Quasi-optimal initial value conditions for the Navier-Stokes equations

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Navier–Stokes Equations (NSt)

$$
\begin{aligned}
u_t - \Delta u + u \cdot \nabla u + \nabla p &= f \quad \text{in } \Omega \times (0,T) \\
\operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,T) \\
u &= 0 \quad \text{on } \partial\Omega \times (0,T) \\
u(0) &= u_0 \quad \text{at } t = 0
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^3$ bounded (or exterior domain), $\partial\Omega \in C^{1,1}$,

$0 < T \leq \infty$, $\nu = 1$

Theorem (Existence of Weak Leray-Hopf Solutions)

Let $u_0 \in L^2_\sigma(\Omega)$ *and* $f = \mathit{div}\, F$ *with* $F \in L^2(0,T;L^2)$ *be given.*

Then there exists at least one weak Leray-Hopf solution

$$u \in \mathcal{LH}_T = L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_{0,\sigma}(\Omega))$$

of (NSt) in the sense of distributions. Moreover,

$u \in C^0_w([0,T);L^2_\sigma(\Omega))$, *i.e.,* u *is weakly continuous in* $L^2_\sigma(\Omega)$,

and u *satisfies the strong energy inequality (SEI), i.e.,*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^{t} \|\nabla u\|_2^2 \, d\tau \leq \frac{1}{2}\|u(t_0)\|_2^2 + \int_{t_0}^{t} \langle f, u\rangle d\tau$$

for a.a. $0 \leq t_0 \leq T$ *(including* $t_0 = 0$*) and all* $t \in (t_0, T)$.

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for a.a. $0 \le t_0 \le T$ *(including* $t_0 = 0$*) and all* $t \in (t_0, T)$.

Note A weak solution satisfies

$$u \in L^s(0,T;L^q_\sigma(\Omega)) \quad \text{for all} \quad s \ge 2,\, q \ge 2 : \frac{2}{s} + \frac{3}{q} = \frac{3}{2}$$

Strong Solutions

Definition A weak solution $u \in \mathcal{LH}_T$ is called a *strong solution* if additionally

$$u \in L^s(0,T; L^q_\sigma(\Omega)) \quad \text{for some} \quad s > 2,\, q > 3 : \frac{2}{s} + \frac{3}{q} = 1$$

Theorem (Uniqueness) Let u be a strong solution of (NSt). Then u is unique among all weak solutions satisfying *(EI)* (only $t_0 = 0$ is required).

Theorem (Regularity) Let u be a strong solution of (NSt) and $f \in C_0^\infty(\overline{\Omega} \times (0,T))$, $\partial\Omega \in C^\infty$. Then $u \in C^\infty(\overline{\Omega} \times (0,T))$.

Initial Values – Optimal Condition (Sohr, Varnhorn, F. ('09))

Let $\Omega \subset \mathbb{R}^3$ bounded with $\partial\Omega \in C^{1,1}$ and $u_0 \in L^2_\sigma(\Omega)$, let $2 < s,\ 3 < q$ and $\frac{2}{s} + \frac{3}{q} = 1$.

1. The condition

$$\int_0^\infty \|e^{-\tau A_2} u_0\|_q^s \, d\tau < \infty$$

is *necessary and sufficient* for the existence of a unique strong solution $u \in L^s(0,T;L^q)$, $0 < T < \infty$, of (NSt).

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2. $\exists \varepsilon_* = \varepsilon_*(\Omega, q) > 0$: If

$$\int_0^T \|e^{-\tau A_2} u_0\|_q^s \, d\tau \leq \varepsilon_*$$

   then there exists a unique strong solution u on $[0,T)$ with $u(0) = u_0$.

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Exterior domain: Komo-Fa., Analysis 33 (2013)
General domain: L^2-theory (Sohr, Varnhorn, Fa. (2009)) or $\tilde{L}^q$-spaces $\tilde{L}^q = L^q \cap/+ L^2$ (Riechwald, PhD, Darmstadt 2011)

Ideas of Proof I

(i) A weak solution u has the representation

$$u(t) = E(t) - \int_0^t A^{1/2} e^{-(t-\tau)A} (A^{-1/2} P\mathsf{div}\,)(u \otimes u)(\tau)\, d\tau$$

where $E(t) = e^{-tA} u_0$ and the formal operator $A^{-1/2} P\mathsf{div}$ is bounded on each $L^q(\Omega)$, $1 < q < \infty$

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(ii) The operator $\mathcal{F}$ for $\tilde{u} = u - E$,

$$\mathcal{F}(\tilde{u})(t) = -\int_0^t A^{1/2} e^{-(t-\tau)A} (A^{-1/2} P\mathrm{div}\,)(\tilde{u}+E) \otimes (\tilde{u}+E))(\tau)\, d\tau,$$

defines a strict contraction in a closed ball of $L^s(0,T'; L^q(\Omega))$ for sufficiently small $0 < T' < \infty$ (need ε_* and T' small)

Ideas of Proof II

(iii) Show that $\tilde{u} = \mathcal{F}(\tilde{u})$ defines a weak solution $u = \tilde{u} + E$ of (NSt) in $\mathcal{LH}_{T'}$ with initial value u_0

Step 1: Show that $\nabla \tilde{u} \in L^2(0,T;L^2(\Omega))$:
Use Yosida approximation operators to get for $\tilde{u}_n = J_n \tilde{u}$ that

$$\left\| A^{\frac{1}{2}} \tilde{u}_n \right\|_{2,2;T} \leq c \left\| u \right\|_{q,s;T} \left(\left\| A^{\frac{1}{2}} \tilde{u}_n \right\|_{2,2;T} + \left\| \nabla E \right\|_{2,2;T} \right)$$

Choose T small for absorption $\Rightarrow$ uniformly in $n \in \mathbb{N}$

$$\left\| A^{\frac{1}{2}} \tilde{u}_n \right\|_{2,2;T} \leq 2c \left\| u \right\|_{q,s;T} \left\| \nabla E \right\|_{2,2;T} < \infty.$$

$n \to \infty \Rightarrow$

$$A^{\frac{1}{2}} \tilde{u}, \nabla \tilde{u}, \nabla u \in L^2(0,T';L^2(\Omega))$$

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Step 2: Show that $uu \in L^2(0, T; L^2) \Rightarrow$ u is a weak Leray-Hopf solution satisfying the energy equality on $[0, T]$ □

Besov space notation by H. Amann (2002):

$$\mathbb{B}^{2/s}_{q',s'}(\Omega) = B^{2/s}_{q',s'}(\Omega) \cap L^{q'}_\sigma(\Omega) = \{v \in B^{2/s}_{q',s'}(\Omega) : \operatorname{div} v = 0, N \cdot v|_{\partial\Omega} = 0\}$$

Define

$$\mathbb{B}^{-2/s}_{q,s}(\Omega) = \big(\mathbb{B}^{2/s}_{q',s'}(\Omega)\big)^* = \big(L^{q'}_\sigma, \mathcal{D}(A_{q'})\big)^*_{1-1/s',s'} = \big(\mathcal{D}(A_{q'})^*, L^q_\sigma\big)_{1-1/s,s}$$

using real interpolation, duality $\Rightarrow$

$$\|u_0\|_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} \sim \|A^{-1}u_0\|_{(L^q_\sigma(\Omega),\mathcal{D}(A_q))_{1-1/s,s}} \sim \Big(\int_0^\delta \|e^{-\tau A}u_0\|_q^s \, d\tau\Big)^{1/s}$$

for bounded Ω and with δ
$\Rightarrow$ integrability of $\|e^{-\tau A}u_0\|_q^s$, $u_0 \in L^2_\sigma(\Omega)$, is crucial only near 0

Applications

Apply criteria using $\mathbb{B}_{q,s}^{-2/s}(\Omega)$ along a given weak solution at all t_0 (or at almost all t_0 (?)) and identify the local strong solution starting at t_0 with the weak solution via Serrin's Uniqueness Theorem
$\Rightarrow$ left-hand or right-hand local/global regularity and uniqueness, see Sohr, Varnhorn, F. (Indiana Univ. Math. J. 2007, JMFM 2008, 2012, 2014, Ann. Univ. Ferrara 2009, 2014)

Extension to Quasi-optimal Initial Values

$$\mathcal{D}(A_2^{1/4}) \subset L^3_\sigma(\Omega) \subset L^{3,r}_\sigma(\Omega) \subset \mathbb{B}^{-2/s_q}_{q,s_q}(\Omega) = \mathbb{B}^{-1+3/q}_{q,s_q}(\Omega)$$

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where $\frac{2}{s_q} + \frac{3}{q} = 1$.
For $s_q < s < \infty$ we proceed with

$$\mathbb{B}^{-1+3/q}_{q,s_q}(\Omega) \subset \mathbb{B}^{-1+3/q}_{q,s}(\Omega) \subset \mathbb{B}^{-1+3/q}_{q,\infty}(\Omega)$$

where $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$, $0 < \alpha < \frac{1}{2}$.

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where $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$, $0 < \alpha < \frac{1}{2}$.

Here $\mathbb{B}^{-1+3/q}_{q,s}(\Omega) = \big(\mathcal{D}(A_{q'})^*, L^q_\sigma(\Omega)\big)_{1-\alpha-1/s,s}$ and

$$\|u_0\|_{\mathbb{B}^{-1+3/q}_{q,s}} \sim \|e^{-\tau A}u_0\|_{L^s_\alpha(L^q)} := \int_0^\infty \left(\tau^\alpha \|e^{-\tau A}u_0\|_q\right)^s d\tau < \infty$$

$$\sim \int_0^\delta \left(\tau^\alpha \|e^{-\tau A}u_0\|_q\right)^s d\tau < \infty$$

How far do the previous results hold when $s_q < s < \infty$?

We are looking for an $L^s_\alpha(L^q)$-strong solution u, i.e., u is a Leray-Hopf weak solution and

$$\int_0^T \left(\tau^\alpha \|u(\tau)\|_q\right)^s d\tau < \infty$$

where $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < \frac{1}{2}$ with $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$

Main Theorem (Giga, Hsu, F. ('14))

Let $u_0 \in L^2_\sigma(\Omega)$ and s, q, α be given.

(1) There exists an $\epsilon_* = \epsilon_*(q, s, \alpha, \Omega) > 0$: If

$$\|e^{-\tau A} u_0\|_{L^s_\alpha(0,T;L^q)} \leq \epsilon_*,$$

then (NSt) has a unique $L^s_\alpha(L^q)$-strong solution with data u_0 on $[0, T)$.

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(2) The condition

$$\int_0^\infty (\tau^\alpha \|e^{-\tau A} u_0\|_q)^s \, d\tau < \infty$$

is sufficient and necessary for the existence of a unique $L^s_\alpha(L^q)$-strong solution $u \in L^s_\alpha(0, T; L^q)$ with data u_0.

(3) The $L^s_\alpha(L^q)$-strong solution satisfies the energy equality on $[0, T)$.

Ideas of Proof I

(i) Construction of the $L^s_\alpha(L^q)$-strong solution: Find $\tilde{u} = u - E$, $E(t) = e^{-tA}u_0$, as fixed point $\tilde{u} = \mathcal{F}(\tilde{u})$ of the operator

$$\mathcal{F}(\tilde{u})(t) = -\int_0^t A^{1/2}e^{-(t-\tau)A}(A^{-1/2}P\mathrm{div}\,)(\tilde{u}+E)\otimes(\tilde{u}+E))(\tau)\,d\tau.$$

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Lemma: (Weighted Hardy-Littlewood-Sobolev inequality)
Let $0 < \lambda < 1$, $\alpha_2 \le \alpha_1$, $1 < s_1 \le s_2 < \infty$, $-\frac{1}{s_j} < \alpha_j < 1 - \frac{1}{s_j}$, $j = 1, 2$ and $\frac{1}{s_1} + (\lambda + \alpha_1 - \alpha_2) = 1 + \frac{1}{s_2}$. Then

$$I_\lambda f(t) = \int_{\mathbb{R}} (t-\tau)^{-\lambda} f(\tau)\, d\tau$$

defines a bounded operator $I_\lambda : L^{s_1}_{\alpha_1}(\mathbb{R}) \to L^{s_2}_{\alpha_2}(\mathbb{R})$.

Ideas of Proof II

Lemma: $\nabla u \in L^2(0,T;L^2)$, hence $u \in L^2(0,T;L^6)$

Proof Weighted Hardy-Littlewood-Sobolev inequality $\Rightarrow$

$$\left\| A^{\frac{1}{2}} J_n u \right\|_{2,2,;T} \leq 2c \left\| u \right\|_{L^s_\alpha(L^q)} \left\| \nabla E \right\|_{2,2;T} < \infty. \qquad \square$$

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As a direct consequence of Hölder's inequality and $\tilde{u} = \mathcal{F}(\tilde{u})$ (this argument requires $\alpha > 0\,(!)$)

$$\|\tilde{u}(t)\|_2 \leq c\|u\|_{L^s_\alpha(0,t;L^q)}\|\nabla u\|_{L^2(L^2)}$$

$\Rightarrow$ $u \in L^\infty(0,T;L^2)$ and $\|\tilde{u}(t)\|_2 \to 0$ as $t \to 0$

$\Rightarrow$ $\qquad u(t) \to u_0 \text{ in } L^2(\Omega) \text{ as } t \to 0$

$\Rightarrow$ u is a weak solution with $u(0) = u_0$

Ideas of Proof III

Lemma: $u \in L^4_{\alpha/(2+8\alpha)}(0,T;L^4(\Omega))$.

Proof With $\beta = \frac{1}{2+8\alpha}$, $q_1, s_1 \ldots$, $\frac{2}{s_1} + \frac{3}{q_1} = \frac{3}{2}$, and an interpolation inequality

$$\int_0^T \tau^{4\alpha\beta} \|u\|_4^4 \, d\tau \le \|u\|^{4\beta}_{L^s_\alpha(L^q)} \|u\|^{4(1-\beta)}_{L^{s_1}(L^{q_1})} < \infty. \quad \square$$

Ideas of Proof III

Lemma: $u \in L^4_{\alpha/(2+8\alpha)}(0,T;L^4(\Omega))$.

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$\Rightarrow u \in L^4(\varepsilon,T;L^4(\Omega)) \Rightarrow u \otimes u \in L^2(\varepsilon,T;L^2)$ can be considered as right-hand side of a Stokes system on (ε,T)

$\Rightarrow u$ satisfies the energy equality on (ε,T):

$$\frac{1}{2}\|u(t)\|_2^2 + \int_\varepsilon^t \|\nabla u\|_2^2 \, d\tau = \frac{1}{2}\|u(\varepsilon)\|_2^2$$

Since $u(\varepsilon) \to u_0$ in $L^2(\Omega)$ as $\varepsilon \to 0$,

$$u \ \text{ satisfies the energy equality on } [0,T).$$

Are $L^s_\alpha(L^q)$-strong solutions unique?

The proof of Serrin's uniqueness theorem

does not work when $\alpha > 0$!

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Uniqueness Theorem: Let $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < \frac{1}{2}$ with $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$ and $u_0 \in L^2_\sigma(\Omega) \cap \mathbb{B}^{-1+\frac{3}{q}}_{q,s}(\Omega)$.
Then the unique $L^s_\alpha(0, T; L^q)$-strong solution u is *unique* on $[0, T)$ *in the class of all well-chosen weak solutions.*

Assumptions

(i) Let $u_{0n} \rightharpoonup u_0$ in $L^2_\sigma(\Omega)$ and $u_{0n} \to u_0$ in $\mathbb{B}_{q,s}^{-1+\frac{3}{q}}(\Omega)$ as $n \to \infty$

(ii) Let $(J_n) \subset \mathcal{L}\big(L^q_\sigma(\Omega), D(A_q^{1/2})\big)$ such that

$$\|J_n\|_{\mathcal{L}(L^q_\sigma)} + \|\frac{1}{n}A_q^{1/2}J_n\|_{\mathcal{L}(L^q_\sigma)} \leq C_q \text{ and } J_n u \overset{n\to\infty}{\to} u \text{ in } L^q_\sigma(\Omega)$$

Consider the approximate (NSt)

$$\begin{cases} \partial_t u_n - \Delta u_n + (J_n u_n) \cdot \nabla u_n + \nabla p_n = 0, \quad \operatorname{div} u_n = 0 \\ \qquad\qquad\qquad\qquad u_n|_{\partial\Omega} = 0, \quad u_n(0) = u_{0n} \end{cases}$$

Definition of well-chosen weak solutions I

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Example Yosida approximation $J_n = \big(I + \frac{1}{n}A^{1/2}\big)^{-1}$, $u_{0n} = J_n u_0$

Consequences $\exists!$ weak solution $u_n \in \mathcal{LH}_\infty$:

$$\|u_n\|_{L^\infty(L^2)} + \|u_n\|_{L^2(H^1)} \le C\|u_{0n}\|_2 \le C < \infty$$

Definition of well-chosen weak solutions II

There exists $(u_{n_k}) \subset (u_n)$ and $v \in \mathcal{LH}_\infty$:

$$u_{n_k} \rightharpoonup v \text{ in } L^2(H_0^1), \;\; u_{n_k} \overset{*}{\rightharpoonup} v \text{ in } L^\infty(L^2), \;\; u_{n_k} \to v \text{ in } L^2(L^2).$$

$\Rightarrow u_{n_k}(t_0) \to v(t_0)$ in $L^2(\Omega)$ for a.a. $t_0 \in (0,T)$.

$$\Rightarrow v \text{ is a weak solution of (NSt) with } v(0) = u_0$$

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Definition The solution v constructed as above is called a *well-chosen weak solution* of (NSt) with $v(0) = u_0$.

Remark By the uniqueness theorem (to be proved) the well-chosen weak solution v with $v(0) = u_0$ is unique;
$\Rightarrow v$ does *not* depend on the choice of $u_{0n} \to u_0$ and the choice of (weakly) convergent subsequences!

Proof of uniqueness theorem for well-chosen weak solutions I

(i) Construction of a local strong $L^s_\alpha(L^q)$-solution u' on $[0,T')$:
$\exists\ \varepsilon_* > 0$, and since $u_{0n} \to u_0$ in $\mathbb{B}^{-1+3/q}_{q,s}(\Omega)$, we find $0 < T' \le T$:

$$\int_0^{T'} \left(\tau^\alpha \|e^{-\tau A} u_{0n}\|_q\right)^s d\tau < \varepsilon_*$$

$\Rightarrow$ the approximate (NSt) has a unique solution
$u_n \in L^s_\alpha(0,T';L^q)$ with bound

$$\|u_n\|_{L^s_\alpha(0,T';L^q)} \le C\varepsilon_* \quad \forall n$$

$\Rightarrow \exists$ subsequence (u'_{n_k}) of (u_{n_k}): $u'_{n_k} \rightharpoonup u'$ in $L^s_\alpha(0,T';L^q)$
$\Rightarrow u' = v$ is a strong $L^s_\alpha(L^q)$-solution on $[0,T')$ with $u'(0) = u_0$.

Proof of uniqueness theorem for well-chosen weak solutions II

Up to now we got 3 solutions:

$$u \in L^s_\alpha(0,T;L^q), \quad u' \in L^s_\alpha(0,T';L^q), \quad v \in \mathcal{LH}_T$$

Uniqueness in $L^s_\alpha(0,T';L^q) \Rightarrow u = u' = v$ on $(0,T')$

$\Rightarrow u_n \rightharpoonup u' = u$ in $L^s_\alpha(0,T';L^q)$ (whole sequence (u_n)),
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$\Rightarrow v$ satisfies (EE) on $[0,T']$ and, of course, (SEI) on $[0,T)$
$\Rightarrow v$ satisfies (EI) on $[T',T)$. Serrin's Uniqueness Theorem $\Rightarrow$

$$u = v \quad \text{on} \quad [T',T)$$

□

The limit $s = \infty$

Is the space $B^{-1+3/q}_{q,\infty}(\Omega)$ too large? For $\Omega = (0, 2\pi)^3_{\text{per}}$

$$\exists\, u_0 \in \bigcap_{1<q\leq\infty} B^{-1+3/q}_{q,\infty}(\Omega) \cap L^2_\sigma(\Omega)$$

such that each Leray-Hopf weak solution u with $u(0) = u_0$ satisfies

$$\|u(t) - u_0\|_{B^{-1+3/q}_{q,\infty}} \geq \delta > 0$$

with δ independent of u, although

$$\|u(t) - u_0\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to 0,$$

cf. Cheskidov-Shvydkoy, Proc. AMS 2010

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What about $B^{-1+3/q}_{q,\infty}(\Omega)^{\mathrm{o}} := \big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)^{\mathrm{o}}_{1-\alpha,\infty}$?