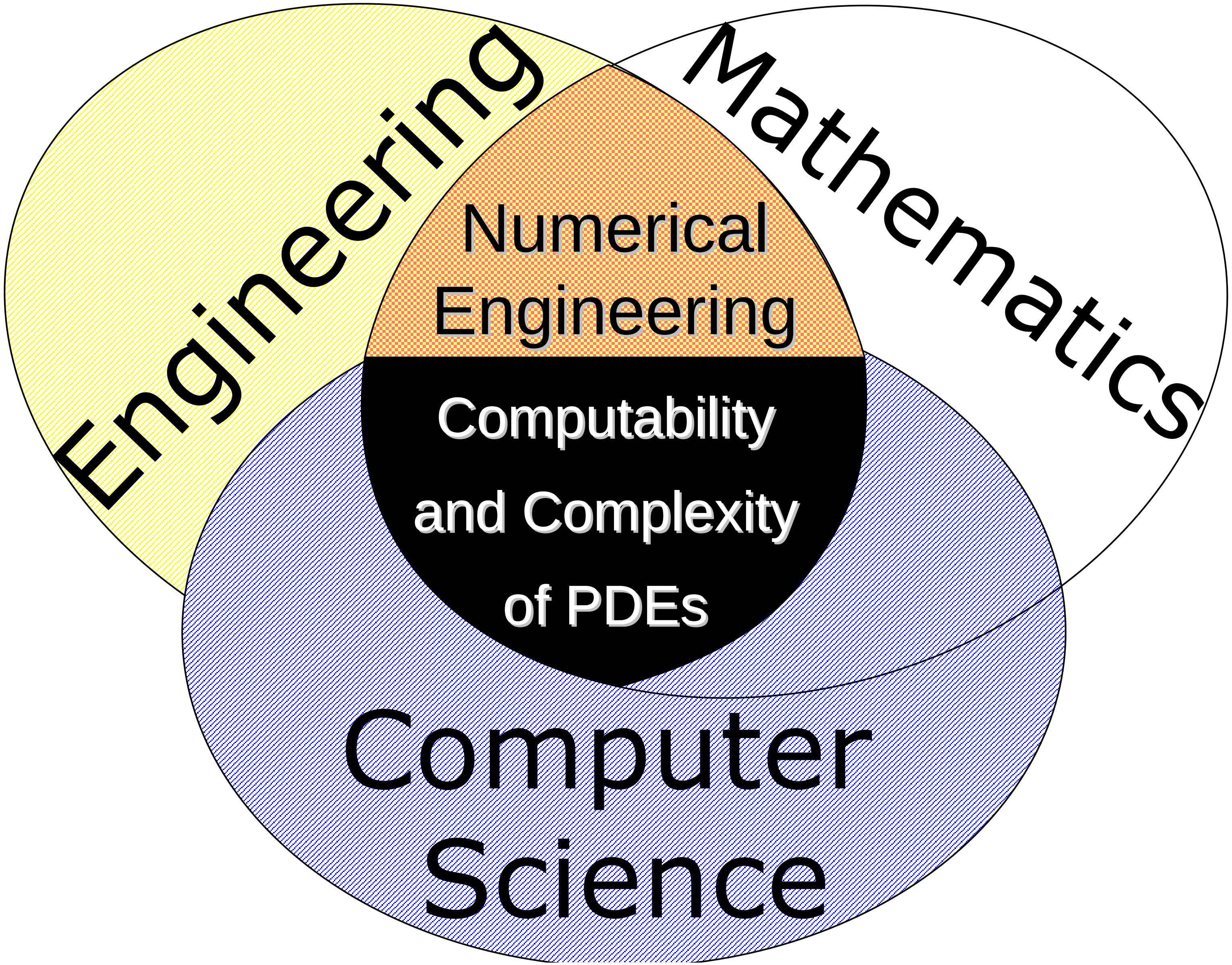
Martin ZIEGLER, Mar.13, IRTG1529, WASEDA

Computability and Complexity Theory of PDEs

Algorithmic Foundations of Numerics



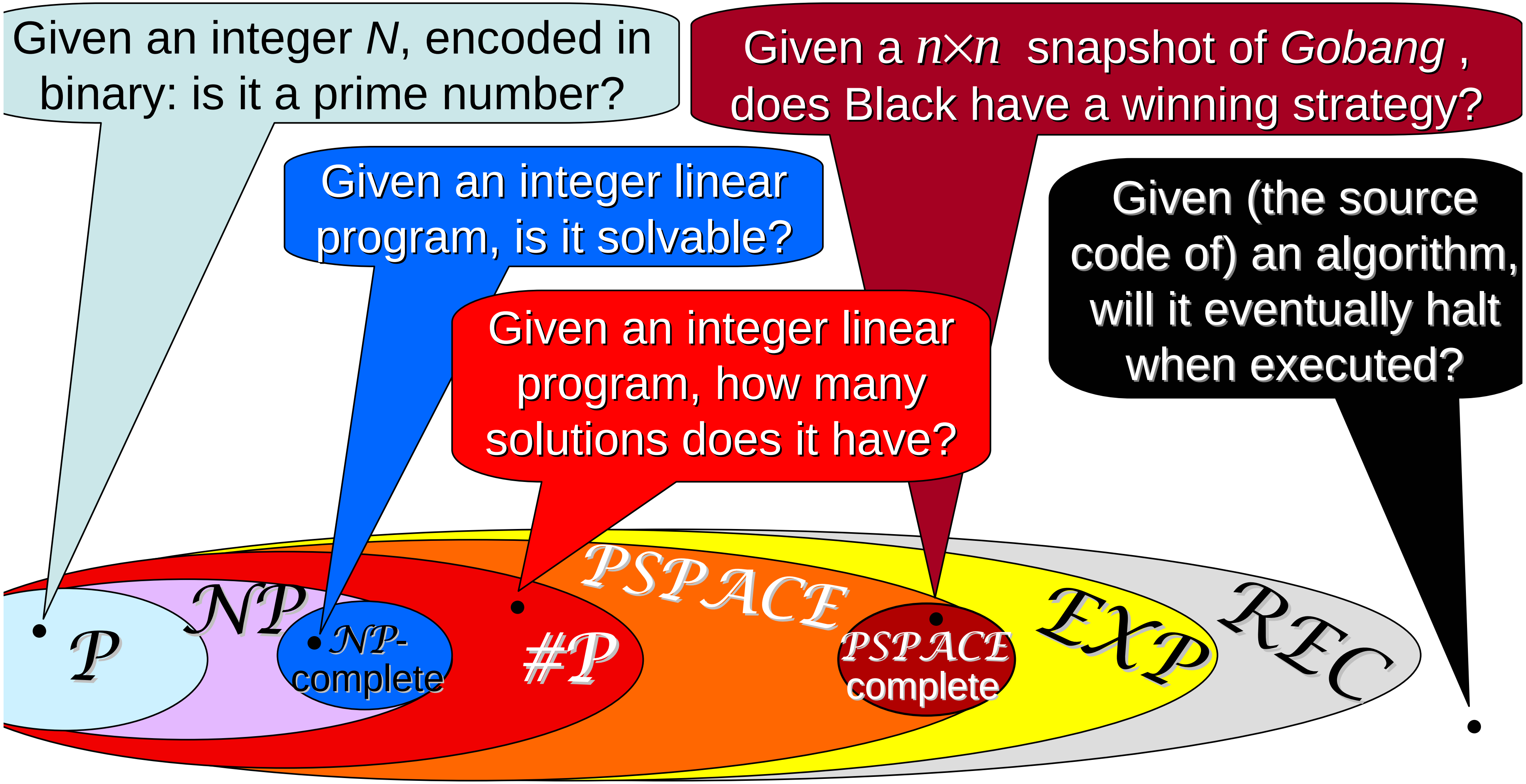
Overview

1. Theory of Computing
   over countable, discrete sets
2. Theory of Computing
   over separable metric spaces
3. Un-/computability of PDEs
4. Complexity Theory of PDEs
5. Encodings of function spaces
6. Summary and Perspectives

Theory of Computing over countable, discrete sets

Input: finite bit string / encoded integer, graph…; **Output: 0/1**

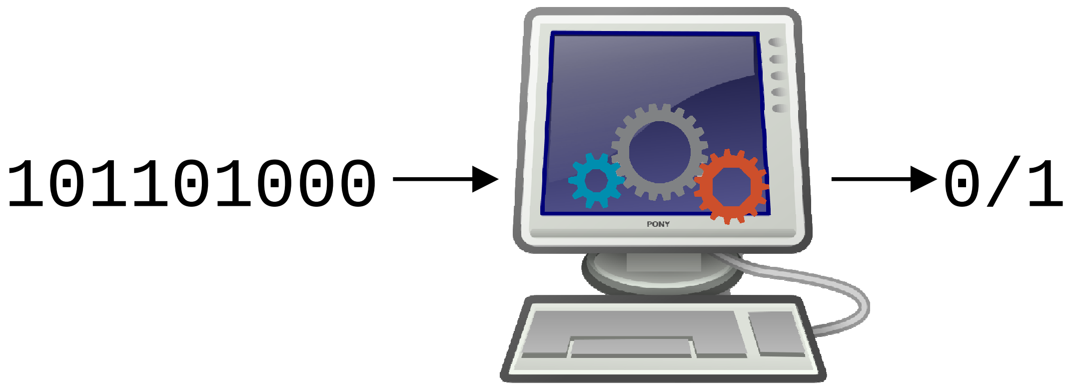
Runtime asymptotic growth w.r.t. binary input length $n \to \infty$

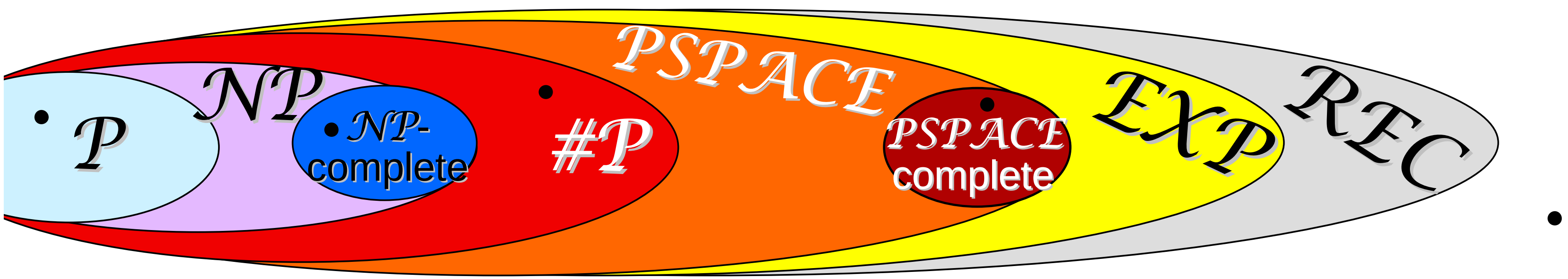


Theory of Computing over countable, discrete sets

Input: finite bit string / encoded integer, graph…; **Output: 0/1**

Runtime asymptotic growth w.r.t. <u>binary input length *n*</u>→∞





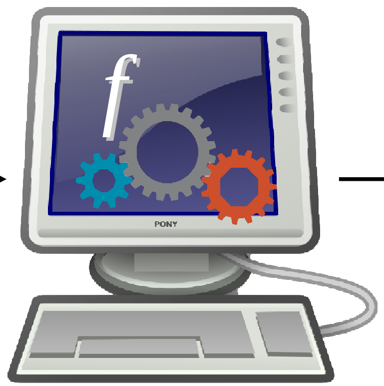
Theory of Computing over separable metric spaces X,Y

Input: finite bit string / encoded integer, graph…; **Output: 0/1**

Runtime asymptotic growth w.r.t. binary input length $n\to\infty$

$x\in X$

10110100010111… →



→ 0100011011…

$y=f(x)\in Y$

independent of x: for compact X

Definition: a) **Input:** infinite bit string encoding some $x\in X$ by a sequence of approximations

b) **Output:** infinite bit string encoding $y=f(x)\in Y$ by a sequence of approximations

c) **Runtime** asymptotically w.r.t. binary output length $n\to\infty$

Theory of Computing over **separable metric spaces**

Fact: • (i)–(iv) computably equivalent; stricly stronger than v)
• Algorithm can convert (ii) ↔ (iii) ↔ (iv); but not → (i)
• Conversion runtime (ii) ↔ (iii) is polynomial, (iv) → (ii) not
• $+, \times, \exp, \ln, \ldots$ computable in polynom. time w.r.t. (ii), (iii)

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i) sequence of binary coefficients $(b_n) \subseteq \{0,1\}$ s.t. $r=\sum_n b_n 2^{-n}$
ii) sequence of *signed* bin. coefficients $(c_n) \subseteq \{\mathbf{-1},0,1\}$ s.t. $r=\sum_n c_n 2^{-n}$
iii) sequence of dyadic approximations $(a_n) \subseteq \mathbb{Z}$ s.t. $|r-a_n/2^n| \leq 2^{-n}$
iv) unbounded sequences $(a_n), (b_n), (c_n) \subseteq \mathbb{Z}$ s.t. $|r-a_n/b_n| \leq 1/c_n$
v) sequence $(a_n) \subseteq \mathbb{Z}$ s.t. $a_n/2^n \to r$ as $n \to \infty$.

Definition: a) Input: infinite bit string
encoding some $x \in X$ by a sequence of approximations
b) Output: infinite bit string
encoding $y=f(x) \in Y$ by a sequence of approximations
c) Runtime asymptotically w.r.t. binary output length $n \to \infty$

Theory of <u>Encodings</u> of separable metric spaces

Fact: • (i)–(iv) computably equivalent;
• Algorithm can convert (ii) $\leftrightarrow$ (iii) $\leftrightarrow$ (iv);
• Conversion runtime (ii) $\leftrightarrow$ (iii) is polynomial,
• $+, \times, \exp, \ln, \ldots$ computable in polynom. time w.r.t. (ii), (iii)

Research programme: Take a space X with operations $f,g,\ldots$
• Devise 'reasonable' encodings of X over infinite binar.strings
• that render given operations $f,g,\ldots$ (polyn.time) computable.
• Compare&classify encodings w.r.t. computable equivalence, w.r.t. computable conversion, polynomial-time convertability

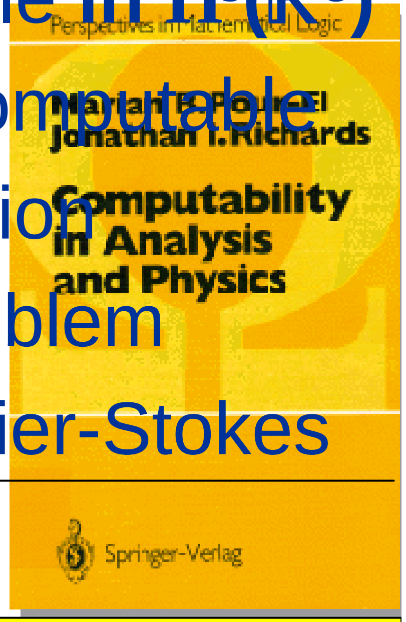
Examples: a) $\mathbb{R}$ with $+, \times, \exp, \ln, \lim, <$ — w.r.t. computability
b) Finite/countable Cartesian products, quotients, unions, …
c) Compact Euclidean sets d) space $C(X,Y)$ of cont. func.s
e) Analytic functions f) Gevrey's Hierarchy [Kawamura, **Z**, …]

Computability of PDEs

- Weierstrass function → [Myhill'71] computable $h\in C^1(\mathbb{R})$ with uncomputable derivative $h'(1)$ and $h\equiv 0$ on $[0,½]$
- [Pour-El&Richards'81] 3D wave equation with initial condition $f(\underline{x}):=h(|\underline{x}|)$, $g(\underline{x}):\equiv 0$ → **un**computable $u(1,\underline{0})$

$$u(t,0) = \frac{d}{dt}\big(h(t)\cdot t\big) = h'(t)\cdot t + h(t)$$

- [Weihrauch&Zhong'02] Wave eq. computable **in** $\mathbf{H}^s(\mathbb{R}^3)$
- [Weihrauch&Zhong'05] solution to KdV is computable
- [Weihrauch&Zhong'06] Schrödinger's equation
- [Weihrauch&Zhong'07] abstract Cauchy problem
- [Sun&Zhong&**Z.**'15] L^2-computability of Navier-Stokes



Church-Turing Hypothesis (Kleene):

Anything 'computed' by a physical device
can also be approximated by a Turing machine

$$\partial^2/\partial t^2\, u(t,\underline{x}) = \Delta u(t,\underline{x}),\quad u(0,\underline{x})=f(\underline{x}),\quad \partial/\partial t\, u(0,\underline{x})=g(\underline{x})$$

Non-Uniform Complexity of Operators on C[0;1]

Fix polyn.time computable (⇒ continuous) $f:[0;1]\to[0;1]$

- Max: $f \to \mathrm{Max}(f): x \to \max\{\, f(t): t \le x \,\}$
  $\mathrm{Max}(f)$ computable in exponent. time; polyn.time-computable iff $\mathcal{P}=\mathcal{NP}$

even when restricting to $f\in C^\infty$

- $\int: f \to \int f: (x \to \int_0^x f(t)\,dt)$
  $\int f$ computable in exponential time; polynom-time computable iff $\mathcal{P}=\#\mathcal{P}$

polytime if f is analytic

[Friedman&Ko'82ff]

- odesolve: $C^1([0;1]\times[-1;1])\ni f \to z\text{:}\;\; \dot z(t)=f(t,z),\;\; z(0)=0.$
  $\mathcal{PSPACE}$-"complete"

[Kawamura,Ota,Rösnick,**Z.**'14]

- Solution to Poisson's Equation is classical and $\#\mathcal{P}$-"complete"

$$\Delta u = f \;\text{ on } B_2(\mathbf{0},1) \qquad u = 0 \text{ on } \partial B_2(\mathbf{0},1)$$

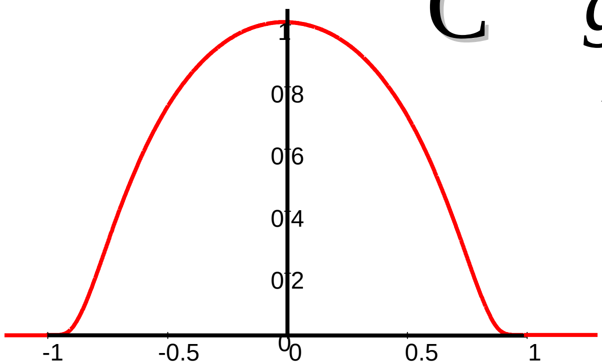
[Kawamura, Steinberg, **Z.**'13]

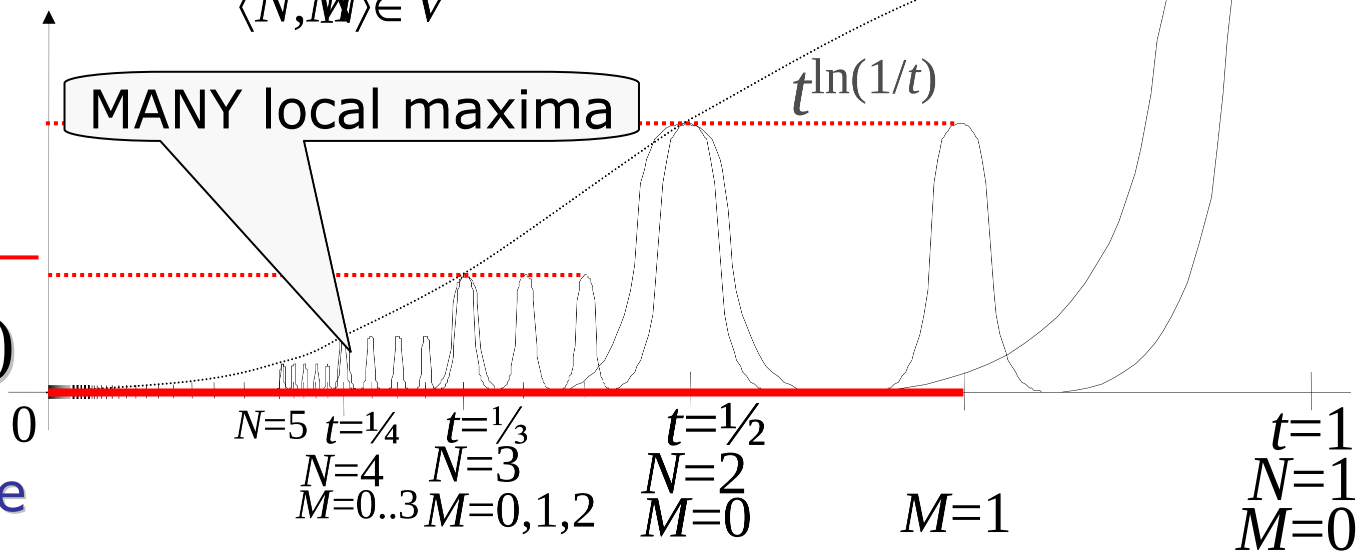
$\mathcal{P} \subseteq \mathcal{NP} \subseteq \#\mathcal{P} \subseteq \mathcal{PSPACE} \subseteq \mathcal{EXP}$

'Max is $\mathcal{NP}$-hard'

$\mathcal{NP} \ni L = \{\, N\in\mathbb{N} \mid \exists M<N: \langle N,M\rangle \in V \,\},\ V\in\mathcal{P}$ polytime

$C^\infty\quad g_V: t \to \sum_{\langle N,M\rangle\in V} \varphi(2tN^2-2N^2-M)/N^{\ln N}$





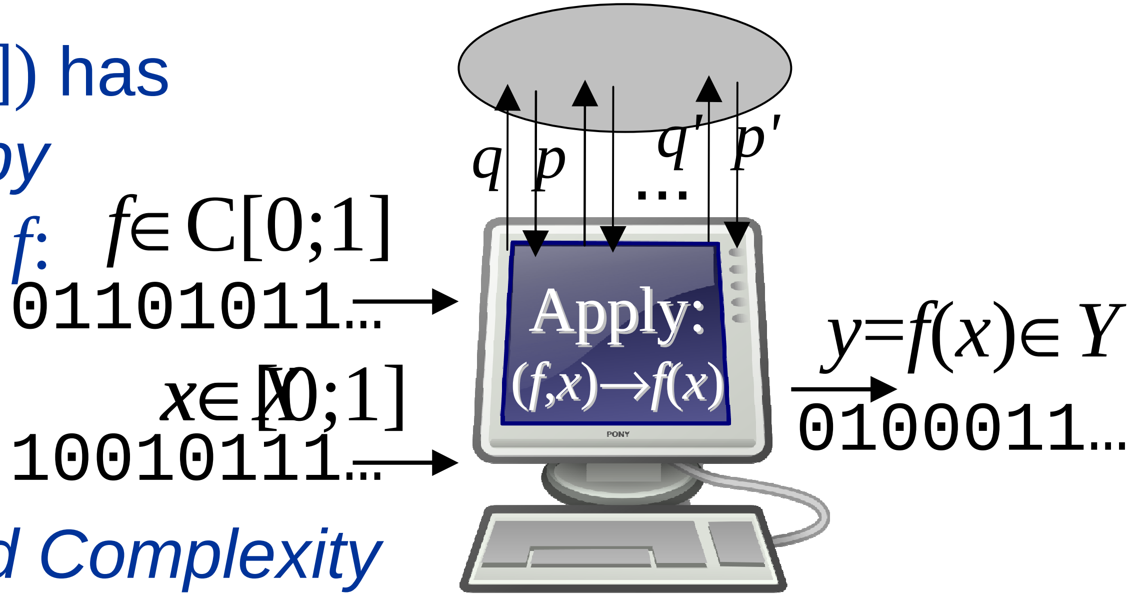
$\varphi(t) = \exp(-t^2/1-t^2)$

C^∞ 'pulse' function
polytime computable

To every $L\in\mathcal{NP}$ there exists a polytime computable C^∞ function $g_L:[0,1]\to\mathbb{R}$ s.t.: $[0,1]\ni t\to\max g_L|_{[0,t]}$ again polytime iff $L\in\mathcal{P}$

Uniform Complexity of Operators
Encoding Function Spaces

- There <u>is no</u> encoding of the (compact) set $\mathrm{Lip}_1([0;1],[0;1])$ rendering application $(f,x)\to f(x)$ polynomial-time computable:
- Because $\mathrm{Lip}_1([0;1],[0;1])$ has too large a *metric entropy* for <u>sequential</u> access to f: encoding over strings
- Give <u>oracle</u> access to f, cmp. *Information-Based Complexity*



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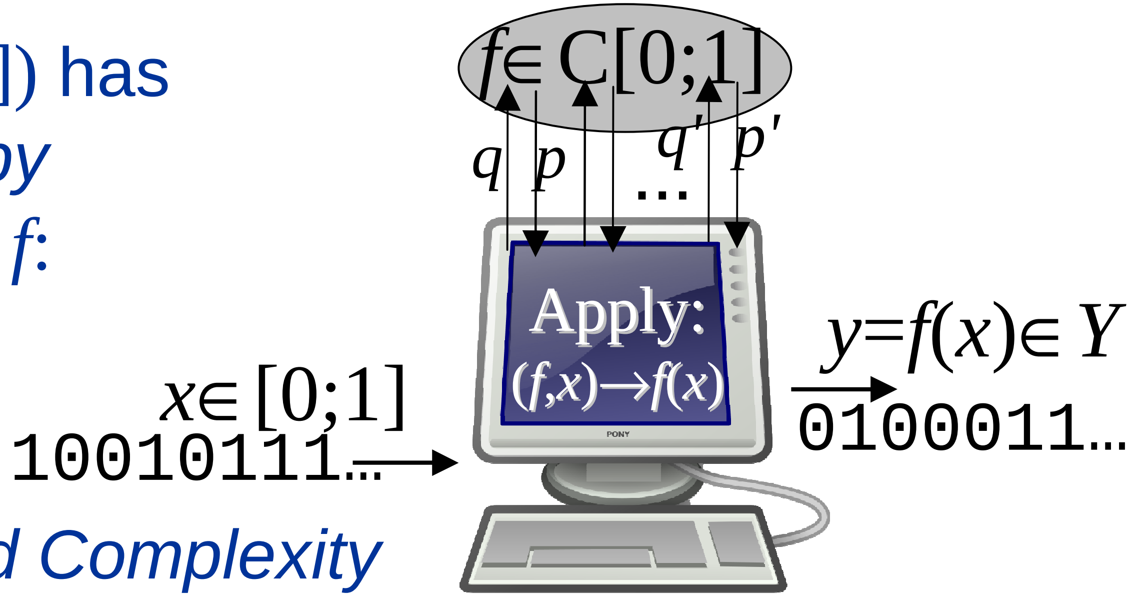

- [Kawamura&Cook'10]: A 2nd-order representation is an encoding over graded Baire space of binary string *functions*
- $C[0;1]$ is not (even σ-)compact, no runtime bounds in n only:
- [Kawamura&Cook'10]: 2nd-order polynomial time on C[0;1]: bounded by a polynomial in n and a modulus of continuity of f

Summary and Perspectives

- Rigorous algorithmic foundation of numerical calculations:

- Systematic study of encodings of separable metric spaces
- comparison/classification w.r.t. (polyn.-time) computability

- Purportedly "easy" computational problems over the reals actually correspond to known algorithmically hard ones: Millennium Prize $\mathcal{P}/\mathcal{NP}$ or even undecidable Halting problem.

- Previous results on computability of solutions to PDEs
- to refine from the perspective of computational complexity
- uniformly, i.e. with initial data "given" by encoding as oracle
- require non-classical function spaces for well-posedness

- Canonical encoding of L^p and Sobolev spaces?
- Counterpart to *modulus of continuity* as runtime parameter?