Vortex stretching and local anisotropic diffusion in the 3D NSE

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3D Navier-Stokes equations (NSE) – describing a flow of 3D incompressible viscous fluid – read

$$u_t + (u \cdot \nabla)u = -\nabla p + \nu \triangle u,$$

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taking the curl yields the vorticity formulation,

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \triangle \omega,$$

where $\omega = \operatorname{curl} u$ is the vorticity of the fluid

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$\longrightarrow$ *locally anisotropic* dissipation there is strong numerical evidence that the regions of intense vorticity organize in coherent vortex structures, and in particular, in elongated vortex filaments, cf.

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an in-depth analysis of creation and dynamics of vortex tubes in 3D turbulent incompressible flows was presented in [Constantin, Procaccia and Segel, 1995]; see also

[Galanti, Gibbon and Heritage, 1997; Gibbon, Fokas and Doering, 1999; Ohkitani, 2009; Hou, 2009]

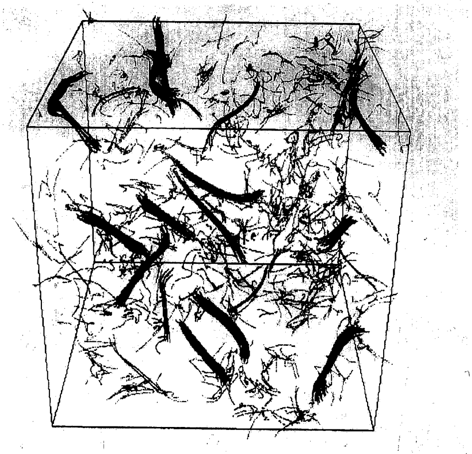

Figure 11: Intermittent vortex filaments in a 3D computational simulation of equilibrium homogeneous turbulence. (She *et al.*, 1991)

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(ii) *local existence* of a sparse/thin direction

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and later in [Beirao da Veiga and Berselli, 2002] where the Lipschitz condition was replaced by $\frac{1}{2}$-Hölder localized vortex-stretching term can be written [G., 2009] as

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(\omega \cdot \nabla)u \cdot \phi\omega\,(x) &= \phi^{\frac{1}{2}}\,(x)\,\frac{\partial}{\partial x_i} u_j(x)\,\phi^{\frac{1}{2}}\,(x)\,\omega_i(x)\,\omega_j(x) \\
&= -c\,P.V. \int_{B(x_0,2r)} \epsilon_{jkl}\,\frac{\partial^2}{\partial x_i \partial y_k}\,\frac{1}{|x-y|}\,\phi^{\frac{1}{2}}\,\omega_l\,dy\;\phi^{\frac{1}{2}}\,(x)\,\omega_i(x)\,\omega_j(x) + \text{ LOT} \\
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geometric cancelations in the highest order-term VST were utilized in [G., 2009] to obtain a spatiotemporal localization of $\frac{1}{2}$-Hölder coherence of the vorticity direction regularity criterion

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the following regularity class – a scaling-invariant improvement of $\frac{1}{2}$-Hölder coherence – is included,

$$\int_{t_0-(2R)^2}^{t_0} \int_{B(x_0,2R)} |\omega(x,t)|^2 \; \rho^2_{\frac{1}{2},2R}(x,t) dx\, dt < \infty; \tag{2}$$

$$\rho_{\gamma,r}(x,t) = \sup_{y \in B(x,r), y \neq x} \frac{|\sin\varphi\big(\xi(x,t),\xi(y,t)\big)|}{|x-y|^\gamma}$$

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a corresponding *a priori* bound had been previously obtained in [Constantin, 1990],

$$\int_0^T \int_{\mathbb{R}^3} |\omega(x,t)||\nabla\xi(x,t)|^2 \, dx\, dt \le \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2 \, dx$$

(see also [Constantin, Procaccia and Segel, 1995].)

the studies of the coherence of the vorticity direction up to the boundary-regularity criteria (for slip boundary conditions) were presented in [Beirao da Veiga and Berselli, 2002] and [Beirao da Veiga, 2013]

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assuming the type I blow-up (at most scaling-invariant blow-up rate),

$$|u(x,t)| \le \frac{C}{(T-t)^{\frac{1}{2}}},$$

Giga and Miura [2011] showed that if the vorticity direction possesses a *uniform modulus of continuity*, no singularity can form at $t = T$

(cf. [Giga, Hsu and Maekawa, 2014], for the case of the half-space)

* * *

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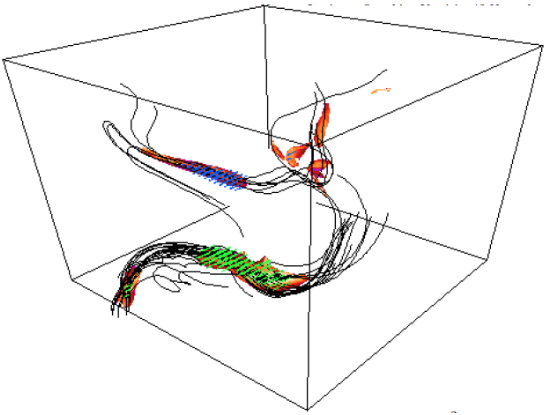

[Holm and Kerr, 2002]

Definition

Let x_0 be a point in $\mathbb{R}^3$, $r > 0$, S an open subset of $\mathbb{R}^3$ and δ in $(0,1)$.

The set S is linearly δ-sparse around x_0 at scale r in weak sense if there exists a unit vector d in S^2 such that

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then the following holds [G., *Nonlinearity* 2013]

Theorem (local anisotropic diffusion)

Suppose that a solution u is regular on an interval $(0, T^*)$.

Assume that either

(i) there exists t in $(0, T^*)$ such that $t + \frac{1}{d_0^2 \|\omega(t)\|_\infty} \geq T^*$, or

(ii) $t + \frac{1}{d_0^2 \|\omega(t)\|_\infty} < T^*$ for all t in $(0, T^*)$, and there exists ϵ in $(0, T^*)$ such that for any t in $(T^* - \epsilon, T^*)$, there exists $s = s(t)$ in $\left[t + \frac{1}{4 d_0^2 \|\omega(t)\|_\infty}, t + \frac{1}{d_0^2 \|\omega(t)\|_\infty}\right]$ with the property that for any spatial point x_0, there exists a scale $r = r(x_0)$, $0 < r \leq \frac{1}{2 d_0^2 \|\omega(t)\|_\infty^{\frac{1}{2}}}$, such that the super-level set $\Omega_s(M)$ is linearly δ-sparse around x_0 at scale r in weak sense; here, $\delta = \delta(x_0)$ is an arbitrary value in $(0, 1)$, $h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$, $\alpha = \alpha(\delta) \geq \frac{1-h}{h}$, and $M = M(\delta) = \frac{1}{d_0^\alpha} \|\omega(t)\|_\infty$.

Then, there exists $\gamma > 0$ such that ω is in $L^\infty\Big((T^* - \epsilon, T^* + \gamma); L^\infty\Big)$, i.e., T^* is not a singular time. (d_0 is a suitable absolute constant.)

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(iii) a consequence of the general harmonic measure majorization principle:

let D be open and K closed in $\mathbb{C}$, f analytic in $D \setminus K$, $|f| \leq M$, and $|f| \leq m$ on K.

then

$$|f(z)| \leq m^\theta \, M^{1-\theta}$$

for any z in $D \setminus K$, where $\theta = h(z, D, K)$ is the harmonic measure of K with respect to D evaluated at z

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(iv) a result on extremal properties of the harmonic measure in the unit disk $\mathbb{D}$ [Solynin, 1999]

consider a flow near the first (possible) singular time T^*, and define *the region of intense vorticity* at time $s(t) < T^*$ to be the region in which the vorticity magnitude exceeds a fraction of $\|\omega(t)\|_\infty$; this corresponds to the set

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let us for a moment accept this as a plausible geometric blow up scenario; the length scale associated with the diameters of the cross-sections can then be estimated *indirectly*, by estimating the rate of the decrease of the total volume of the region of intense vorticity $\Omega_{s(t)}\Big(\frac{1}{c_1}\|\omega(t)\|_\infty\Big)$

taking the initial vorticity to be a bounded measure, Constantin [1990] showed that the L^1-norm of the vorticity is *a priori* bounded over any finite time-interval; a desired estimate on the total volume of the region of intense vorticity follows simply from Tchebyshev inequality,

$$\text{Vol}\left(\Omega_{s(t)}\left(\frac{1}{c_1}\|\omega(t)\|_\infty\right)\right) \le \frac{c_2^0}{\|\omega(t)\|_\infty} \quad (c_2^0 > 1)$$

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this implies the decrease of the diameters of the cross-section of at least $\dfrac{c_3^0}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$ $(c_3^0 > 1)$, which is exactly the scale of *local one-dimensional sparseness* of the region of intense vorticity [the scale of local anisotropic diffusion] needed to prevent the formation of singularities presented in the previous theorem

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in other words, the NSE problem in this scenario becomes *critical*

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$\longrightarrow$ the vortex stretching in this scenario acts as the mechanism bridging (literally) the scaling gap in the regularity problem

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introduced in the study of turbulent transport rates in 3D incompressible fluid flows:

[Dascaliuc and G., *Comm. Math. Phys.* 2011, 2012, 2013; *C. R. Math. Acad. Sci. Paris* 2012]

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the parameters K_1 and K_2 represent the maximal *global* and *local multiplicities*, respectively

for a physical density of interest f, consider – localized to the cover elements $B(x_i, R)$ (per unit mass) – local quantities $\hat{f}_{x_i,R}$,

$$\hat{f}_{x_i,R} = \frac{1}{R^3} \int_{B(x_i,2R)} f(x) \psi^{\delta}_{x_i,R}(x)\, dx$$

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the key feature of $\{\langle F \rangle_R\}_{0<R\le R_0}$ is that $\langle F \rangle_R$ being *stable* – i.e., nearly-independent on a particular choice of the cover (with the fixed local multiplicity K_2) – indicates there are *no significant sign fluctuations* at scales comparable or greater than R

the key feature of $\{\langle F\rangle_R\}_{0<R\leq R_0}$ is that $\langle F\rangle_R$ being *stable* – i.e., nearly-independent on a particular choice of the cover (with the fixed local multiplicity K_2) – indicates there are *no significant sign fluctuations* at scales comparable or greater than R

on the other hand, if f does exhibit significant sign fluctuations at scales comparable or greater than R, suitable *rearrangements* of the cover elements up to the maximal multiplicity – emphasizing first the positive and then the negative parts of f – will result in $\langle F\rangle_R$ experiencing a wide range of values, from positive through zero to negative, respectively (the larger K_2, the finer detection..)

for a non-negative density f, the ensemble averages are all comparable to each other throughout the full range of scales, $0 < R \leq R_0$; in particular, they are all comparable to the simple average over the macro scale domain for a non-negative density f, the ensemble averages are all comparable to each other throughout the full range of scales, $0 < R \leq R_0$; in particular, they are all comparable to the simple average over the macro scale domain

$$\frac{1}{K_1} F_0 \leq \langle F \rangle_R \leq K_2 F_0 \tag{3}$$

for all $0 < R \leq R_0$, where

$$F_0 = \frac{1}{R_0^3} \int f(x) \psi_0^\delta(x)\, dx$$

denote the time-averaged localized vortex-stretching terms per unit mass associated to the cover element $B(x_i, R)$ by $VST_{x_i,R,t}$,

$$VST_{x_i,R,t} = \frac{1}{t}\int_0^t \frac{1}{R^3}\int (\omega \cdot \nabla)u \cdot \omega \; \phi_i \, dx \, ds \tag{4}$$

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the quantity of interest is the ensemble average of $\{VST_{x_i,R,t}\}_{i=1}^n$,

$$\langle VST\rangle_{R,t} = \frac{1}{n}\sum_{i=1}^{n} VST_{x_i,R,t} \tag{5}$$

$B(x_i, R)$-localized enstrophy level dynamics is as folows

$$\begin{aligned} \int_0^t \int (\omega \cdot \nabla) u \cdot \phi_i \, \omega \, dx \, ds &= \int \frac{1}{2} |\omega(x,t)|^2 \psi_i(x) \, dx + \int_0^t \int |\nabla \omega|^2 \phi_i \, dx \, ds \\ &- \int_0^t \int \frac{1}{2} |\omega|^2 \big((\phi_i)_s + \triangle \phi_i \big) \, dx \, ds \\ &- \int_0^t \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds, \end{aligned} \tag{6}$$

for any t in $(2T/3, T)$, and $1 \leq i \leq n$

denote by $E_{0,t}$ time-averaged enstrophy per unit mass associated with the macro scale domain $B(0, 2R_0) \times (0, t)$,

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and by $\sigma_{0,t}$ a corresponding Kraichnan-type scale,

$$\sigma_{0,t} = \left(\frac{E_{0,t}}{P_{0,t}} \right)^{\frac{1}{2}}$$

then the following holds [Dascaliuc and G., *J. Math. Phys.* 2012] then the following holds [Dascaliuc and G., *J. Math. Phys.* 2012]

Theorem (vortex stretching/Taylor *vs.* v. Karman)

Let u be a global-in-time local Leray solution on $\mathbb{R}^3 \times (0, \infty)$, regular on $(0, T)$. Suppose that, for some $t \in (2T/3, T)$,

$$C \max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\} \, \sigma_{0,t}^{\frac{1}{2}} < R_0 \tag{7}$$

where $M_0 = \sup_t \int_{B(0,2R_0)} |u|^2 < \infty$, and $C > 1$ a suitable constant depending only on the cover parameters.
Then,

$$\frac{1}{C} P_{0,t} \leq \langle VST \rangle_{R,t} \leq C \, P_{0,t} \tag{8}$$

for all R satisfying

$$C \max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\} \, \sigma_{0,t}^{\frac{1}{2}} \leq R \leq R_0. \tag{9}$$

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(i) suppose that T is the first (possible) singular time, and that the macro scale domain contains some of the spatial singularities (at time T); this, paired with the assumption that u is a global-in-time local Leray solution implies

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(iii) the power of $\frac{1}{2}$ on $\sigma_{0,t}$ is a correction originating in the localized transport term

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2 different (and somewhat complementary) results

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2 different (and somewhat complementary) results

[Bradshaw and G., J. Math. Fluid Mech. 2013]

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the idea is to try to get a uniform-in-time estimate on

$$\int \psi\, w_k \log w_k \, dx \qquad \text{or} \qquad \int \psi\, w \log w \, dx$$

$\left(w_k = \sqrt{1+\omega_k^2},\ w = \sqrt{1+|\omega|^2} \right)$

[Bradshaw and G., Indiana Univ. Math. J. 2014] (vectorial approach)

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the maximal function of a distribution f is defined as

$$M_h f(x) = \sup_{t>0} |f * h_t(x)|$$

where h is a fixed, normalized test function supported in the unit ball, and h_t denotes $t^{-n} h(\cdot/t)$

a distribution f is in the Hardy space $\mathcal{H}^1$ if $\|f\|_{\mathcal{H}^1} = \|M_h f\|_1 < \infty$

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the local maximal function is defined as,

$$m_h f(x) = \sup_{0<t<1} |f * h_t(x)|, \ x \in \mathbb{R}^n,$$

a distribution f is in the local Hardy space $\mathfrak{h}^1$ if $\|f\|_{\mathfrak{h}^1} = \|m_h f\|_1 < \infty$

Div-Curl Lemma (Coifman, Lions, Meyer, Semmes)

suppose that E and B are L^2-vector fields satisfying div $E =$ curl $B = 0$ (in the sense of distributions). then,

$$\|E \cdot B\|_{\mathcal{H}^1} \leq c(n)\, \|E\|_{L^2} \|B\|_{L^2}$$

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the classical space of bounded mean oscillations, BMO is defined as follows

$$BMO = \left\{ f \in L^1_{loc} : \sup_{x \in \mathbb{R}^n, r>0} \Omega\big(f, I(x,r)\big) < \infty \right\}$$

where $\Omega\big(f, I(x,r)\big) = \dfrac{1}{|I(x,r)|} \displaystyle\int_{I(x,r)} |f(x) - f_I| \, dx$ is the mean oscillation of the function f with respect to its mean $f_I = \frac{1}{|I(x,r)|} \int_{I(x,r)} f(x) \, dx$, over the cube $I(x,r)$ centered at x with the side-length r

a local version of BMO, usually denoted by bmo, is defined by finiteness of the following expression,

$$\|f\|_{bmo} = \sup_{x \in \mathbb{R}^n, 0<r<\delta} \Omega\big(f, I(x,r)\big) + \sup_{x \in \mathbb{R}^n, r \geq \delta} \frac{1}{|I(x,r)|} \int_{I(x,r)} |f(y)|\, dy,$$

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if $f \in L^1$, we can focus on small scales, e.g., $0 < r < \frac{1}{2}$. let ϕ be a positive, non-decreasing function on $(0, \frac{1}{2})$, and consider the following version of local weighted spaces of bounded mean oscillations,

$$\|f\|_{\widetilde{bmo}_\phi} = \|f\|_{L^1} + \sup_{x \in \mathbb{R}^n, 0<r<\frac{1}{2}} \frac{\Omega\big(f, I(x,r)\big)}{\phi(r)}$$

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of special interest will be the spaces $\widetilde{bmo} = \widetilde{bmo}_1$, and $\widetilde{bmo}_{\frac{1}{|\log r|}}$

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a sharp pointwise multiplier theorem.

let h be in $\widetilde{bmo}$, and g in $L^\infty \cap \widetilde{bmo}_{\frac{1}{|\log r|}}$. then

$$\|g\,h\|_{\widetilde{bmo}} \le c(n) \left(\|g\|_\infty + \|g\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}} \right) \|h\|_{\widetilde{bmo}}$$

more precisely, the space of pointwise $\widetilde{bmo}$ multipliers coincides with $L^\infty \cap \widetilde{bmo}_{\frac{1}{|\log r|}}$

let M denote the Hardy-Littlewood maximal operator

Coifman and Rochberg

$$\|\log Mf\|_{BMO} \leq c(n)$$

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this estimate remains valid if we replace Mf with $\mathcal{M}f = \left(M\sqrt{|f|}\right)^2$. the advantage of working with $\mathcal{M}$ is that the L^2-maximal theorem implies the following estimate

$$\|\mathcal{M}f\|_1 \leq c(n)\|f\|_1$$

(a bound that does not hold for the original maximal operator M.)

for any τ in $[0, T)$,

$$I(\tau) \equiv \int \psi(x)\, w(x,\tau) \log w(x,\tau)\, dx \le I(0) + c \int_0^{\tau} \int_x \omega \cdot \nabla u \cdot \psi\, \xi \, \log w \, dx\, dt$$

$$+ \text{ \textit{a priori} bounded}$$

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in order to take the advantage of the Coifman-Rochberg's estimate, we decompose the logarithmic factor as

$$\log w = \log \frac{w}{\mathcal{M} w} + \log \mathcal{M} w$$

denoting $\int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \, \xi \, \log w \, dx \, dt$ by J, this yields $J = J_1 + J_2$ where

$$J_1 = \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \, \xi \, \log \frac{w}{\mathcal{M} w} \, dx \, dt$$

and

$$J_2 = \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \, \xi \, \log \mathcal{M} w \, dx \, dt$$

for J_1, we use the *pointwise* inequality

$$w \log \frac{w}{\mathcal{M}w} \leq \mathcal{M}w - w$$

(a consequence of the pointwise inequality $\mathcal{M}f \geq f$, and the inequality $e^{x-1} \geq x$ for $x \geq 1$)

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this leads to

$$J_1 \leq \int_0^T \int_x |\nabla u| \Big(\mathcal{M}w - w \Big) \psi \, dx \, dt$$

which is *a priori* bounded by the Cauchy-Schwarz and the L^2-maximal theorem for J_2, we have the following string of inequalities

$$
\begin{aligned}
J_2 &\leq c \int_0^\tau \|\omega \cdot \nabla u\|_{\mathfrak{h}^1} \|\psi\,\xi \log \mathcal{M}w\|_{bmo}\, dt \\
&\leq c \int_0^\tau \|\omega \cdot \nabla u\|_{\mathcal{H}^1} \|\psi\,\xi \log \mathcal{M}w\|_{\widetilde{bmo}}\, dt \\
&\leq c \int_0^\tau \|\omega\|_2 \|\nabla u\|_2 \Big(\|\psi\,\xi\|_\infty + \|\psi\,\xi\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}}\Big)\Big(\|\log \mathcal{M}w\|_{BMO} + \|\log \mathcal{M}w\|_1\Big)\, dt \\
&\leq c \sup_{t\in(0,T)} \Big\{\Big(1 + \|\psi\,\xi\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}}\Big)\Big(\|\log \mathcal{M}w\|_{BMO} + \|\log \mathcal{M}w\|_1\Big)\Big\} \int_t \int_x |\nabla u|^2 \\
&\leq c \Big(1 + \sup_{t\in(0,T)} \|\psi\,\xi\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}}\Big) \Big(1 + \sup_{t\in(0,T)} \|\omega\|_1\Big) \int_t \int_x |\nabla u|^2
\end{aligned}
$$

by $\mathfrak{h}^1 - bmo$ duality, the Div-Curl Lemma, the pointwise $\widetilde{bmo}$-multiplier theorem, the Coifman-Rochberg's estimate, and the L^1-bound on the modified maximal operator $\mathcal{M}$

this implies the following result [Bradshaw and G., Indiana Univ. Math. J. 2014]

Theorem (breaking the criticality/the log-log-chaos sphere)

Let u be a Leray solution to the 3D NSE. Assume that the initial vorticity ω_0 is in $L^1 \cap L^2$, and that $T > 0$ is the first (possible) blow-up time. Suppose that

$$\sup_{t \in (0,T)} \|(\psi\xi)(\cdot,t)\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}} < \infty.$$

Then,

$$\sup_{t \in (0,T)} \int \psi(x)\, w(x,t) \log w(x,t)\, dx < \infty.$$

`good news:`

$\widetilde{bmo}_\phi$ contains discontinuous functions if and only if $\int_0^{\frac{1}{2}} \frac{\phi(r)}{r}\, dr = \infty$

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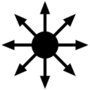
$\xi(\mathsf{sing}_x, T) \ \sim$ 

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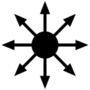
$\xi(\mathrm{sing}_x, T) \ \sim$ 

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[in particular, 'crossing of the vortex lines' is not an obstruction] conclusion:

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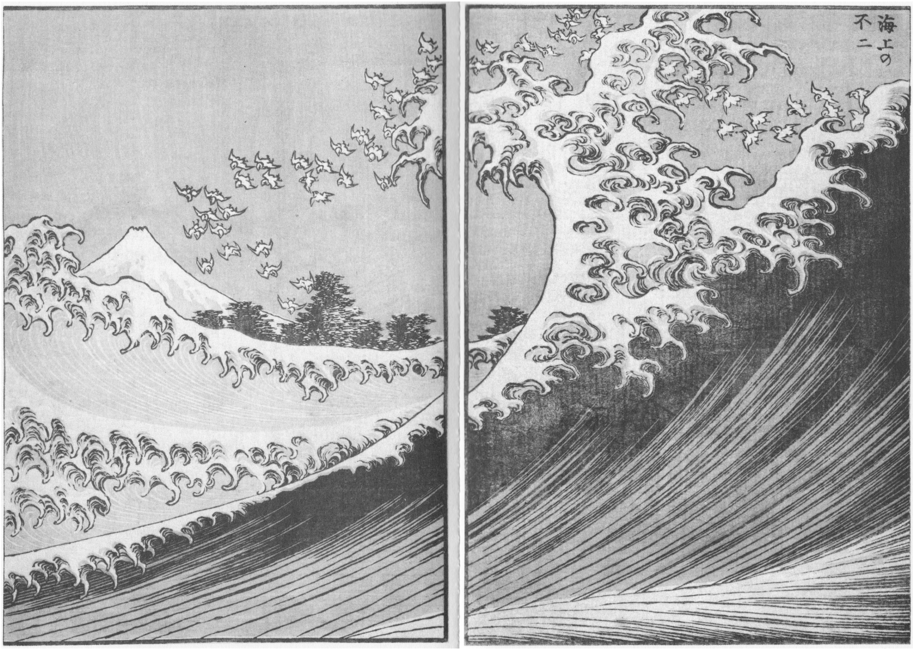
$$\left\|\xi(\cdot,T)\right\| < \infty \implies \lambda_{\omega(t)}(\beta) = O\left(\frac{1}{\beta \log \beta}\right) \implies \text{anisotropic diffusion wins}$$


冨嶽三十六景 神奈川沖浪裏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