Vortex stretching and local anisotropic diffusion in the 3D NSE

Zoran Grujić

University of Virginia

The 11th Japanese-German International Workshop on Mathematical Fluid Dynamics, Waseda University, Tokyo, Japan, March 10, 2015
3D Navier-Stokes equations (NSE) – describing a flow of 3D incompressible viscous fluid – read

\[ u_t + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \]

supplemented with the incompressibility condition \( \text{div} \, u = 0 \), where \( u \) is the velocity of the fluid, \( p \) is the pressure, and \( \nu \) is the viscosity.
3D Navier-Stokes equations (NSE) – describing a flow of 3D incompressible viscous fluid – read

\[ u_t + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \]

supplemented with the incompressibility condition \( \text{div} \ u = 0 \), where \( u \) is the velocity of the fluid, \( p \) is the pressure, and \( \nu \) is the viscosity

taking the curl yields the vorticity formulation,

\[ \omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega, \]

where \( \omega = \text{curl} \ u \) is the vorticity of the fluid
\[ \omega = \text{curl} \, u \quad \rightarrow \quad \Delta u = - \text{curl} \, \omega \]
\[ \omega = \text{curl } u \quad \Rightarrow \quad \nabla u = -\text{curl } \omega \]

\[ u(x) = c \int \nabla \frac{1}{|x - y|} \times \omega(y) \, dy \]

Zoran Grujić  Vortex stretching and local anisotropic diffusion in the 3D NSE
\[ \omega = \text{curl} \, u \quad \rightarrow \quad \Delta u = -\text{curl} \, \omega \]

\[ u(x) = c \int \nabla \frac{1}{|x - y|} \times \omega(y) \, dy \]

\[ \frac{\partial}{\partial x_i} u_j(x) = c \, \text{P.V.} \int \epsilon_{jkl} \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x - y|} \omega_l(y) \, dy \]
vortex-stretching has been viewed as the principal physical mechanism responsible for the vigorous creation of small scales in turbulent flows.
vortex-stretching has been viewed as the principal physical mechanism responsible for the vigorous creation of small scales in turbulent flows

d this goes back to G.I. Taylor:

vortex-stretching has been viewed as the principal physical mechanism responsible for the vigorous creation of small scales in turbulent flows.

this goes back to G.I. Taylor:


the production part is relatively well-understood; amplification of the vorticity via the process of vortex stretching is essentially a consequence of the conservation of the angular momentum in the incompressible flows.
vortex-stretching has been viewed as the principal physical mechanism responsible for the vigorous creation of small scales in turbulent flows

this goes back to G.I. Taylor:


the production part is relatively well-understood; amplification of the vorticity via the process of vortex stretching is essentially a consequence of the conservation of the angular momentum in the incompressible flows.

Taylor vs. v. Karman \( \langle V ST \rangle > 0 \) vs. \( \langle V ST \rangle = 0 \)
vortex-stretching has been viewed as the principal physical mechanism responsible for the vigorous creation of small scales in turbulent flows

this goes back to G.I. Taylor:


the production part is relatively well-understood; amplification of the vorticity via the process of vortex stretching is essentially a consequence of the conservation of the angular momentum in the incompressible flows.

Taylor vs. v. Karman $\langle VST \rangle > 0$ vs. $\langle VST \rangle = 0$

perhaps this is scale-dependent [?]
the precise physics and mathematics behind the vortex stretching-induced dissipation is much less transparent..
the precise physics and mathematics behind the vortex stretching-induced dissipation is much less transparent.

for his part, Taylor – based primarily on the wind tunnel measurements of turbulent flow past a (uniform) grid – concluded the following:

“\textquote{It seems that the stretching of vortex filaments must be regarded as the principal mechanical cause of the high rate of dissipation which is associated with turbulent motion.}”
the precise physics and mathematics behind the vortex stretching-induced dissipation is much less transparent..

for his part, Taylor – based primarily on the wind tunnel *measurements* of turbulent flow past a (uniform) grid – concluded the following:

“it seems that the stretching of vortex filaments must be regarded as the principal mechanical cause of the high rate of dissipation which is associated with turbulent motion.”

→ *locally anisotropic* dissipation
there is strong numerical evidence that the regions of intense vorticity organize in coherent vortex structures, and in particular, in elongated vortex filaments, cf.

[Siggia, 1981; Ashurst, Kerstein, Kerr and Gibson, 1987; She, Jackson and Orszag, 1991; Jimenez, Wray, Saffman and Rogallo, 1993; Vincent and Meneguzzi, 1994]
there is strong numerical evidence that the regions of intense vorticity organize in coherent vortex structures, and in particular, in elongated vortex filaments, cf. [Siggia, 1981; Ashurst, Kerstein, Kerr and Gibson, 1987; She, Jackson and Orszag, 1991; Jimenez, Wray, Saffman and Rogallo, 1993; Vincent and Meneguzzi, 1994]

an in-depth analysis of creation and dynamics of vortex tubes in 3D turbulent incompressible flows was presented in [Constantin, Procaccia and Segel, 1995]; see also [Galanti, Gibbon and Heritage, 1997; Gibbon, Fokas and Doering, 1999; Ohkitani, 2009; Hou, 2009]
Figure 11: Intermittent vortex filaments in a 3D computational simulation of equilibrium homogeneous turbulence. (She et al., 1991)
there are two imminent signatures of the filamentary geometry
there are two imminent signatures of the filamentary geometry

(i) local coherence of the vorticity direction
there are two imminent signatures of the filamentary geometry

(i) *local coherence* of the vorticity direction

(ii) *local existence* of a sparse/thin direction
geometric depletion of the nonlinearity

rigorous study of the anisotropic dissipation induced by local coherence of the vorticity direction was pioneered by Constantin [Constantin, 1994]

\[ |\sin \phi(\xi(x,t), \xi(y,t))| \leq L |x-y| \]

and later in [Beirao da Veiga and Berselli, 2002] where the Lipschitz condition was replaced by \(1/2\)-Hölder.
geometric depletion of the nonlinearity

rigorous study of the anisotropic dissipation induced by local coherence of the vorticity direction was pioneered by Constantin [Constantin, 1994]

based on a singular integral representation for the stretching factor in evolution of the vorticity magnitude featuring a geometric kernel depleted by local coherence of the vorticity direction
geometric depletion of the nonlinearity

rigorous study of the anisotropic dissipation induced by local coherence of the vorticity direction was pioneered by Constantin [Constantin, 1994]

based on a singular integral representation for the stretching factor in evolution of the vorticity magnitude featuring a geometric kernel depleted by local coherence of the vorticity direction

this was utilized in [Constantin and Fefferman, 1993] to show that as long as

\[ |\sin \varphi(\xi(x,t), \xi(y,t))| \leq L|x - y| \]

holds in the regions of intense vorticity, no finite-time blow up can occur; \( \xi = \frac{\omega}{|\omega|} \)
rigorous study of the anisotropic dissipation induced by local coherence of the vorticity direction was pioneered by Constantin [Constantin, 1994]

based on a singular integral representation for the stretching factor in evolution of the vorticity magnitude featuring a geometric kernel depleted by local coherence of the vorticity direction

this was utilized in [Constantin and Fefferman, 1993] to show that as long as $|\sin \varphi(\xi(x,t), \xi(y,t))| \leq L|x - y|$ holds in the regions of intense vorticity, no finite-time blow up can occur; $\xi = \frac{\omega}{|\omega|}$

and later in [Beirao da Veiga and Berselli, 2002] where the Lipschitz condition was replaced by $\frac{1}{2}$-Hölder
localized vortex-stretching term can be written [G., 2009] as

\[(\omega \cdot \nabla)u \cdot \phi \omega (x) = \phi^{\frac{1}{2}}(x) \frac{\partial}{\partial x_i} u_j(x) \phi^{\frac{1}{2}}(x) \omega_i(x) \omega_j(x)\]

\[= -c \text{P.V.} \int_{B(x_0,2r)} \epsilon_{jkl} \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x - y|} \phi^{\frac{1}{2}} \omega_l \, dy \, \phi^{\frac{1}{2}}(x) \omega_i(x) \omega_j(x) + \text{LOT}\]

\[= \text{VST} + \text{LOT} \]  

(1)
localized vortex-stretching term can be written [G., 2009] as

\[
(\omega \cdot \nabla) u \cdot \phi \omega (x) = \phi^{\frac{1}{2}}(x) \frac{\partial}{\partial x_i} u_j(x) \phi^{\frac{1}{2}}(x) \omega_i(x) \omega_j(x)
\]

\[
= -c \text{ P.V.} \int_{B(x_0,2r)} \epsilon_{jkl} \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x - y|} \phi^{\frac{1}{2}} \omega_l \, dy \, \phi^{\frac{1}{2}}(x) \omega_i(x) \omega_j(x) + \text{LOT}
\]

\[
= \text{VST} + \text{LOT}
\]

(1)

*geometric cancelations* in the highest order-term VST were utilized in [G., 2009] to obtain a spatiotemporal localization of \( \frac{1}{2} \)-Hölder coherence of the vorticity direction regularity criterion.
and later in [G. and Guberović, 2010] to introduce a family of *scaling-invariant* regularity classes featuring a balance between coherence of the vorticity direction and the vorticity magnitude.
and later in [G. and Guberović, 2010] to introduce a family of *scaling-invariant* regularity classes featuring a balance between coherence of the vorticity direction and the vorticity magnitude

the following regularity class – a scaling-invariant improvement of $\frac{1}{2}$-Hölder coherence – is included,

$$\int_{t_0}^{t_0-(2R)^2} \int_{B(x_0,2R)} |\omega(x,t)|^2 \rho^{\frac{1}{2},2R}(x,t) dx \, dt < \infty; \quad (2)$$

$$\rho_{\gamma,r}(x,t) = \sup_{y \in B(x,r), y \neq x} \frac{|\sin \varphi(\xi(x,t), \xi(y,t))|}{|x - y|^{\gamma}}$$
and later in [G. and Guberović, 2010] to introduce a family of scaling-invariant regularity classes featuring a balance between coherence of the vorticity direction and the vorticity magnitude

the following regularity class – a scaling-invariant improvement of $\frac{1}{2}$-Hölder coherence – is included,

$$\int_{t_0}^{t_0-(2R)^2} \int_{B(x_0,2R)} |\omega(x,t)|^2 \rho_{\frac{1}{2},2R}(x,t) dx dt < \infty; \quad (2)$$

$$\rho_{\gamma,r}(x,t) = \sup_{y \in B(x,r), y \neq x} \frac{|\sin \varphi(\xi(x,t),\xi(y,t))|}{|x-y|^\gamma}$$

a corresponding a priori bound had been previously obtained in [Constantin, 1990],

$$\int_0^T \int_{\mathbb{R}^3} |\omega(x,t)||\nabla \xi(x,t)|^2 dx dt \leq \frac{1}{2} \int_{\mathbb{R}^3} |u_0(x)|^2 dx$$

(see also [Constantin, Procaccia and Segel, 1995].)
the studies of the coherence of the vorticity direction up to the boundary-regularity criteria (for slip boundary conditions) were presented in [Beirao da Veiga and Berselli, 2002] and [Beirao da Veiga, 2013]

\[
|u(x,t)| \leq C(T-t)^{1/2}
\]

Giga and Miura [2011] showed that if the vorticity direction possesses a uniform modulus of continuity, no singularity can form at \( t = T \) (cf. [Giga, Hsu and Maekawa, 2014], for the case of the half-space)
the studies of the coherence of the vorticity direction up to the boundary-regularity criteria (for slip boundary conditions) were presented in [Beirao da Veiga and Berselli, 2002] and [Beirao da Veiga, 2013]

* * *

Zoran Grujić
Vortex stretching and local anisotropic diffusion in the 3D NSE
the studies of the coherence of the vorticity direction up to the boundary-regularity criteria (for slip boundary conditions) were presented in [Beirao da Veiga and Berselli, 2002] and [Beirao da Veiga, 2013]

* * *

assuming the type I blow-up (at most scaling-invariant blow-up rate),

$$|u(x, t)| \leq \frac{C'}{(T - t)^{\frac{1}{2}}}.$$ 

Giga and Miura [2011] showed that if the vorticity direction possesses a uniform modulus of continuity, no singularity can form at $t = T$

(cf. [Giga, Hsu and Maekawa, 2014], for the case of the half-space)
essentially, the unhappy scenario is 'crossing of the vortex lines' – the vorticity direction becomes discontinuous (in some sense) – as we approach the singularity [Holm and Kerr, 2002].
essentially, the unhappy scenario is ‘crossing of the vortex lines’ – the \textit{vorticity direction} becomes \textit{discontinuous} (in some sense) – as we approach the singularity
essentially, the unhappy scenario is ‘crossing of the vortex lines’ – the \textit{vorticity direction} becomes \textit{discontinuous} (in some sense) – as we approach the singularity.
local anisotropic diffusion and vortex stretching

**Definition**

Let \( x_0 \) be a point in \( \mathbb{R}^3 \), \( r > 0 \), \( S \) an open subset of \( \mathbb{R}^3 \) and \( \delta \) in \( (0, 1) \).

The set \( S \) is linearly \( \delta \)-sparse around \( x_0 \) at scale \( r \) in weak sense if there exists a unit vector \( d \) in \( S^2 \) such that

\[
\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.
\]
local anisotropic diffusion and vortex stretching

Definition

Let \( x_0 \) be a point in \( \mathbb{R}^3 \), \( r > 0 \), \( S \) an open subset of \( \mathbb{R}^3 \) and \( \delta \) in \((0, 1)\).

The set \( S \) is linearly \( \delta \)-sparse around \( x_0 \) at scale \( r \) in weak sense if there exists a unit vector \( d \) in \( S^2 \) such that

\[
\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.
\]

denote by \( \Omega_t(M) \) the vorticity super-level set at time \( t \); more precisely,

\[
\Omega_t(M) = \{ x \in \mathbb{R}^3 : |\omega(x, t)| > M \}\]
local anisotropic diffusion and vortex stretching

**Definition**

Let $x_0$ be a point in $\mathbb{R}^3$, $r > 0$, $S$ an open subset of $\mathbb{R}^3$ and $\delta$ in $(0, 1)$.

The set $S$ is linearly $\delta$-sparse around $x_0$ at scale $r$ in weak sense if there exists a unit vector $d$ in $S^2$ such that

$$\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.$$

Denote by $\Omega_t(M)$ the vorticity super-level set at time $t$; more precisely,

$$\Omega_t(M) = \{x \in \mathbb{R}^3 : |\omega(x, t)| > M\}$$

then the following holds [G., *Nonlinearity* 2013]
Theorem (local anisotropic diffusion)

Suppose that a solution $u$ is regular on an interval $(0, T^*)$.

Assume that either

(i) there exists $t$ in $(0, T^*)$ such that $t + \frac{1}{d_0^2 \| \omega(t) \|_{\infty}} \geq T^*$, or

(ii) $t + \frac{1}{d_0^2 \| \omega(t) \|_{\infty}} < T^*$ for all $t$ in $(0, T^*)$, and there exists $\epsilon$ in $(0, T^*)$ such that for any $t$ in $(T^* - \epsilon, T^*)$, there exists $s = s(t)$ in $\left[ t + \frac{1}{4d_0^2 \| \omega(t) \|_{\infty}}, t + \frac{1}{d_0^2 \| \omega(t) \|_{\infty}} \right]$ with the property that for any spatial point $x_0$, there exists a scale $r = r(x_0)$, $0 < r \leq \frac{1}{2d_0^2 \| \omega(t) \|_{\infty}^{\frac{1}{2}}}$, such that the super-level set $\Omega_s(M)$ is linearly $\delta$-sparse around $x_0$ at scale $r$ in weak sense; here, $\delta = \delta(x_0)$ is an arbitrary value in $(0, 1)$,

$h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$, $\alpha = \alpha(\delta) \geq \frac{1-h}{h}$, and $M = M(\delta) = \frac{1}{d_0^\alpha} \| \omega(t) \|_{\infty}$.

Then, there exists $\gamma > 0$ such that $\omega$ is in $L^\infty \left( (T^* - \epsilon, T^* + \gamma); L^\infty \right)$, i.e., $T^*$ is not a singular time. ({$d_0$} is a suitable absolute constant.)
a remark

it suffices to assume the sparseness condition at (suitably chosen) \textit{finitely many} times \slash \textit{intermittently} in time
a remark

it suffices to assume the sparseness condition at (suitably chosen) *finitely many times* / *intermittently* in time

main ingredients in the proof
a remark

it suffices to assume the sparseness condition at (suitably chosen) *finitely many* times / *intermittently* in time

main ingredients in the proof

(i) a local-in-time lower bound on the radius of spatial analyticity in $L^\infty$

(ii) translational and rotational symmetries

(iii) a consequence of the general harmonic measure majorization principle: let $D$ be open and $K$ closed in $C$, $f$ analytic in $D \setminus K$, $|f| \leq M$, and $|f| \leq m$ on $K$. then $|f(z)| \leq m \theta M^{1-\theta}$ for any $z$ in $D \setminus K$, where $\theta = h(z,D,K)$ is the harmonic measure of $K$ with respect to $D$ evaluated at $z$

(iv) a result on extremal properties of the harmonic measure in the unit disk [Solynin, 1999]

Zoran Grujić

Vortex stretching and local anisotropic diffusion in the 3D NSE
a remark

it suffices to assume the sparseness condition at (suitably chosen) \textit{finitely many} times / \textit{intermittently} in time

main ingredients in the proof

(i) a local-in-time lower bound on the radius of spatial analyticity in $L^\infty$

(ii) translational and rotational symmetries

\textbf{Zoran Grujić} 
Vortex stretching and local anisotropic diffusion in the 3D NSE
a remark

it suffices to assume the sparseness condition at (suitably chosen) finitely many times
/ intermittently in time

main ingredients in the proof

(i) a local-in-time lower bound on the radius of spatial analyticity in \( L^\infty \)

(ii) translational and rotational symmetries

(iii) a consequence of the general harmonic measure majorization principle:

let \( D \) be open and \( K \) closed in \( \mathbb{C} \), \( f \) analytic in \( D \setminus K \), \( |f| \leq M \), and \( |f| \leq m \) on \( K \). then

\[
|f(z)| \leq m^\theta M^{1-\theta}
\]

for any \( z \) in \( D \setminus K \), where \( \theta = h(z, D, K) \) is the harmonic measure of \( K \) with respect to \( D \) evaluated at \( z \)
a remark

it suffices to assume the sparseness condition at (suitably chosen) *finitely many times*/ *intermittently* in time

main ingredients in the proof

(i) a local-in-time lower bound on the radius of spatial analyticity in $L^\infty$

(ii) translational and rotational symmetries

(iii) a consequence of the general harmonic measure majorization principle:

let $D$ be open and $K$ closed in $\mathbb{C}$, $f$ analytic in $D \setminus K$, $|f| \leq M$, and $|f| \leq m$ on $K$. then

$$|f(z)| \leq m^\theta M^{1-\theta}$$

for any $z$ in $D \setminus K$, where $\theta = h(z, D, K)$ is the harmonic measure of $K$ with respect to $D$ evaluated at $z$

(iv) a result on extremal properties of the harmonic measure in the unit disk $\mathbb{D}$ [Solynin, 1999]
consider a flow near the first (possible) singular time $T^*$, and define the region of intense vorticity at time $s(t) < T^*$ to be the region in which the vorticity magnitude exceeds a fraction of $\|\omega(t)\|_\infty$; this corresponds to the set

$$\Omega_{s(t)}\left(\frac{1}{c_1} \|\omega(t)\|_\infty \right),$$

for some $c_1 > 1$
consider a flow near the first (possible) singular time $T^*$, and define the region of intense vorticity at time $s(t) < T^*$ to be the region in which the vorticity magnitude exceeds a fraction of $\|\omega(t)\|_{\infty}$; this corresponds to the set

$$\Omega_{s(t)} \left( \frac{1}{c_1} \|\omega(t)\|_{\infty} \right),$$

for some $c_1 > 1$

denote a suitable macro scale associated with the flow by $R_0$; the picture painted by numerical simulations indicates that the region of intense vorticity comprises – in statistically significant sense – of vortex filaments with the lengths comparable to $R_0$. 
consider a flow near the first (possible) singular time $T^*$, and define the region of intense vorticity at time $s(t) < T^*$ to be the region in which the vorticity magnitude exceeds a fraction of $\|\omega(t)\|_\infty$; this corresponds to the set

$$\Omega_{s(t)}\left(\frac{1}{c_1}\|\omega(t)\|_\infty\right),$$

for some $c_1 > 1$

denote a suitable macro scale associated with the flow by $R_0$; the picture painted by numerical simulations indicates that the region of intense vorticity comprises – in statistically significant sense – of vortex filaments with the lengths comparable to $R_0$

let us for a moment accept this as a plausible geometric blow up scenario; the length scale associated with the diameters of the cross-sections can then be estimated indirectly, by estimating the rate of the decrease of the total volume of the region of intense vorticity $\Omega_{s(t)}\left(\frac{1}{c_1}\|\omega(t)\|_\infty\right)$
taking the initial vorticity to be a bounded measure, Constantin [1990] showed that the $L^1$-norm of the vorticity is \textit{a priori} bounded over any finite time-interval; a desired estimate on the total volume of the region of intense vorticity follows simply from Tchebyshev inequality,

$$\text{Vol} \left( \Omega_{s(t)} \left( \frac{1}{c_1} \| \omega(t) \|_{\infty} \right) \right) \leq \frac{c_0^2}{\| \omega(t) \|_{\infty}} \quad (c_0^2 > 1)$$
taking the initial vorticity to be a bounded measure, Constantin [1990] showed that the $L^1$-norm of the vorticity is \textit{a priori} bounded over any finite time-interval; a desired estimate on the total volume of the region of intense vorticity follows simply from Tchebyshev inequality,

$$\text{Vol} \left( \Omega_{s(t)} \left( \frac{1}{c_1} \| \omega(t) \|_\infty \right) \right) \leq \frac{c_2^0}{\| \omega(t) \|_\infty} \quad (c_2^0 > 1)$$

this implies the decrease of the diameters of the cross-section of at least \( \frac{c_3^0}{\| \omega(t) \|_\infty^{\frac{1}{2}}} \) \((c_3^0 > 1)\), which is exactly the scale of \textit{local one-dimensional sparseness} of the region of intense vorticity [the scale of local anisotropic diffusion] needed to prevent the formation of singularities presented in the previous theorem.
taking the initial vorticity to be a bounded measure, Constantin [1990] showed that the $L^1$-norm of the vorticity is *a priori* bounded over any finite time-interval; a desired estimate on the total volume of the region of intense vorticity follows simply from Tchebyshev inequality,

$$\text{Vol} \left( \Omega_{s(t)} \left( \frac{1}{c_1} \| \omega(t) \|_\infty \right) \right) \leq \frac{c_2^0}{\| \omega(t) \|_\infty} \quad (c_2^0 > 1)$$

this implies the decrease of the diameters of the cross-section of at least $\frac{c_3^0}{\| \omega(t) \|_\infty^{\frac{1}{2}}}$ ($c_3^0 > 1$), which is exactly the scale of *local one-dimensional sparseness* of the region of intense vorticity [the scale of local anisotropic diffusion] needed to prevent the formation of singularities presented in the previous theorem

in other words, the NSE problem in this scenario becomes *critical*
it is instructive to check the scaling in the \textit{geometrically worst case scenario}, no sparseness – the super level set being clumped in a ball
it is instructive to check the scaling in the geometrically worst case scenario, no sparseness – the super level set being clumped in a ball

in this case, the criticality requires

$$\lambda \psi(t)(\beta) = O\left(\frac{1}{\beta^{3/2}}\right)$$

uniformly in \((T^* - \epsilon, T^*)\); here, \(\lambda\) denotes the distribution function
it is instructive to check the scaling in the geometrically worst case scenario, no sparseness – the super level set being clumped in a ball

in this case, the criticality requires

\[ \lambda_{\omega(t)}(\beta) = O\left(\frac{1}{\beta^{3/2}}\right) \]

uniformly in \((T^* - \epsilon, T^*)\); here, \(\lambda\) denotes the distribution function

this is a scaling-invariant condition – back to super-criticality

\[ O\left(\frac{1}{\beta^{3/2}}\right) \text{ vs. } O\left(\frac{1}{\beta^1}\right) \]
it is instructive to check the scaling in the geometrically worst case scenario, no sparseness – the super level set being clumped in a ball

in this case, the criticality requires

$$\lambda_\omega(t)(\beta) = O\left(\frac{1}{\beta^{3/2}}\right)$$

uniformly in \((T^* - \epsilon, T^*)\); here, \(\lambda\) denotes the distribution function

this is a scaling-invariant condition – back to super-criticality

$$O\left(\frac{1}{\beta^{3/2}}\right) \text{ vs. } O\left(\frac{1}{\beta^1}\right)$$

→ the vortex stretching in this scenario acts as the mechanism bridging (literally) the scaling gap in the regularity problem
back to the vortex-stretching term $S\omega \cdot \omega$ ($S$ is the symmetric part of $\nabla u$)
back to the vortex-stretching term $S\omega \cdot \omega$ ($S$ is the symmetric part of $\nabla u$)

one way to identify the range of (axial) scales at which the dynamics of creation and persistence of vortex filaments takes place is to identify the range of scales of positivity of $S\omega \cdot \omega$
back to the vortex-stretching term $S\omega \cdot \omega$ ($S$ is the symmetric part of $\nabla u$)

one way to identify the range of (axial) scales at which the dynamics of creation and persistence of vortex filaments takes place is to identify the range of scales of positivity of $S\omega \cdot \omega$

exploit a new spatial multi-scale averaging method designed to detect sign fluctuations of a quantity of interest across physical scales
back to the vortex-stretching term $S\omega \cdot \omega$ ($S$ is the symmetric part of $\nabla u$)

one way to identify the range of (axial) scales at which the dynamics of creation and persistence of vortex filaments takes place is to identify the range of scales of positivity of $S\omega \cdot \omega$

exploit a new spatial multi-scale averaging method designed to detect sign fluctuations of a quantity of interest across physical scales

introduced in the study of turbulent transport rates in 3D incompressible fluid flows:

let $B(0, R_0)$ be a macro-scale domain

a physical scale $R$, $0 < R \leq R_0$, is realized via suitable ensemble averaging of the localized quantities with respect to ‘$(K_1, K_2)$-covers at scale $R$’
let $B(0, R_0)$ be a macro-scale domain

a *physical scale* $R, \ 0 < R \leq R_0$, is realized via suitable ensemble averaging of the localized quantities with respect to $\{(K_1, K_2)\}$-covers at scale $R$

let $K_1$ and $K_2$ be two positive integers, and $0 < R \leq R_0$; a cover $\{B(x_i, R)\}_{i=1}^{n}$ of $B(0, R_0)$ is a $(K_1, K_2)$-cover at scale $R$ if

$$\left( \frac{R_0}{R} \right)^3 \leq n \leq K_1 \left( \frac{R_0}{R} \right)^3,$$

and any point $x$ in $B(0, R_0)$ is covered by at most $K_2$ balls $B(x_i, 2R)$

---

Zoran Grujić | Vortex stretching and local anisotropic diffusion in the 3D NSE
let $B(0, R_0)$ be a macro-scale domain

a physical scale $R$, $0 < R \leq R_0$, is realized via suitable ensemble averaging of the localized quantities with respect to ‘$(K_1, K_2)$-covers at scale $R$’

let $K_1$ and $K_2$ be two positive integers, and $0 < R \leq R_0$; a cover $\left\{ B(x_i, R) \right\}_{i=1}^n$ of $B(0, R_0)$ is a $(K_1, K_2)$-cover at scale $R$ if

$$\left( \frac{R_0}{R} \right)^3 \leq n \leq K_1 \left( \frac{R_0}{R} \right)^3,$$

and any point $x$ in $B(0, R_0)$ is covered by at most $K_2$ balls $B(x_i, 2R)$

the parameters $K_1$ and $K_2$ represent the maximal global and local multiplicities, respectively
for a physical density of interest \( f \), consider – localized to the cover elements \( B(x_i, R) \) (per unit mass) – local quantities \( \hat{f}_{x_i, R} \),

\[
\hat{f}_{x_i, R} = \frac{1}{R^3} \int_{B(x_i, 2R)} f(x) \psi^\delta_{x_i, R}(x) \, dx
\]

for some \( 0 < \delta \leq 1 \).
for a physical density of interest \( f \), consider – localized to the cover elements \( B(x_i, R) \) (per unit mass) – local quantities \( \hat{f}_{x_i,R} \),

\[
\hat{f}_{x_i,R} = \frac{1}{R^3} \int_{B(x_i, 2R)} f(x) \psi^\delta_{x_i,R}(x) \, dx
\]

for some \( 0 < \delta \leq 1 \)

denote by \( \langle F \rangle_R \) the \textit{ensemble average} given by

\[
\langle F \rangle_R = \frac{1}{n} \sum_{i=1}^{n} \hat{f}_{x_i,R}
\]
for a physical density of interest \( f \), consider – localized to the cover elements \( B(x_i, R) \) (per unit mass) – local quantities \( \hat{f}_{x_i,R} \),

\[
\hat{f}_{x_i,R} = \frac{1}{R^3} \int_{B(x_i,2R)} f(x) \psi^{\delta}_{x_i,R}(x) \, dx
\]

for some \( 0 < \delta \leq 1 \)

denote by \( \langle F \rangle_R \) the ensemble average given by

\[
\langle F \rangle_R = \frac{1}{n} \sum_{i=1}^{n} \hat{f}_{x_i,R}
\]
the key feature of \( \{ \langle F \rangle_R \}_{0 < R \leq R_0} \) is that \( \langle F \rangle_R \) being stable – i.e., nearly-independent on a particular choice of the cover (with the fixed local multiplicity \( K_2 \)) – indicates there are no significant sign fluctuations at scales comparable or greater than \( R \)
the key feature of \( \{ \langle F \rangle_R \}_{0 < R \leq R_0} \) is that \( \langle F \rangle_R \) being stable – i.e., nearly-independent on a particular choice of the cover (with the fixed local multiplicity \( K_2 \)) – indicates there are no significant sign fluctuations at scales comparable or greater than \( R \).

On the other hand, if \( f \) does exhibit significant sign fluctuations at scales comparable or greater than \( R \), suitable rearrangements of the cover elements up to the maximal multiplicity – emphasizing first the positive and then the negative parts of \( f \) – will result in \( \langle F' \rangle_R \) experiencing a wide range of values, from positive through zero to negative, respectively (the larger \( K_2 \), the finer detection..)
for a non-negative density \( f \), the ensemble averages are all comparable to each other throughout the full range of scales, \( 0 < R \leq R_0 \); in particular, they are all comparable to the simple average over the macro scale domain.
for a non-negative density $f$, the ensemble averages are all comparable to each other throughout the full range of scales, $0 < R \leq R_0$; in particular, they are all comparable to the simple average over the macro scale domain

$$\frac{1}{K_1} F_0 \leq \langle F \rangle_R \leq K_2 F_0$$

(3)

for all $0 < R \leq R_0$, where

$$F_0 = \frac{1}{R_0^3} \int f(x) \psi_0^\delta(x) \, dx$$
denote the time-averaged localized vortex-stretching terms per unit mass associated to the cover element $B(x_i, R)$ by $VST_{x_i, R, t}$,

$$VST_{x_i, R, t} = \frac{1}{t} \int_0^t \frac{1}{R^3} \int (\mathbf{\omega} \cdot \nabla) \mathbf{u} \cdot \mathbf{\omega} \phi_i \, dx \, ds$$  (4)
denote the time-averaged localized vortex-stretching terms per unit mass associated to the cover element $B(x_i, R)$ by $VST_{x_i, R, t}$,

\[ VST_{x_i, R, t} = \frac{1}{t} \int_0^t \frac{1}{R^3} \int (\omega \cdot \nabla) u \cdot \omega \phi_i \, dx \, ds \]  

the quantity of interest is the ensemble average of $\{VST_{x_i, R, t}\}_{i=1}^n$,

\[ \langle VST \rangle_{R, t} = \frac{1}{n} \sum_{i=1}^n VST_{x_i, R, t} \]
\[ B(x_i, R)\text{-localized enstrophy level dynamics is as follows} \]

\[ \int_{0}^{t} \int (\omega \cdot \nabla)u \cdot \phi_i \omega \, dx \, ds = \int \frac{1}{2} |\omega(x, t)|^2 \psi_i(x) \, dx + \int_{0}^{t} \int |\nabla \omega|^2 \phi_i \, dx \, ds \]
\[ - \int_{0}^{t} \int \frac{1}{2} |\omega|^2 ((\phi_i)_s + \Delta \phi_i) \, dx \, ds \]
\[ - \int_{0}^{t} \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds, \] (6)

for any \( t \) in \((2T/3, T)\), and \( 1 \leq i \leq n \)
denote by $E_{0,t}$ time-averaged enstrophy per unit mass associated with the macro scale domain $B(0, 2R_0) \times (0, t)$,

$$E_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{1/2} \, dx \, ds,$$
denote by $E_{0,t}$ time-averaged enstrophy per unit mass associated with the macro scale domain $B(0, 2R_0) \times (0,t)$,

$$E_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{1/2} \, dx \, ds,$$

by $P_{0,t}$ a modified time-averaged palinstrophy per unit mass,

$$P_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int |\nabla \omega|^2 \phi_0 \, dx \, ds + \frac{1}{t} \frac{1}{R_0^3} \int \frac{1}{2} |\omega(x,t)|^2 \psi_0(x) \, dx$$

(the modification is due to the shape of the temporal cut-off $\eta$),
denote by $E_{0,t}$ time-averaged enstrophy per unit mass associated with the macro scale domain $B(0, 2R_0) \times (0, t)$,

$$E_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{1/2} \, dx \, ds,$$

by $P_{0,t}$ a modified time-averaged palinstrophy per unit mass,

$$P_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int |\nabla \omega|^2 \phi_0 \, dx \, ds + \frac{1}{t} \frac{1}{R_0^3} \int \frac{1}{2} |\omega(x, t)|^2 \psi_0(x) \, dx$$

(the modification is due to the shape of the temporal cut-off $\eta$),

and by $\sigma_{0,t}$ a corresponding Kraichnan-type scale,

$$\sigma_{0,t} = \left( \frac{E_{0,t}}{P_{0,t}} \right)^{1/2}$$
then the following holds [Dascaliuc and G., *J. Math. Phys.* 2012]
then the following holds [Dascaliuc and G., *J. Math. Phys.* 2012]

**Theorem (vortex stretching/Taylor vs. v. Karman)**

Let \( u \) be a global-in-time local Leray solution on \( \mathbb{R}^3 \times (0, \infty) \), regular on \( (0, T) \).
Suppose that, for some \( t \in (2T/3, T) \),

\[
C \max\{M_0^{1/2}, R_0^{1/2}\} \sigma_{0,t}^{1/2} < R_0
\]

where \( M_0 = \sup_t \int_{B(0, 2R_0)} |u|^2 < \infty \), and \( C > 1 \) a suitable constant depending only on the cover parameters.

Then,

\[
\frac{1}{C} P_{0,t} \leq \langle VST \rangle_{R,t} \leq C P_{0,t}
\]

for all \( R \) satisfying

\[
C \max\{M_0^{1/2}, R_0^{1/2}\} \sigma_{0,t}^{1/2} \leq R \leq R_0.
\]
a couple of remarks
a couple of remarks

(i) suppose that $T$ is the first (possible) singular time, and that the macro scale domain contains some of the spatial singularities (at time $T$); this, paired with the assumption that $u$ is a global-in-time local Leray solution implies

$$\sigma_{0,t} \to 0, \ t \to T^-$$

hence, the condition (7) in the theorem is automatically satisfied for any $t$ near the singular time $T$
a couple of remarks

(i) suppose that $T$ is the first (possible) singular time, and that the macro scale domain contains some of the spatial singularities (at time $T$); this, paired with the assumption that $u$ is a global-in-time local Leray solution implies

$$\sigma_{0,t} \to 0, \ t \to T^-$$

hence, the condition (7) in the theorem is automatically satisfied for any $t$ near the singular time $T$

(ii) $P_{0,t} \to \infty, \ t \to T^- \quad \Rightarrow \quad$ the vortex stretching intensifies as we approach the singularity
a couple of remarks

(i) suppose that $T$ is the first (possible) singular time, and that the macro scale
domain contains some of the spatial singularities (at time $T$); this, paired with the
assumption that $u$ is a global-in-time local Leray solution implies

$$\sigma_{0,t} \to 0, \; t \to T^-$$

hence, the condition (7) in the theorem is automatically satisfied for any $t$ near the
singular time $T$

(ii) $P_{0,t} \to \infty, \; t \to T^- \quad \longrightarrow \quad$ the vortex stretching intensifies as we approach the
singularity

(iii) the power of $\frac{1}{2}$ on $\sigma_{0,t}$ is a correction originating in the localized transport term
any hope for *breaking the criticality* in this setting?
any hope for *breaking the criticality* in this setting?

one can try to get a bit of extra decay on the distribution function of the vorticity
any hope for *breaking the criticality* in this setting?

one can try to get a bit of extra decay on the distribution function of the vorticity

2 different (and somewhat complementary) results

any hope for *breaking the criticality* in this setting?

one can try to get a bit of extra decay on the distribution function of the vorticity

2 different (and somewhat complementary) results

the idea is to try to get a uniform-in-time estimate on

\[ \int \psi w_k \log w_k \, dx \quad \text{or} \quad \int \psi w \log w \, dx \]

\[ (w_k = \sqrt{1 + \omega_k^2}, \ w = \sqrt{1 + |\omega|^2}) \]
the maximal function of a distribution $f$ is defined as
$$M_h f(x) = \sup_{t > 0} |f \ast h_t(x)|$$
where $h$ is a fixed, normalized test function supported in the unit ball, and $h_t$ denotes $t^{-n} h(\cdot/t)$.

A distribution $f$ is in the Hardy space $H^1$ if
$$\|f\|_{H^1} = \|M_h f\|_1 < \infty$$

The local maximal function is defined as,
$$m_h f(x) = \sup_{0 < t < 1} |f \ast h_t(x)|, \quad x \in \mathbb{R}^n,$$

A distribution $f$ is in the local Hardy space $h^1$ if
$$\|f\|_{h^1} = \|m_h f\|_1 < \infty$$
the maximal function of a distribution $f$ is defined as

$$M_h f(x) = \sup_{t > 0} |f \ast h_t(x)|$$

where $h$ is a fixed, normalized test function supported in the unit ball, and $h_t$ denotes $t^{-n} h(\cdot/t)$

a distribution $f$ is in the Hardy space $\mathcal{H}^1$ if $\|f\|_{\mathcal{H}^1} = \|M_h f\|_1 < \infty$
the maximal function of a distribution $f$ is defined as

$$M_h f(x) = \sup_{t > 0} |f \ast h_t(x)|$$

where $h$ is a fixed, normalized test function supported in the unit ball, and $h_t$ denotes $t^{-n} h(\cdot/t)$

a distribution $f$ is in the Hardy space $\mathcal{H}^1$ if $\|f\|_{\mathcal{H}^1} = \|M_h f\|_1 < \infty$

the local maximal function is defined as,

$$m_h f(x) = \sup_{0 < t < 1} |f \ast h_t(x)|, \quad x \in \mathbb{R}^n,$$

a distribution $f$ is in the local Hardy space $\mathfrak{h}^1$ if $\|f\|_{\mathfrak{h}^1} = \|m_h f\|_1 < \infty$
Div-Curl Lemma (Coifman, Lions, Meyer, Semmes)

suppose that \( E \) and \( B \) are \( L^2 \)-vector fields satisfying \( \text{div} \, E = \text{curl} \, B = 0 \) (in the sense of distributions). then,

\[
\| E \cdot B \|_{\mathcal{H}^1} \leq c(n) \| E \|_{L^2} \| B \|_{L^2}
\]
Div-Curl Lemma (Coifman, Lions, Meyer, Semmes)

suppose that \( E \) and \( B \) are \( L^2 \)-vector fields satisfying \( \text{div} \ E = \text{curl} \ B = 0 \) (in the sense of distributions). then,

\[
\| E \cdot B \|_{H^1} \leq c(n) \| E \|_{L^2} \| B \|_{L^2}
\]

the classical space of bounded mean oscillations, \( BMO \) is defined as follows

\[
BMO = \left\{ f \in L^1_{\text{loc}} : \sup_{x \in \mathbb{R}^n, r > 0} \Omega(f, I(x, r)) < \infty \right\}
\]

where \( \Omega(f, I(x, r)) = \frac{1}{|I(x, r)|} \int_{I(x, r)} |f(x) - f_I| \, dx \) is the mean oscillation of the function \( f \) with respect to its mean \( f_I = \frac{1}{|I(x, r)|} \int_{I(x, r)} f(x) \, dx \), over the cube \( I(x, r) \) centered at \( x \) with the side-length \( r \).
a local version of $BMO$, usually denoted by $bmo$, is defined by finiteness of the following expression,

$$\|f\|_{bmo} = \sup_{x \in \mathbb{R}^n, 0 < r < \delta} \Omega(f, I(x, r)) + \sup_{x \in \mathbb{R}^n, r \geq \delta} \frac{1}{|I(x, r)|} \int_{I(x, r)} |f(y)| \, dy,$$

for some positive $\delta$. 

[The rest of the document contains additional text, possibly related to vortex stretching and local anisotropic diffusion in the 3D NSE.]
a local version of BMO, usually denoted by \( bmo \), is defined by finiteness of the following expression,

\[
\|f\|_{bmo} = \sup_{x \in \mathbb{R}^n, 0 < r < \delta} \Omega(f, I(x, r)) + \sup_{x \in \mathbb{R}^n, r \geq \delta} \frac{1}{|I(x, r)|} \int_{I(x, r)} |f(y)| \, dy,
\]

for some positive \( \delta \)

if \( f \in L^1 \), we can focus on small scales, e.g., \( 0 < r < \frac{1}{2} \). Let \( \phi \) be a positive, non-decreasing function on \( (0, \frac{1}{2}) \), and consider the following version of local weighted spaces of bounded mean oscillations,

\[
\|f\|_{\tilde{bmo}_\phi} = \|f\|_{L^1} + \sup_{x \in \mathbb{R}^n, 0 < r < \frac{1}{2}} \frac{\Omega(f, I(x, r))}{\phi(r)}
\]
a local version of $BMO$, usually denoted by $bmo$, is defined by finiteness of the following expression,

$$
\|f\|_{bmo} = \sup_{x \in \mathbb{R}^n, 0 < r < \delta} \Omega(f, I(x, r)) + \sup_{x \in \mathbb{R}^n, r \geq \delta} \frac{1}{|I(x, r)|} \int_{I(x, r)} |f(y)| \, dy,
$$

for some positive $\delta$

if $f \in L^1$, we can focus on small scales, e.g., $0 < r < \frac{1}{2}$. Let $\phi$ be a positive, non-decreasing function on $(0, \frac{1}{2})$, and consider the following version of local weighted spaces of bounded mean oscillations,

$$
\|f\|_{\tilde{bmo}_\phi} = \|f\|_{L^1} + \sup_{x \in \mathbb{R}^n, 0 < r < \frac{1}{2}} \frac{\Omega(f, I(x, r))}{\phi(r)}
$$

of special interest will be the spaces $\tilde{bmo} = \tilde{bmo}_1$, and $\tilde{bmo} \frac{1}{|\log r|}$
\((\mathcal{H}^1)^* = BMO\) and \((\mathfrak{h}^1)^* = bmo\); the duality is realized via integration of one object against the other.
\((\mathcal{H}^1)^* = BMO\) and \((\mathcal{H}^1)^* = bmo\); the duality is realized via integration of one object against the other

a sharp pointwise multiplier theorem.

let \(h\) be in \(\tilde{bmo}\), and \(g\) in \(L^\infty \cap \tilde{bmo} \frac{1}{|\log r|}\). then

\[
\|gh\|_{\tilde{bmo}} \leq c(n) \left( \|g\|_\infty + \|g\|_{\tilde{bmo} \frac{1}{|\log r|}} \right) \|h\|_{\tilde{bmo}}
\]

more precisely, the space of pointwise \(\tilde{bmo}\) multipliers coincides with \(L^\infty \cap \tilde{bmo} \frac{1}{|\log r|}\)
let $M$ denote the Hardy-Littlewood maximal operator

Coifman and Rochberg

$$\| \log Mf \|_{BMO} \leq c(n)$$

for any locally integrable function $f$ (the bound is completely independent of $f$.)
let $M$ denote the Hardy-Littlewood maximal operator

Coifman and Rochberg

$$\| \log Mf \|_{BMO} \leq c(n)$$

for any locally integrable function $f$ (the bound is completely independent of $f$.)

this estimate remains valid if we replace $Mf$ with $\mathcal{M}f = (M\sqrt{|f|})^2$. the advantage of working with $\mathcal{M}$ is that the $L^2$-maximal theorem implies the following estimate

$$\| \mathcal{M}f \|_1 \leq c(n)\|f\|_1$$

(a bound that does not hold for the original maximal operator $M$.)
for any $\tau$ in $[0, T)$,

$$I(\tau) \equiv \int \psi(x) w(x, \tau) \log w(x, \tau) \, dx \leq I(0) + c \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log w \, dx \, dt\] + \textit{a priori} \text{ bounded}$$
for any $\tau$ in $[0, T)$,

$$I(\tau) \equiv \int \psi(x) w(x, \tau) \log w(x, \tau) \, dx \leq I(0) + c \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log w \, dx \, dt$$

$$+ \text{ a priori \ bounded}$$

in order to take the advantage of the Coifman-Rochberg’s estimate, we decompose the logarithmic factor as

$$\log w = \log \frac{w}{M w} + \log M w$$
denoting $\int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log w \, dx \, dt$ by $J$, this yields $J = J_1 + J_2$ where

$$J_1 = \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log \frac{w}{\mathcal{M}w} \, dx \, dt$$

and

$$J_2 = \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi \xi \log \mathcal{M}w \, dx \, dt$$
for $J_1$, we use the \textit{pointwise} inequality

$$w \log \frac{w}{\mathcal{M}w} \leq \mathcal{M}w - w$$

(a consequence of the pointwise inequality $\mathcal{M}f \geq f$, and the inequality $e^{x-1} \geq x$ for $x \geq 1$)
for $J_1$, we use the pointwise inequality

$$w \log \frac{w}{Mw} \leq Mw - w$$

(a consequence of the pointwise inequality $Mf \geq f$, and the inequality $e^{x-1} \geq x$ for $x \geq 1$)

this leads to

$$J_1 \leq \int_{0}^{\tau} \int_{x} |\nabla u| (Mw - w) \psi \, dx \, dt$$

which is a priori bounded by the Cauchy-Schwarz and the $L^2$-maximal theorem
for $J_2$, we have the following string of inequalities

\[
J_2 \leq c \int_0^T \| \omega \cdot \nabla u \|_{L^1} \| \psi \xi \log Mw \|_{bmo} dt
\]

\[
\leq c \int_0^T \| \omega \cdot \nabla u \|_{L^1} \| \psi \xi \log Mw \|_{\tilde{bmo}} dt
\]

\[
\leq c \int_0^T \| \omega \|_2 \| \nabla u \|_2 \left( \| \psi \xi \|_\infty + \| \psi \xi \|_{\tilde{bmo}} \frac{1}{\| \log r \|} \right) \left( \| \log Mw \|_{BMO} + \| \log Mw \|_1 \right) dt
\]

\[
\leq c \sup_{t \in (0,T)} \left\{ \left( 1 + \| \psi \xi \|_{\tilde{bmo}} \frac{1}{\| \log r \|} \right) \left( \| \log Mw \|_{BMO} + \| \log Mw \|_1 \right) \right\} \int_t^T \int_x |\nabla u|^2
\]

\[
\leq c \left( 1 + \sup_{t \in (0,T)} \| \psi \xi \|_{\tilde{bmo}} \frac{1}{\| \log r \|} \right) \left( 1 + \sup_{t \in (0,T)} \| \omega \|_1 \right) \int_t^T \int_x |\nabla u|^2
\]

by $L^1 - bmo$ duality, the Div-Curl Lemma, the pointwise $\tilde{bmo}$-multiplier theorem, the Coifman-Rochberg’s estimate, and the $L^1$-bound on the modified maximal operator $M$. 

Zoran Grujić
Vortex stretching and local anisotropic diffusion in the 3D NSE
this implies the following result [Bradshaw and G., Indiana Univ. Math. J. 2014]

**Theorem (breaking the criticality/the log-log-chaos sphere)**

Let $u$ be a Leray solution to the 3D NSE. Assume that the initial vorticity $\omega_0$ is in $L^1 \cap L^2$, and that $T > 0$ is the first (possible) blow-up time. Suppose that

$$\sup_{t \in (0, T)} \| (\psi \xi)(\cdot, t) \|_{bmo} \frac{1}{|\log r|} < \infty.$$ 

Then,

$$\sup_{t \in (0, T)} \int \psi(x) w(x, t) \log w(x, t) \, dx < \infty.$$
good news:
\( \tilde{bmo}_\phi \) contains discontinuous functions if and only if
\[
\int_0^1 \frac{\phi(r)}{r} \, dr = \infty
\]
good news:
\[ \widetilde{\text{bmo}}_\phi \text{ contains discontinuous functions if and only if } \int_0^\infty \frac{\phi(r)}{r} \, dr = \infty \]

in particular, \( \text{bmo} \frac{1}{|\log r|} \) contains bounded functions with the singularities of, say,

\[ \sin \log |\log(\text{something algebraic})| \]-type
good news:

\( \tilde{bmo}_\phi \) contains discontinuous functions if and only if

\[
\int_0^1 \frac{\phi(r)}{r} dr = \infty
\]

in particular, \( \tilde{bmo} \left( \frac{1}{|\log r|} \right) \) contains bounded functions with the singularities of, say,

\[
\sin \log |\log(\text{something algebraic})| \text{-type}
\]

\( \xi \) can (as it approaches \( T \)) oscillate among infinitely many points on the unit sphere –

\[
\xi(\text{sing}_x, T) \sim \quad \text{– and still yield extra-log decay of the distribution function of } \omega
\]
good news:
\( \widetilde{bmo}_\phi \) contains discontinuous functions if and only if
\[
\int_0^1 \frac{\phi(r)}{r} \, dr = \infty
\]

in particular, \( \widetilde{bmo} \frac{1}{| \log r |} \) contains bounded functions with the singularities of, say,
\[
\sin \log | \log(\text{something algebraic}) | - \text{type}
\]
\( \xi \) can (as it approaches \( T \)) oscillate among infinitely many points on the unit sphere –
\[
\xi(\text{sing}_x, T) \sim \quad \text{(Diagram of oscillation)}
\]
– and still yield extra-log decay of the distribution function of \( \omega \)

[in particular, ‘crossing of the vortex lines’ is not an obstruction]
conclusion:
conclusion:

* for a physically, numerically and mathematical analysis-motivated scenario
conclusion:

* for a physically, numerically and mathematical analysis-motivated scenario

\[ \|\xi(\cdot, T)\| < \infty \implies \lambda_{\omega(t)}(\beta) = O\left(\frac{1}{\beta \log \beta}\right) \implies \text{anisotropic diffusion wins} \]
Vortex stretching and local anisotropic diffusion in the 3D NSE