. If $t_1 \le t_2 \le t_3$ then

$$\|\chi_{[t_1,t_2)}(v - v(t_2))\|_{V^p}^p + \|\chi_{[t_2,t_3)}(v - v(t_3))\|_{V^p}^p \le \|\|\chi_{[t_1,t_2)}(v - v(t_3))\|_{V^p}^p.$$

We choose

$$t_{k,0} = -\infty, t_{k,1} = \infty, u^0 = \lim_{t\to-\infty} u(t)$$

$$t_{k,2j} = t_{k-1,j}$$

$$t_{k,2j-1} = \sup \Big\{ t : t < t_{k,2j}, \|\chi_{(-\infty,t)}(v - v(t))\|_{V^p}^p \ge \|\chi_{(-\infty,t_{k,2j})}(v - v(t_{k,2j}))\|$$

$$u^k(t) = \sum_j (v(t_{k,2j-1}) - v(t_{k,2(j-1)}))\chi_{[t_{k,2j-1},t_{k,2j})}.$$

Proof of interpolation II

Let

$$u = \sum_{k=1}^{k_0} u^k, \qquad w = v - u$$

Then

$$\Big\| v - \sum_{j=0}^{k_0} u^j \Big\|_{sup} \le 2^{-k_0/p}$$

$$\|u^k\|_{sup} \le 2^{-k/p}, \qquad \#\tau(u^k) \le 2^k.$$

$$\|u^k\|_{U^r} \le 2^{-k(\frac{1}{p}-\frac{1}{r})}.$$

and we arrive at

$$\|u\|_{U^p} \le k_0, \|w\|_{U^q} \le \frac{1}{1-2^{\frac{1}{q}-\frac{1}{p}}} 2^{-k_0(\frac{1}{p}-\frac{1}{q})}.$$

Duality

Recall

$$B(v,u) = \sum v(t_j)(u(t_j) - u(t_{j-1})).$$

Theorem

Let $\frac{1}{q} + \frac{1}{p} = 1$*,* $1 < p, q < \infty$*. The bilinear map defines a unique bilinear map* $B : V^p \times U^q$ *such that*

$$V^p \ni v \to (u \to B(v,u)) \in (U^q)^*$$

is an isometric isomorphism.

Proof of Duality

Proof.

For atoms (after an integration/summation by parts, with $t_N = \infty$)

$$B(v,a) \le \sum |v(t_{j+1}) - v(t_j)||a(t_j)| \le \|v\|_{V^p}.$$

This gives the bound. If $L \in (U^q)^*$ define

$$v(t) = L([t,\infty)).$$

□

This is a generalization of Young's integral (1912).

$$\int_a^b uv'dt$$

with $u \in V^p \cap C, v \in V^q \cap C$, $\frac{1}{p} + \frac{1}{q} > 1$.

Lemma

Step functions are dense in V^p. Test functions are weak dense in V^p.*

Proof.

Let $\tilde{V}^p$ be the closure of the step functions in V^p. Let $X \subset U^q$ be the set of all functions for which $B(v,u) = 0$ for all $v \in \tilde{V}^p$. Since $u(t) = -B(\chi_{[t,\infty)}, u)$ the set X is trivial. Then $\{0\} = X^* = V^p/\tilde{V}^p$. Let $\tilde{V}^p$ be the weak closure of C_0^∞ and let $(\tilde{V}^p)^\perp \subset U^q$ be the functions which are orthogonal. It suffices to show that $u(0) = 0$ for all functions in $(\tilde{V}^p)^\perp$. This requires a simple explicit construction. □

Duality 2

Theorem

The space U^q is the dual space to

$$V^p_C := \{v \in V^p \cap C(\mathbb{R}) : v(t) \to 0 \text{ as } t \to \infty\}$$

$C_0^\infty \subset U^p$ is weak$*$ dense.

In particular is suffices to test by smooth functions.

Proof.

It suffices to find a representation of a linear functional L.We reverse time. Then $U^p_C = U^p \cap C \subset V^p_C$. Thus $L \in (U^p_C)^*$. By Hahn Banach there is an extension $\tilde{L} \in (U^p)^*$. By the duality theorem it can be represented by a function $g \in V^q$ which we can choose to be right continuous. Now we integrate by parts. The weak* density is almost obvious. □

Duality 3

We can consider sequence spaces u_p and v_p on sequences $(u_j)_{j\in\mathbb{N}}$. Let v_p^0 be the subspace of sequences converging to 0. Then

- $(v_p^0)^* = u_p$
- $u_p^* = v_p$
- $v_p^0 \subset v_p$ has codimension 1

The space v_2^0 has been introduced by James (1951) because of this property. These spaces played a role in the study of Banach spaces by Pisier.

Relation to function spaces

An almost trivial computation implies

$$\|u\|_{V^p([0,1])} \le \|u\|_{\dot{C}^{1/p}([0,1])}.$$

Lemma

Let $\phi \in C_0^\infty(\mathbb{R})$ *with* $\int \phi = 1$. *Then*

$$\|f * \phi\|_{L^p} \le c\|f\|_{V^p}.$$

Moreover

$$B^{1/p}_{p,1} \subset U^p \subset V^p_{rc} \subset B^{1/p}_{p,\infty}.$$

In particular

$$\|u^\Lambda\|_{L^p} \le c\Lambda^{-1/p}\|u^\Lambda\|_{V^p}.$$

Solving ODEs

We consider the initial value problem $u_t = f$, $u(0) = u_0$.

Theorem

Suppose that f is a distribution and

$$\|f\|_{DV^p} = \sup\left\{ \int f\phi dx : \phi \in C_0^\infty, \|\phi\|_{U^q} \le 1 \right\} < \infty.$$

Then there exists a unique solution $u \in V^p$ with

$$\|u\|_{V^p} \le \|f\|_{DV^p} + |u_0|$$

$$\|f\|_{DU^p} = \sup\left\{ \int f\phi dx : \phi \in C_0^\infty, \|\phi\|_{V^q} \le 1 \right\} < \infty.$$

Then there exists a unique solution $u \in U^p$ for $t \ge 0$ with

$$\|u\|_{U^p} \le \|f\|_{DU^p} + |u_0|.$$

Probability, rough path theory

The Brownian B_t motion satisfies for all $p, q > 2$

$$\Big\| \|B_t \chi_{[0,1]}\|_{V^p} \Big\|_{L^q} \le c_{p,q}$$

The space V^p are invariant under reparametrization, and reparametrizations of the Brownian motion are in V^p.

Stochastic differential equation lead to integrals

$$\int f dg$$

where typically g is the Brownian motion and f is a local martingal. The integrals are pathwise defined if $g \in U^2$. This is not the case, and we need the Ito- or the Stratonovitch integral to integrate the Brownian motion.

T. Lyons has observed that one may enhance the Young integral by defining the Levy process by stochastic integration, and then a rough path integral depending only on the path and the Levy area process.
Hairer and Gubinelli have extended these ideas to partial differential equations.

Modifications

- Functions spaces on bounded intervals: Extend functions in V^p by zero to the right, and constant to the left, and functions in U^p constant to the right, and by zero to the left.
- Values in Hilbert/Banach spaces. If $q \ge p$ by Minkowski's inequality

$$\|v\|_{V^q(L^p)} \le \|v\|_{L^(V^q)}$$

$$\|u\|_{L^q(U^p)} \le \|u\|_{U^p(L^q)}.$$

- Pull back a unitary evolution

$$\|u\|_{U^p_P} = \|e^{-itP(D)}u\|_{U^p}$$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The spaces U^p and V^p

- V^p: Bounded p variation.
- p- atom: $a = \sum \phi_j \chi_{[t_j,t_{j+1})}$, $\sum |\phi_j|^p = 1$.
- U^p: $u = \sum \lambda_j a_j$.
- $T : U^p \to X$, $\|T\|_{L(U^p,X)} = \sup \|Ta\|$.
- Duality: $V^p \times U^q \ni (v,u) \to B(v,u) = \int v du$ defines an isometric isomorphism $V^q \to (U^p)^*$ and $U^p \to (V^q_C)^*$.
- Embeddings
$$B^{1/p}_{p,1} \subset U^p \subset V^p_{rc} \subset B^{1/p}_{p,\infty}.$$
- High modulation estimate
$$\|u^{>\Lambda}\|_{L^p} \leq c\Lambda^{-\frac{1}{p}} \|u\|_{V^p}$$
- Step functions are dense. Test functions are weak* dense.

Adaptation to operator

- Values in L^2.

$$\sup_t \|u(t)\|_{L^2} \le \|u\|_{V^p} \le c\|u\|_{U^p} \le c\|u\|_{BV}.$$

- Consider

$$i\partial_t u + Au = 0$$

- Pull back

$$\|u\|_{U^p_A} = \|e^{-itA}u(t)\|_{U^p}$$

$$\|v\|_{V^p_A} = \|e^{-itA}v(t)\|_{V^p}.$$

Solving differential equations

To solve

$$i\partial_t u + Au = f$$

in V^p prove

$$\int_0^\infty \langle f, \phi \rangle_{L^2} dt \le C_1$$

for $\phi \in C_0^\infty$ with $\|\phi\|_{U^q} \le 1$. Then there exists a unique solution u (distributional with values in L^2) with

$$\|u\|_{V^p} \le \|u_0\|_{L^2} + C_1.$$

Similarly with U^p.

Strichartz estimates

The linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

has a fundamental solution

$$g_t(x) = ((4\pi i t)^{1/2})^{-n} e^{-\frac{|x|^2}{4it}}$$

with Fourier transform

$$\hat{g}_t(x) = e^{it|\xi|^2}$$

hence

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \qquad \|u(t)\|_{L^\infty} \le |4\pi t|^{-n/2} \|u_0\|_{L^1}$$

Strichartz estimates

1. Interpolation: $\|u(t)\|_{L^{p'}} \le |4\pi t|^{-n(\frac{1}{p}-\frac{1}{2})}\|u_0\|_{L^p(\mathbb{R}^n)}$.
2. Duhamel's formula, weak Young (Keel-Tao)

$$\|u\|_{L^{r'}_t L^{p'}(\mathbb{R}^n)} \le c\|i\partial_t u + \Delta u\|_{L^r_t L^p(\mathbb{R}^n)}$$

if $r,p \ge 2$, $(r,p,n) \ne (2,1,2)$ and

$$2\Big(\frac{1}{r}-\frac{1}{2}\Big)+n\Big(\frac{1}{p}-\frac{1}{2}\Big)=1$$

3. A TT^* argument gives

$$\|u\|_{L^\infty L^2} + \|u\|_{L^{r'}L^{p'}} \le c\left(\|u_0\|_{L^2} + \|i\partial_t u + \Delta u\|_{L^r L^p}\right)$$

Strichartz estimates

The pointwise decay follows typically by stationary phase. For the Airy equation

$$u_t + u_{xxx} = 0$$

there is a fundamental solution (up to constants)

$$g(t,x) = t^{-1/3}\,\mathrm{Ai}(x/t^{1/3})$$

where

$$\mathrm{Ai}(x) = \int e^{i(x\xi+\xi^3)}$$

The Lemma of van der Corput implies that Ai is bounded. Stationary phase implies that half a derivative is bounded.
The half wave equation

$$i\partial_t u \pm |D| u = 0$$

(and hence the wave equation) has a characteristic set with $n-1$ nonvanishing curvatures. This implies a shift of the exponents compared to the Schrödinger equation.

Strichartz estimates and embedding for U^p and V^p

Let A be a selfadjoint operator on L^2. Then

$$i\partial_t u + Au = 0$$

generates the unitary group $S(t) = e^{itA}$.

Definition

$$U^p_A = S(t)U^p, V^p_A = S(t)V^p$$

$$\|u\|_{U^p_A} = \|S(-t)u(t)\|_{U^p_A}, \quad \|u\|_{V^p_A} = \|S(-t)u(t)\|_{V^p_A}.$$

Theorem

Suppose the unitary group admits Strichartz estimates with the exponents r, p. *Then*

$$\|u\|_{L^{r'} L^{p'}} \le c\|u\|_{U^{r'}_A}$$

and

$$\|u\|_{V^r_A} \le c\|(i\partial_t + \Delta)u\|_{L^r L^p}$$

Proof

It suffices to prove the first estimate for atoms, and hence for free solutions.
Similarly for the second part we choose a partition. Then the estimate reduces to proving them for a fixed interval of the partition. But this is equivalent to the Strichartz estimate for free solutions.

The Fourier transform of free waves

Consider

$$i\partial_t u - \phi(D)u = 0$$

where ϕ is a real valued function, and $\phi(D)$ is the Fourier multiplier. Then

$$i\partial_t \hat{u} = \phi(\xi)\hat{u}$$

and

$$\hat{u}(t,\xi) = e^{-it\phi(\xi)}\hat{u}(0,\xi)$$

Thus

$$\int \mathcal{F}_{t,x} u \psi \, d\xi d\tau = \int_{\mathbb{R}^n} \psi(\phi(\xi),\xi) u_0(\xi) d\tau d\xi$$

hence

$$\mathcal{F}_{t,x} u = \hat{u}_0(\xi)\delta_{\tau-\phi}.$$

Convolution estimates

Let Σ_j, $j = 1, 2$ be two hypersurfaces defined by $\Xi_j(\xi) = 0$. We search estimates

$$\|uv\|_{L^2} \le C\|\hat{u}\|_{L^2(\delta_{\Xi_1})}\|\hat{v}\|_{L^2(\delta_{\Xi_2})}$$

where by and abuse of notation the Fourier transform of $\hat{u}$ resp $\hat{v}$ is $\hat{u}\delta_{\Xi_1}$ resp. $\hat{u}\delta_{\Xi_1}$. By Plancherel this reduces to

$$\|(\hat{u}\delta_{\Xi_1}) * (\hat{v}\delta_{\Xi_2})\|_{L^2} \le C\|\hat{u}\|_{L^2(\delta_{\Xi_1})}\|\hat{v}\|_{L^2(\delta_{\Xi_2})}.$$

The calculation

We approximate the Dirac function by smooth functions. Then, with nonnegative functions h_1 and h_2,

$$\begin{aligned}
&\|uh_1 * vh_2\|_{L^2}^2 \\
=&\int \left| \int u(\xi-\eta) h_1^{1/2}(\xi-\eta) h_2^{1/2}(\eta) v(\eta) h_2^{\frac{1}{2}}(\eta) h_1^{1/2}(\xi-\eta) d\eta \right|^2 d\xi \\
\le& \int \int |u(\eta)|^2 h_1(\eta) h_2(\xi-\eta) d\eta \int |v(\eta)|^2 h_2(\eta) h_1(\xi-\eta) d\eta d\xi \\
\le& C_h^2 \|h_1^{\frac{1}{2}} u\|_{L^2}^2 \|h^{1/2} v\|_{L^2(\delta_{\Xi_2})}^2
\end{aligned}$$

where

$$C_h^2 = \sup_{\xi_1,\xi_2} \int h_1(\xi-\xi_1) h_2(\xi-\xi_2) d\xi.$$

The calculation

We set $h_i = j_k \circ \Xi_i$ for a Dirac sequence and obtain in the limit by the coarea formula with respect to h_i

$$C^2 = \sup_{\xi_1 \in \Sigma_1, \xi_2 \in \Sigma_2} \int \delta_{\Xi_1}(\xi - \xi_2)\delta_{\Xi_2}(\xi - \xi_1) d\xi. \tag{1}$$

For this limit we used the coarea formula: $\phi : U \to V \subset \mathbb{R}^m$, $m < n$.

$$\int_U \det(D\phi D\phi^T)^{1/2} f(x) dm^n(x) = \int_V \int_{\phi^{-1}(y)} f(x) d\mathcal{H}^{n-m} dm^m(y)$$

A reduction

Consider

$$\Xi_1(\tau,\xi) = \tau - \phi_1(\xi), \qquad \Xi_2(\tau,\xi) = \tau - \phi_2(\xi)$$

In this case the formula (1) can be considerably simplified.

Lemma

$$\int_{-\infty}^{\infty} \delta_{\Xi_1}\delta_{\Xi_2} dt = \delta_{\phi_1-\phi_2}$$

Proof.

By the calculation of Gram determinants

$$\delta_{\Xi_1}\delta_{\Xi_2} = \delta_{\Xi_1}\delta_{\Xi_2-\Xi_1}$$

and hence

$$\delta_{\tau-\phi_1}\delta_{\tau-\phi_2} = \delta_{\tau-\phi_1}\delta_{\phi_2-\phi_1}$$

Now the formula follows by an application of Fubini's theorem.

Local smoothing

Consider $d=1$, $\Xi_1 = \tau - \xi^3$, $\Xi_2 = \tau$. The equation $\xi_1^3 = (\xi - \xi_2)^3$ with the unique solution $\xi = \xi_2 - \xi_1$. The gradients are 0 and $3\xi_1^2$. Hence

$$\int (\xi f)\delta_{\tau-\xi^3} * (g\delta_\tau) d\eta \frac{1}{\sqrt{3}} \le \|f\|_{L^2_{\delta(\tau-\xi^3)}} \|g\|_{L^2(\mathbb{R})}.$$

This gives the local smoothing estimate below.

Theorem

Let u be the solution to the Airy equation with initial data u_0 given by the Fourier transform. Then

$$\sup_x \left(\int |u_x(t,x)|^2 dt \right)^{1/2} \le \frac{1}{\sqrt{3}} \|u_0\|_{L^2}$$

Local smoothing

Proof.

We apply the previous formula with u and a sequence $g_j(x)$ so that g_j^2 is Dirac measure. Then

$$\|(\partial_x u(x,t))g_j(x)\|_{L^2} \to \left(\int |\partial_x u(t,0)|^2 dx\right)^{1/2}$$

and

$$\|(\partial_x u)g_j\|_{L^2(\mathbb{R}\times\mathbb{R})} \leq c\|u(0)\|_{L^2(\mathbb{R})}$$

In the limit we obtain the bound at $x=0$, and by translation we obtain the general bound. □

One space dimension

Consider $\Xi_1 = \Xi_2 = \tau - \xi^3$.

Theorem

Suppose that u and v satisfy the Airy equation. Then

$$\| ||D_1|^2 - |D_2|^2|^{\frac{1}{2}} uv \|_{L^2} \le c \|u_0\|_{L^2} \|v_0\|_{L^2}.$$

Proof.

This is roughly a gain of one derivative. The proof requires going through the proof of the bilinear estimate with a bilinear multiplier,

$$\int_{\eta_1+\eta_2=\eta} |\eta_1^2 - \eta_2^2|^{1/2} |\hat{u}(t,\eta_1)\hat{u}(t,\eta_2)| d\eta_1.$$

This exactly compensates for C without further changes. □

Schrödinger equations

Theorem

Let $a, b \in \mathbb{R}\backslash 0$ *and*

$$ia\partial_t u + \Delta u = 0$$

$$ib\partial_t v + \Delta v = 0$$

Suppose that the Fourier transform of u *is supported in* $B_R(\xi_0)$. *Then, if* $a = b$

$$\| |D_1 - D_2|^{1/2}(uv) \|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \le cR^{\frac{n-1}{2}} \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}.$$

Proof of bilinear estimates for Schrödinger 1

Proof.

Let $\tau_1 = |\xi_1|^2$ and $\tau_2 = |\xi_2|^2$. The equations

$$\tau - |\xi_2|^2 - |\xi - \xi_2|^2 = 0 = \tau - |\xi_1|^2 - |\xi - \xi_1|^2$$

lead to

$$\langle \xi, \xi_2 - \xi_1 \rangle = 0.$$

We integrate out τ in the definition of C and get with

$$\phi(\xi) = 2\langle \xi, \xi_2 - \xi_1 \rangle$$

$$C^2 \leq \sup_{\xi_1, \xi_2} \int_{B_R(\xi_0 - \xi_2)} \delta_{2\langle \xi, \xi_2 - \xi_1 \rangle} d\xi = 2|\xi_2 - \xi_1|^{-1} |B_1| R^{1-n}.$$

□

Comment

It is worthwhile to explore the case $a \neq b$. Then we obtain a codimension 2 parabola.

Kadomtsev-Petviashvili II,2d

The Kadomtsev-Petviashvili II equation is

$$\partial_t u + u_{xxx} + \partial_x^{-1} u_{yy} + \partial_x u^2 = 0.$$

where $(t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and (τ, ξ, η) are the corresponding Fourier variables. The Strichartz estimate is

$$\|u\|_{L^4(\mathbb{R}^3)} \leq c\|u\|_{U^4}.$$

The bilinear estimate is (with u_λ the Littlewood Paley projection to $\lambda \leq |\xi| \leq 2\lambda$)

$$\|u_\mu u_\lambda\|_{L^2} \leq c\Big(\frac{\mu}{\lambda}\Big)^{1/2} \|u_\mu\|_{L^2} \|u_\lambda\|_{L^2}.$$

Here

$$\hat{u}_\lambda = \chi_{\lambda \leq |\xi| < 2\lambda} \hat{u}$$

is a Littlewood-Paley decomposition with respect to x.

Kadomtsev-Petviashvili II, 3d

With two y directions for $\mu \le \lambda/4$

$$\|u_\lambda\|_{L^4(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^2)} \le c\lambda^{1/2}\|u_\lambda(0)\|_{L^2(\mathbb{R}\times\mathbb{R}^2)}$$

and

$$\|u_\mu u_\lambda\|_{L^2} \le c\mu\|u_\mu\|_{L^2}\|u_\lambda\|_{L^2}.$$

Proof of the bilinear estimate

By the reduction

$$C = \sup \int \delta_{\phi(\xi_1,\eta_1)+\phi(\xi-\xi_1,\eta-\eta_1)-\phi(\xi_2,\eta_2)-\phi(\xi-\xi_2,\eta-\eta_2)}$$

with $\mu \le |\xi_1| \le 2\mu$, $\lambda \le |\xi_2| \le 2\lambda$ and $\phi = \xi^3 - \frac{|\eta|^2}{\xi}$. The set is given by

$$\phi(\xi_1,\eta_1) + \phi(\xi-\xi_1,\eta-\eta_1) = \phi(\xi_2,\eta_2) + \phi(\xi-\xi_2,\eta-\eta_2)$$

An algebraic calculation gives

$$\begin{aligned}
\phi(\xi_2,\eta_2) - \phi(\xi_1,\eta_1) - \Phi(\xi_1-\xi_2,\eta_1-\eta_2) + 3(\xi_1-\xi_2)(\xi-\xi_1)(\xi-\xi_2)& \\
=&\phi(\xi-\xi_1,\eta-\eta_1) - \Phi(\xi-\xi_2,\eta-\eta_2) + \Phi(\xi_1-\xi_2,\eta_1-\eta_2) \\
&+ 3(\xi_1-\xi_2)(\xi-\xi_1)(\xi-\xi_2) \\
=&(\xi_2-\xi_1)(\xi-\xi_1)(\xi-\xi_2)\left(\frac{\left|\frac{\eta-\eta_1}{\xi-\xi_1} - \frac{\eta-\eta_2}{\xi-\xi_2}\right|}{|\xi_1-\xi_2|}\right)^2.
\end{aligned}$$

Proof

This describes a circle in the η variables, or a point, or the empty set, for fixed $\xi_1, \xi_2, \xi, \eta_1$ and η_2.
We apply Fubini's theorem and integrate in the η variables. This integral does not depend on the the radius:

$$\int_{\mathbb{R}^2} \delta_{a(|x|^2 - R^2)} dx = |a|^{-1} \pi.$$

We are left with

$$\frac{\pi}{2|\xi_2 - \xi_1|} \int_{|\xi - \xi_2| \le \mu} |\xi - \xi_2| |\xi - \xi_1| d\xi \le \pi \mu^2.$$

Connection to U^2

There is a general argument that deduces bilinear estimates with respect to U^2.

Theorem

Suppose that u and v are solutions to the dispersive equations

$$i\partial_t u - p_1(D)u = 0 = i\partial_t v - p_2(D)v$$

$$\Big\| \int k(\xi,\eta)\hat u(t,\xi-\eta)\hat v(t,\eta) \Big\|_{L^2} \le c\|u_0\|_{L^2}\|v_0\|_{L^2}$$

Then

$$\Big\| \int k(\xi,\eta)\hat u(t,\xi-\eta)\hat v(t,\eta) \Big\|_{U^2_{p_1}} \le c\|u_0\|_{U^2_{p_1}}\|v_0\|_{U^2_{p_2}}.$$


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The spaces U^p and V^p

- V^p: Bounded p variation.
- p- atom: $a = \sum \phi_j \chi_{[t_j, t_{j+1})}$, $\sum |\phi_j|^p = 1$.
- U^p: $u = \sum \lambda_j a_j$.
- $T : U^p \to X$, $\|T\|_{L(U^p, X)} = \sup \|Ta\|$.
- Duality: $V^p \times U^q \ni (v, u) \to B(v, u) = \int v du$ defines an isometric isomorphism $V^q \to (U^p)^*$ and $U^p \to (V^q_C)^*$.
- Embeddings

$$B^{1/p}_{p,1} \subset U^p \subset V^p_{rc} \subset B^{1/p}_{p,\infty}.$$

- High modulation estimate

$$\|u^{>\Lambda}\|_{L^p} \leq c\Lambda^{-\frac{1}{p}} \|u\|_{V^p}$$

- Step functions are dense. Test functions are weak* dense.

Adaptation to operator

- Values in L^2.

$$\sup_t \|u(t)\|_{L^2} \le \|u\|_{V^p} \le c\|u\|_{U^p} \le c\|u\|_{BV}.$$

- Consider

$$i\partial_t u + Au = 0$$

- Pull back

$$\|u\|_{U^p_A} = \|e^{-itA}u(t)\|_{U^p}$$

$$\|v\|_{V^p_A} = \|e^{-itA}v(t)\|_{V^p}.$$

Solving differential equations

To solve

$$i\partial_t u + Au = f$$

in V^p prove

$$\int_0^\infty \langle f, \phi \rangle_{L^2} dt \leq C_1$$

for $\phi \in C_0^\infty$ with $\|\phi\|_{U^q} \leq 1$. Then there exists a unique solution u (distributional with values in L^2) with

$$\|u\|_{V^p} \leq \|u_0\|_{L^2} + C_1.$$

Similarly with U^p.

Strichartz estimates

The linear Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

has a fundamental solution

$$g_t(x) = ((4\pi i t)^{1/2})^{-n} e^{-\frac{|x|^2}{4it}}$$

with Fourier transform

$$\hat{g}_t(x) = e^{it|\xi|^2}$$

hence

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \qquad \|u(t)\|_{L^\infty} \le |4\pi t|^{-n/2} \|u_0\|_{L^1}$$

Strichartz estimates and bilinear estimates

Strichartz estimates for free waves

$$\| |D|^s u \|_{L^p_t L^q} \le c \|u(0)\|_{L^2}$$

imply

$$\| |D|^s u \|_{L^p_t L^q} \le c \|u(0)\|_{U^p}$$

Bilinear estimates for free solutions

$$\Big\| \int_{\eta_1+\eta_2=\eta} k(\eta_1,\eta_2)\hat{u}(t,\eta_1)\hat{v}(t,\eta_2) d\eta_1 \Big\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \le c\|u(0)\|_{L^2}\|v(0)\|_{L^2}$$

imply

$$\Big\| \int_{\eta_1+\eta_2=\eta} k(\eta_1,\eta_2)\hat{u}(t,\eta_1)\hat{v}(t,\eta_2) d\eta_1 \Big\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \le c\|u\|_{U^2}\|v\|_{U^2}.$$

A toy problem

Consider in $\mathbb{R} \times \mathbb{R}^2 \ni (t,x)$

$$i\partial_t u + \Delta u = \partial_{x_1} \bar{u}^2$$

with initial condition $u(0,x) = u_0(x)$.

Theorem

There exists $\varepsilon > 0$ such that for all u_0 with $\|u_0\|_{L^2} < \varepsilon$ there exists a unique global in time solution u. It scatters at ∞: The limit

$$\lim_{t\to\infty} e^{-it\Delta} u(t)$$

exists in L^2.

Step 1: Littlewood-Paley decomposition, duality

Let $\lambda \in 2^{\mathbb{Z}}$ and $\hat{u}_\lambda = \chi_{\lambda \le |\xi| < 2\lambda} \hat{u}$. Let

$$\|u\|_X = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \|u_\lambda\|_{V^2}^2 \right)^{1/2} .$$

Then

$$v(t) = \begin{cases} e^{it\Delta} u_0 & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

satisfies

$$\|v\|_X \le \sqrt{2} \|u_0\|_{L^2}.$$

Reduction to a trilinear estimate

We claim

$$\left| \int_{\mathbb{R}\times\mathbb{R}^2} \bar{u}\bar{v}\partial_{x_1}\bar{w}\,dxdt \right| \le c\|u\|_X\|v\|_X\|w\|_X. \tag{2}$$

Then, by duality

$$\left\| \int_0^t e^{i(t-s)\Delta}\partial_{x_1}\bar{u}\bar{v}\,ds \right\|_X \le c\|u\|_X\|v\|_X$$

and the theorem follows by standard arguments.

Littlewood-Paley reduction

We expand

$$u = \sum_{\lambda \in 2^{\mathbb{Z}}} u_\lambda$$

where

$$\hat{u}_\lambda = \chi_{\lambda \le |\xi| < 2\lambda} \hat{u}$$

and expand the integral. We claim

$$\sum_{\mu \le \lambda} \left| \int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dxdt \right| \le c\lambda^{-1} \left(\sum_{\mu \le \lambda} \|u_\mu\|_{V^2}^2 \right)^{1/2} \|v_\lambda\|_{V^2} \|v_\lambda\|_{V^2}. \qquad (3)$$

Dyadic implies full estimate

We expand (with sums over $2^{\mathbb{Z}}$)

$$\int \bar{u}\bar{v}\partial_{x_1}\bar{w}dxdt \leq \sum_{\lambda_1,\lambda_2,\lambda_2} \left| \int \bar{u}_{\lambda_1}\bar{v}_{\lambda_2}\partial_{x_1}\bar{w}_{\lambda_3}.dxdt \right|.$$

Since the integral of the product is the evaluation of the Fourier transform of the triple convolution at 0, there is only a contribution if there are

$$\xi_1 + \xi_2 + \xi_3 = 0, \lambda_j \leq |\xi_j| \leq 2\lambda_j.$$

Then necessarely the two larger numbers of λ_j are of similar size. To simplify the notation we assume that they are equal and we denote them by λ and the smaller number by μ.

Moreover

$$\|\partial_{x_1} w_{\lambda_3}\|_{V^2} \leq 2\lambda_3 \|w_{\lambda_3}\|_{V^2}$$

and we may replace the derivative with a multiplication by λ_3.

The bound

We bound using (3)

$$\sum_\lambda \sum_{\mu \le \lambda} \left| \lambda \int \bar u_\mu \bar v_\lambda \bar w_\lambda dx dt \right| \le c \sum_\lambda \Big(\sum_{\mu \le \lambda} \|u_\mu\|_{V^2}^2 \Big)^{1/2} \|u_\lambda\|_{V^2} \|w_\lambda\|_{V^2}$$
$$\le c \|u\|_X \|v\|_X \|w\|_X$$

and

$$\sum_\lambda \sum_{\mu \le \lambda} \mu \left| \int \bar u_\lambda \bar v_\lambda \bar w_\mu dx dt \right| \le c \sum_\lambda \sum_{\mu \le \lambda} \frac{\mu}{\lambda} \|u_\lambda\|_{V^2} \|v_\lambda\|_{V^2} \|w_\mu\|_{V^2}.$$
$$\le c \|u\|_X \|v\|_X \|w\|_X$$

Modulation

Step 2. We want to bound the left hand side of (3), in particular

$$\left|\int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dxdt\right|.$$

The integral is zero unless there are points in the support which add up to 0. If $\tau_1 = |\xi_1|^2$ and $\tau_2 = |\xi_2|^2$ and $\tau_3 = -\tau_1 - \tau_2$ and $\xi_3 = -\xi_1 - \xi_2$ then

$$\tau_3 - |\xi_3|^2 = -|\xi_1|^2 - |\xi_2|^2 - |\xi_1 + \xi_2|^2$$

Thus, with $\mu \leq \lambda$, in

$$\int \bar{u}_\mu \bar{v}_\lambda \bar{w}_\lambda dx\, dt$$

at least one of the terms has high modulation - i.e. vertical distance $\lambda^2/3$ to the characteristic set, otherwise the integral is zero.

High modulation on low frequency

We denote this term by ^{h} and we have to bound

$$\begin{aligned} \left| \int \bar{u}^h_\mu \bar{v}_\lambda \bar{w}_\lambda dxdt \right| \leq & \|u^h_\mu\|_{L^2} \|(v_\lambda w_\lambda)_\mu\|_{L^2} \\ \leq & \lambda^{-1} \|u_\mu\|_{V^2} \|v_\lambda\|_{L^4} \|w_\lambda\|_{L^4} \\ \leq & \lambda^{-1} \|u_\mu\|_{V^2} \|v_\lambda\|_{U^4} \|w_\lambda\|_{U^4} \end{aligned}$$

This completes the estimate in this case since

$$\|v_\lambda\|_{U^4} \leq c \|v_\lambda\|_{V^2}$$

and

$$\left(\sum_{\mu \leq \lambda} \|(v_\lambda w_\lambda)_\mu\|^2_{L^2} \right)^{1/2} \leq \|v_\lambda w_\lambda\|_{L^2}$$

High modulation on high frequency

Here we bound

$$\left|\int \bar{u}_\mu \bar{v}_\lambda \bar{w}^h_\lambda dxdt\right| \le c\|u_\mu v_\lambda\|_{L^2}\|w^h_\lambda\|_{L^2}.$$

The bilinear estimate gives

$$\|u_\mu v_\lambda\|_{L^2} \le c(\mu/\lambda)^{1/2}\|u_\mu\|_{U^2}\|v_\lambda\|_{U^2}$$

and the Strichartz estimate implies

$$\|u_\mu v_\lambda\|_{L^2} \le c\|u_\mu\|_{U^4}\|v_\lambda\|_{U^4}$$

which is not good enough. How do we replace U^2 by V^2?

Logarithmic interpolation

For $M > 1$ we write

$$u_\mu = u_\mu^1 + u_\mu^2$$

with

$$\frac{1}{M} \|u_\mu^1\|_{U^2} + e^M \|u_\mu^2\|_{U^4} \le c\|u_\mu\|_{V^2}$$

and similarly $v_\lambda = v_\lambda^1 + v_\lambda^2$. Then

$$\|u_\mu v_\lambda\|_{L^2} \le c\left(\left(\frac{\mu}{\lambda}\right)^{1/2} M^2 + Me^{-M} + e^{-2M}\right)\|u_\mu\|_{V^2}\|v_\lambda\|_{V^2}.$$

Now we obtimize M so that

$$\|u_\mu v_\lambda\|_{L^2} \le c\left(\mu/\lambda\right)^{1/2}\ln(1+\lambda/\mu)\right)\|u_\mu\|_{V^2}\|v_\lambda\|_{V^2}.$$

Conclusion

We obtain

$$\lambda\left|\int \bar{u}_\mu \bar{v}_\lambda \bar{w}^h_\lambda dxdt\right| \le c\left((\mu/\lambda)^{1/2}\ln(1+\lambda/\mu)\right)\|u_\mu\|_{V^2}\|v_\lambda\|_{V^2}\|w_\lambda\|_{V^2}.$$

This allows to sum to get the desired estimate.

The Kadomtsev Petviashvili II equation in 2d

$$u_t + u_{xxx} + \partial_x^{-1} u_{yy} + \partial_x u^2 = 0$$

Symmetries: Scaling $\lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$
Galilean: $u(t, x - cy - |c|^2 t, y + 2ct)$
Critical space:

$$|D_x|^{-1/2} u_0 \in L^2.$$

Estimates: Strichartz $\|u\|_{L^4} \le c\|u\|_{U^4}$
Bilinear: $\|u_\mu u_\lambda\|_{L^2} \le (\mu/\lambda)^{1/2} \|u_\mu\|_{U^2} \|u_\lambda\|_{U^2}$.
Here

$$\hat{u}_\lambda = \chi_{\lambda \le |\xi| < 2\lambda} \hat{u}.$$

Theorem (Hadac, Herr, Koch 2008)

There exists $\delta > 0$ so that there is a unique global solution for all initial data with $\||D_x|^{-1/2}u_0\|_{L^2} \le \delta$. The solution scatters.

The proof is almost verbatim the same as for the nonresonant Schrödinger equation.
Bourgain: Wellposed in L^2. No scattering.

The Kadomtsev Petviashvili II equation in 3d

$$u_t + u_{xxx} + \partial_x^{-1} u_{yy} + \partial_x u^2 = 0$$

Critical space: $|D_x|^{1/2} u_0 \in L^2$.
Strichartz estimate: $\|u_\lambda\|_{L^4} \le c\lambda^{1/2} \|u_\lambda\|_{U^4}$
Bilinear: $\|u_\mu u_\lambda\|_{L^2} \le c\mu \|u_\mu\|_{U^2} \|u_\lambda\|_{U^2}$.

Theorem (Koch, Li 2015)

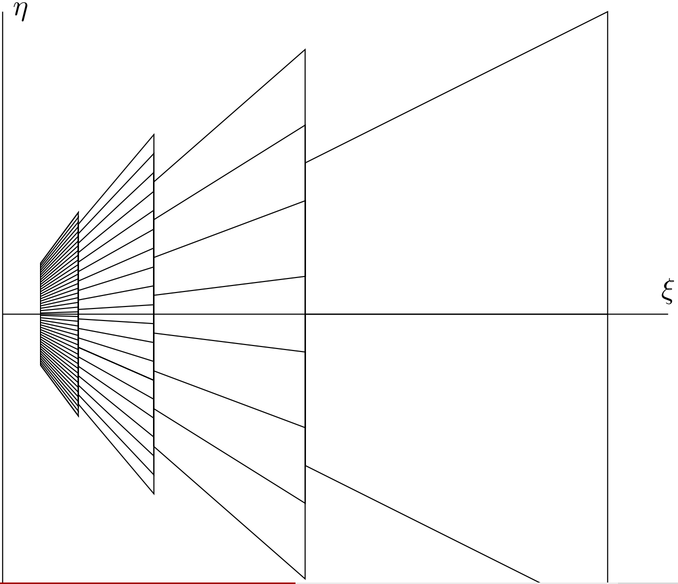
There exists $\delta > 0$ so that the equation is well posed for

$$\|D_x^{1/2} u_0\|_{L^2_*} < \delta$$

and the solution scatters.

The space L^2_* differs from L^2m but it reflect to full symmetry of the equation.

The geometry



Nonlinear Klein Gordon

Theorem (Schottdorf)

Let $n \geq 2$, $s \geq \max\{\frac{1}{2}, \frac{n-2}{2}\}$. Then there exists $\delta > 0$ so that for initial data

$$\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} < \delta$$

there is a unique global solution to the quadratic Klein-Gordon equation

$$u_{tt} - \Delta u + u = u^2$$

$$u(0,x) = u_0 \qquad u_t(0,x) = u_1$$

Nonresonant systems. Without decay assumptions on the initial data.

gKdV

We consider the generalized Korteweg-de Vries equation

$$u_t + u_{xxx} + (u^p)_x = 0, \qquad u(0) = u_0$$

Tools:

- $\| |D|^{\frac{1}{r}} u \|_{L^r_t L^q_x} \le c \|u\|_{U^r}$ for $\frac{2}{r} + \frac{1}{q} = \frac{1}{2}$ $((\infty, 2), (6,6), (4, \infty), (8,4))$ (Strichartz)
- $\|(|D_1|^2 - |D_2|^2)^{1/2}(uv)\|_{L^2(\mathbb{R}\times\mathbb{R})} \le c\lambda^{-1} \|u\|_{U^2} \|v\|_{U^2}$ (bilinear)
- High modulation

Wellposedness for gKdV

- $p = 5$. Scaling: L^2. Wellposedness L^2 (Kenig, Ponce, Vega) ($\dot{B}^0_{2,\infty}$)
- $p = 4$. Scaling $\dot{H}^{-1/6}$. Wellposedness $\dot{H}^{-1/6}$, $\dot{B}^{-1/6}_{2,\infty}$ (Tao, Koch & Marzuola)
- Stability of soliton: Martel & Merle (H^1), scattering at soiton in $\dot{H}^{-1/6}$ (Tao, smallness assumption in $H^1 \cap H^{-1/6}$, Koch & Marzuola smallness in $\dot{H}^{-1/6}$).
- $p = 3$. Scaling $\dot{H}^{-1/2}$. Global wellposedness $H^{1/4}$ (Kenig, Ponce, Vega)
- $p = 2$. Scaling $\dot{H}^{-3/2}$. Global wellposedness $H^{-3/4}$ Christ & Colliander & Tao (local) Nishimoto (local & global) Guo (global).

Nonlinear Schrödinger equation on torus

Consider

$$i\partial_t u + \Delta u = |u|^4 u, \qquad u(0) = u_0 \in H^1(M)$$

on a three dimensional manifold M. Globally wellposed in $\mathbb{R}^3$ (Colliander, Keel, Tao, Staffilani, Takaoka, Tao 2008).

Theorem (Herr & Tataru & Tzvetkov, Pausader & Ionescu, Pausader & Tzvetkov & Wang 12)

Quintic equation wellposed in $H^1(\mathbb{T}^3)$.

Theorem (Herr, Pausader & Tzvetkov & Wang 13)

Quintic equation wellposed in $H^1(\mathbb{S}^3)$.

Theorem (Strunk 14)

Quintic equation wellposed on rectangular torus for small data.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Introduction

In this section we consider the Korteweg-de Vries equation

$$u_t + u_{xxx} - 6uu_x = 0$$

with the soliton solution

$$-2\operatorname{sech}^2(x-4t)$$

and the modified Korteweg-de Vries equation

$$v_t + v_{xxx} - 6v^2 v_x = 0$$

for which the kinks

$$\pm\tanh(x+2t)$$

are special solutions.

Global stability of the kink

Conjecture (Global orbital stability)

Suppose that

$$v_0 - \tanh(x) \in H^1.$$

Then there exists $\gamma < -2$ *and* $y \in C^1(\mathbb{R})$ *so that*

$$\|v(t,.) - \tanh(x - y(t))\|_{H^1((\gamma t,\infty)} \to 0$$

and

$$\lim_{t\to\infty} y' = -2.$$

Soliton resolution

Corollary (Soliton resolution)

Suppose that the conjecture is true, $u_0 \in L^2$ *and* $\varepsilon > 0$. *Let* (λ_j) *be the eigenvalues of*

$$\psi \to -\psi'' + u\psi$$

in increasing order. There exists $N \in \mathbb{N}$ *and functions* $y_j \in C^1$ *so that*

$$y_j' \to 4\lambda_j^2$$

$$\|u(t) - \sum_{j=0}^{N} (-2\lambda_j^2 \operatorname{sech}^2(\lambda_j(x-y)))\|_{L^2(\varepsilon t, \infty)} \to 0$$

Remarks

- Different regimes
- Via inverse scattering under much stronger conditions (implying at most finite number of eigenvalues, decay and integrability).
- Here: Local stability of the kink in L^2 and local asymptotic stability of solitons in H^{-1} (with T. Buckmaster, 2015) .
- Stability of soliton for gKdV: Martel & Merle: Distance in H^1. KdV: Merle and Vega: Distance in L^2.
- Wellposedness in H^{-1} is not known. We obtain global in time estimates in H^{-1} without conserved quantity.
- Baoping Liu 2014: Apriori estimates in H^s for some $-\frac{5}{6} > s > -1$, not uniform in time. (As for cubic nonlinear Schrödinger in $H^{-1/4}$).

The Miura map

The kink and the soliton are connected via the Miura map.

Lemma

Suppose that v satisfies mKdV. Then

$$u(t,x) = v_x(t,x-6t) + v^2(t,x-6t) - 1$$

satisfies the Korteweg-de Vries equation.

Proof.

Calculation. $\square$

Diffeomorphism in function spaces

Example: $v = \tanh(x)$, $v_x + v^2 = 1$, $(-v)_x + (-v)^2 = 1 - 2\,\mathrm{sech}^2(x)$.

$$u(t,x) = v_x(t + 6t,) + v^2(t + 6t, x) - 1$$

The Miura map relates

1. A neighborhood of 0 for KdV
2. A neighborhood of $\tanh$ for mKdV
3. A neighborhood of $-\tanh$ for mKdV and
4. A neighborhood of $-2\,\mathrm{sech}^2$ for KdV

Lax pair

Suppose that u satisfies the KdV equation. Let

$$L_u = -\partial_x^2 + u$$

be the time dependent Schrödinger operator with potential u. Let

$$P = -4\partial_x^3 + 3(u\partial_x + \partial_x u)$$

Then

$$L_t = [P, L].$$

We may factor

$$-\partial_{xx}^2 + u = (\partial_x + v)(-\partial_x + v)$$

provided

$$u = v_x + v^2.$$

Lemma (Perry, Kappeler, Shubin,Topalov)

The potential u is in the range of the Miura map if and only if $L \geq 0$.

Proof.

Let $u = v_x + v^2$. Then

$$\langle L_u \psi, \psi \rangle = \|\psi_x - v\psi\|_{L^2}^2 \geq 0$$

Vice verse, if $L_u \geq 0$ and $t_0 < 0$ then there is a unique nonnegative solution to

$$-\psi_{xx} + u\psi = 0$$

in $[t_0, \infty)$ with $\psi(t_0) = 0$ and $\psi(0) = 1$. This yields a nonnegative solution ψ as $t_0 \to -\infty$. We define

$$v = \partial_x \ln \psi$$

Then

$$\partial_x v + v^2 = \frac{\psi_{xx}}{\psi} = u.$$

Shifts

Let $u \in H^{-1}$. There exists C so that L_{u+C} is positive. This is in the range of the Miura map. We consider the Miura map applied to $\pm(\lambda \tanh(\lambda x) + r)$ and define

$$F_\lambda^+(r) = r_x + (2\lambda \tanh(\lambda x) + r)r$$

$$F_\lambda^-(r) = -r_x + (2\lambda \tanh(\lambda x) + r)r - 2\lambda^2 \operatorname{sech}^2(\lambda x)$$

Theorem

Let $s \geq 0$ and $\lambda >$. Then

$$F_\lambda^+ : H^s to H^{s-1}$$

is analytic. The range of F_λ^+ is the set of all potentials such that the corresponding Schrödinger operator has spectrum in $(-\lambda^2, \infty)$. The null space of its derivative has dimension 1. Let $r_0 \in L^2$. The map

$$(F^+)^{-1}(r_0) \ni r \to \lim \int e^{-\epsilon|x|^2}(r - r_0)dx$$

F^-

Theorem

The map

$$(0,\infty) \times H^s \ni (\lambda, r) \to F_\lambda(r) \in H^{s-1}$$

is an analytic diffeomorphism to its image. Its range consists of all potentials in H^s with at least on negative eigenvalue. The lowest eigenvalue of the potential $F_\lambda(r)$ is $-\lambda^2$.

The spectrum and the Miura map

The proof relies on a study of the Riccati type differential equation

$$v_x + v^2 = u.$$

The linearization at $\tanh(x)$ is

$$w_x + 2\tanh(x) w = f$$

hence

$$w(x) = w(0) e^{-2\int_0^x \tanh(t) dt} + \int_0^x e^{\int_s^x -2\tanh(t) dt} f(s) ds.$$

The operator has a one dimenensional kernel and it is surjective.

The spectrum and the Miura map

Similarly, if

$$w_x - 2\tanh(x)w = f$$

then

$$w(x) = \begin{cases} \int_x^\infty e^{\int_x^s -2\tanh(t)dt} f(s)ds & \text{if } x > 0 \\ \int_x^\infty e^{\int_s^x 2\tanh(t)dt} f(s)ds & \end{cases}$$

which yields a solution only under the compatibility condition of continuity at 0. The operator is injective, with a range of codimension 1.
The nonlinear equation requires further similar considerations.

The spectrum and the Miura map

Let $u \in C(\mathbb{R}, L^2)$ be a solution to the Korteweg-de Vries equation. The Lax pair

$$L_t = [P, L]$$

implies that the spectrum does not change: P defines a unitary evolution $S(s,t)$ and

$$S(s,t)L(t) = L(s)S(s,t).$$

For matrices this implies similarity.

Soliton resolution

We can derive soliton resolution from the conjecture above. Suppose that the smallest eigenvalue is $-\lambda_0^2$. Then there exists a unique v with $v + \lambda_0 \tanh(\lambda_0 x) \in H^1$ and

$$v_x + v^2 = u$$

Let

$$u_1 = -v_x + v^2$$

Since

$$
\begin{aligned}
(-\partial_{xx} + u_1 + \lambda_0^2)(-\partial_x + v) =& (-\partial_x + v)(\partial_x + v)(-\partial_x + v) \\
=& (-\partial_x + v)(-\partial_{xx} + u + \lambda_0^2)
\end{aligned}
$$

the effect on the spectrum is simple: It removes the lowest eigenvalues. ($-\partial_x + v$ is surjective and it has a one dimensional null space. Thus the operator on the right is surjective. Every other eigenfunction is mapped to an eigenfunction).
We iterate and arrive at an operator which has only small eigenvalues.
This procedure commutes with the evolution - since v is unique.
If we wait long then we can isolate and remove the eigenvalues (and solitons) one by one. With the reverse procedure we regain the solitons.

The linear problem

The linearization at the kink in moving coordinates is

$$u_t + u_{xxx} - 4u_x = \partial_x(6\,\mathrm{sech}^2(x)u) + \alpha(t)\,\mathrm{sech}^2(x).$$

The spectrum of the generator is the imaginary axis. It has an eigenvalue 0 with eigen function $\mathrm{sech}^2(x)$ imbedded in the continuous spectrum. The red term is added in order to mode out the motion in the null space. The effect of this mode on the dynamics is simple: It corresponds to translations. Now

$$\frac{d}{dt}\int e^x u^2 dx = -3B(e^{x/2}u) + \alpha\langle e^x\,\mathrm{sech}^2(x), u\rangle$$

where

$$B(f) := \int f_x^2 + \left(\frac{5}{4} - 2\,\mathrm{sech}^2(x) - 4\,\mathrm{sech}^2(x)\tanh(x)\right) f(x)^2 \; dx.$$

The quadratic form

Lemma

$$B(f) + 2\langle f, e^{\cdot/2}\operatorname{sech}^2(\cdot)\rangle^2 \geq \frac{1}{3}\|f\|_{L^2}^2,$$

holds for all $f \in H^1$; moreover we also have the estimate

$$B(f) + 2\langle f, e^{\cdot/2}\operatorname{sech}^2(\cdot)\rangle^2 \geq \frac{1}{20}\|f\|_{H^1}^2.$$

The quadratic form

Proof.

Lieb-Thirring inequality: $-\Delta + V$

$$\sum_j |\lambda_j|^\gamma \le L_{\gamma,n} \int |V_-|^{\gamma+n/2} dx.$$

$$L_{\gamma,1} = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)}$$

$$\sum |\lambda_j|^{\frac{3}{2}} \le \frac{3}{16} \int |V_-|^2 dx = \frac{567}{320}$$

Now we test functions to get upper bounds on λ_0, and get control on the lowest eigen function. *Geometrie in Hilbert space*. □

The linear equation

Consider again

$$u_t + u_{xxx} - 4u_x = \partial_x(6\,\mathrm{sech}^2(x)u) + \alpha(t)\,\mathrm{sech}^2(x)$$

with the orthogonality condition

$$\langle u, e^x \,\mathrm{sech}^2(x)\rangle = 0 \tag{4}$$

This determines α.

Lemma

Let u_0 satisfy

$$\int e^x u_0^2 dx < \infty, \qquad \langle u_0, e^x \,\mathrm{sech}^2(x)\rangle = 0.$$

Then there exists a unique solution in that space which satisfies

$$\int e^x u(t)^2 dx \le e^{-t} \int e^x u_0^2 dx.$$

Nonlinear equation

Ansatz:

$$v = \tanh(x - y(t)) + u$$

Then

$$\begin{aligned} u_t + u_{xxx} - 6u_x - 2\partial_x(u^3 + 3\tanh(x - y(t))u^3 - 3\operatorname{sech}^2(x - y(t))u) \\ =(\dot{y} + 2)\operatorname{sech}^2(x - y). \end{aligned}$$

We choose

$$\eta(x) = \varepsilon + 1 + \tanh((x - A)/2)$$

for some large A and require (4). Then

$$\frac{d}{dt}\int \eta(x - y(t))u^2 dx + 3B((\eta')^{1/2}u) \le l.o.t.$$

This gives stability.

The theorem

Theorem (Buckmaster and K', 2015)

Let $\gamma > 0$. There exists $\varepsilon > 0$ so that if $u_0 \in H^k(\mathbb{R})$, $k \geq -1$,

$$\|u_0 + 2\,\mathrm{sech}^2(x)\|_{H^{-1}} \leq \varepsilon$$

then there exists $y(t) \sim 4t$ so that the solution to KdV satisfies

$$\lim_{t\to\infty} \|u(t) + 2\,\mathrm{sech}^2(x - y(t))\|_{H^k([\gamma t,\infty))} = 0.$$